9.7 (a) Pivoting is necessary, so switch the first and third rows,

\[-8x_1 + x_2 - 2x_3 = -20\]
\[-3x_1 - x_2 + 7x_3 = -34\]
\[2x_1 - 6x_2 - x_3 = -38\]

Multiply the first equation by \(-3/(-8)\) and subtract the result from the second equation to eliminate the \(a_{21}\) term from the second equation. Then, multiply the first equation by \(2/(-8)\) and subtract the result from the third equation to eliminate the \(a_{31}\) term from the third equation.

\[-8x_1 + x_2 - 2x_3 = -20\]
\[-1.375x_2 + 7.75x_3 = -26.5\]
\[-5.75x_2 - 1.5x_3 = -43\]

Pivoting is necessary so switch the second and third row,

\[-8x_1 + x_2 - 2x_3 = -20\]
\[-5.75x_2 - 1.5x_3 = -43\]
\[-1.375x_2 + 7.75x_3 = -26.5\]

Multiply pivot row 2 by \(-1.375/(-5.75)\) and subtract the result from the third row to eliminate the \(a_{32}\) term.

\[-8x_1 + x_2 - 2x_3 = -20\]
\[-5.75x_2 - 1.5x_3 = -43\]
\[8.108696x_3 = -16.21739\]

At this point, the determinant can be computed as

\[D = -8 \times -5.75 \times 8.108696 \times (-1)^2 = 373\]

The solution can then be obtained by back substitution

\[x_3 = \frac{-16.21739}{8.108696} = -2\]
\[x_2 = \frac{-43 + 1.5(-2)}{-5.75} = 8\]
\[x_1 = \frac{-20 + 2(-2) - 1(8)}{-8} = 4\]
(b) Check:

\[ 2(4) - 6(8) - (-2) = -38 \]
\[ -3(4) - (8) + 7(-2) = -34 \]
\[ -8(4) + (8) - 2(-2) = -20 \]
10.3 (a) The coefficient $a_{3i}$ is eliminated by multiplying row 1 by $f_{31} = -0.3$ and subtracting the result from row 2. $a_{3i}$ is eliminated by multiplying row 1 by $f_{31} = 0.1$ and subtracting the result from row 3. The factors $f_{31}$ and $f_{32}$ can be stored in $a_{31}$ and $a_{32}$.

\[
\begin{bmatrix}
10 & 2 & -1 \\
-0.3 & -5.4 & 1.7 \\
0.1 & -0.14815 & 5.3519
\end{bmatrix}
\]

$a_{4i}$ is eliminated by multiplying row 2 by $f_{42} = -0.14815$ and subtracting the result from row 3. The factor $f_{42}$ can be stored in $a_{42}$.

\[
\begin{bmatrix}
10 & 2 & -1 \\
-0.3 & -5.4 & 1.7 \\
0.1 & -0.14815 & 5.3519
\end{bmatrix}
\]

Therefore, the $LU$ decomposition is

\[
[L] = \begin{bmatrix}
1 & 0 & 0 \\
-0.3 & 1 & 0 \\
0.1 & -0.14815 & 1
\end{bmatrix} \quad [U] = \begin{bmatrix}
10 & 2 & -1 \\
0 & -5.4 & 1.7 \\
0 & 0 & 5.3519
\end{bmatrix}
\]

These two matrices can be multiplied to yield the original system. For example, using MATLAB to perform the multiplication gives

\[
>> L = \begin{bmatrix}
1 & 0 & 0 \\
-0.3 & 1 & 0 \\
0.1 & -0.14815 & 1
\end{bmatrix} \\
>> U = \begin{bmatrix}
10 & 2 & -1 \\
0 & -5.4 & 1.7 \\
0 & 0 & 5.3519
\end{bmatrix} \\
>> A=L*U
\]

\[
A =
\begin{bmatrix}
10.0000 & 2.0000 & -1.0000 \\
-3.0000 & -6.0000 & 2.0000 \\
1.0000 & 1.0000 & 5.0000
\end{bmatrix}
\]
10.4 (a) Forward substitution: \([L][D] = [B]\)

\[
\begin{bmatrix}
1 & 0 & 0 & | & d_1 \\
-0.3 & 1 & 0 & | & d_2 \\
0.1 & -0.14815 & 1 & | & d_3 \\
\end{bmatrix}
= 
\begin{bmatrix}
27 \\
-61.5 \\
-21.5 \\
\end{bmatrix}
\]

Solving yields \(d_1 = 27\), \(d_2 = -53.4\), and \(d_3 = -32.1111\).

Back substitution:

\[
\begin{bmatrix}
10 & 2 & -1 & | & x_1 \\
0 & -5.4 & 1.7 & | & x_2 \\
0 & 0 & 5.351852 & | & x_3 \\
\end{bmatrix}
= 
\begin{bmatrix}
27 \\
-53.4 \\
-32.1111 \\
\end{bmatrix}
\]

\[
x_3 = \frac{-32.11111}{5.351852} = -6
\]

\[
x_2 = \frac{-53.4 - 1.7(-6)}{-5.4} = 8
\]

\[
x_1 = \frac{27 + 1(-6) - 2(8)}{10} = 0.5
\]

(b) Forward substitution: \([L][D] = [B]\)

\[
\begin{bmatrix}
1 & 0 & 0 & | & d_1 \\
-0.3 & 1 & 0 & | & d_2 \\
0.1 & -0.14815 & 1 & | & d_3 \\
\end{bmatrix}
= 
\begin{bmatrix}
12 \\
18 \\
-6 \\
\end{bmatrix}
\]

Solving yields \(d_1 = 12\), \(d_2 = 21.6\), and \(d_3 = -4\).

Back substitution:

\[
\begin{bmatrix}
10 & 2 & -1 & | & x_1 \\
0 & -5.4 & 1.7 & | & x_2 \\
0 & 0 & 5.351852 & | & x_3 \\
\end{bmatrix}
= 
\begin{bmatrix}
12 \\
21.6 \\
-4 \\
\end{bmatrix}
\]

\[
x_3 = \frac{-4}{5.351852} = -0.7474
\]
\[ x_2 = \frac{-53.4 - 1.7(-0.7474)}{-5.4} = -4.23529 \]

\[ x_1 = \frac{27 + 1(-0.7474) - 2(-4.23529)}{10} = 1.972318 \]

10.5 The system can be written in matrix form as

\[
[A] = \begin{bmatrix}
2 & -6 & -1 \\
-3 & -1 & 7 \\
-8 & 1 & -2 \\
\end{bmatrix}, \quad
(b) = \begin{bmatrix}
-38 \\
-34 \\
-40 \\
\end{bmatrix}, \quad
[P] = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

Partial pivot:

\[
[A] = \begin{bmatrix}
-8 & 1 & -2 \\
-3 & -1 & 7 \\
2 & -6 & -1 \\
\end{bmatrix}, \quad
[P] = \begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0 \\
\end{bmatrix}
\]

Compute factors:

\[ f_{21} = -3/-8 = 0.375 \quad f_{31} = 2/(-8) = -0.25 \]

Forward eliminate and store factors in zeros:

\[
[A] = \begin{bmatrix}
-8 & 1 & -2 \\
0.375 & -1.375 & 7.75 \\
-0.25 & -5.75 & -1.5 \\
\end{bmatrix}
\]

Pivot again

\[
[A] = \begin{bmatrix}
-8 & 1 & -2 \\
0.25 & 5.75 & 1.5 \\
0.375 & -1.375 & 7.75 \\
\end{bmatrix}, \quad
[P] = \begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
\end{bmatrix}
\]

Compute factors:

\[ f_{32} = -1.375/(-5.75) = 0.23913 \]
Forward eliminate and store factor in zero:

\[
[LU] = \begin{bmatrix}
-8 & 1 & -2 \\
-0.25 & -5.75 & -1.5 \\
0.375 & 0.23913 & 8.1087
\end{bmatrix}
\]

Therefore, the \(LU\) decomposition is

\[
[L] = \begin{bmatrix}
1 & 0 & 0 \\
-0.25 & 1 & 0 \\
0.375 & 0.23913 & 1
\end{bmatrix}, \quad [U] = \begin{bmatrix}
-8 & 1 & -2 \\
0 & -5.75 & -1.5 \\
0 & 0 & 8.1087
\end{bmatrix}
\]

Forward substitution. First pre-multiply right-hand side vector \([b]\) by \([P]\) to give

\[
[P] [b] = \begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix} \begin{bmatrix}
-38 \\
-34 \\
-40
\end{bmatrix} = \begin{bmatrix}
-40 \\
-38 \\
-34
\end{bmatrix}
\]

Therefore,

\[
\begin{bmatrix}
1 & 0 & 0 \\
-0.25 & 1 & 0 \\
0.375 & 0.23913 & 1
\end{bmatrix} [d] = \begin{bmatrix}
-40 \\
-38 \\
-34
\end{bmatrix}
\]

which can be solved for

\[
d_1 = -40
\]

\[
d_2 = -38 - 0.25(-40) = -48
\]

\[
d_3 = -34 - 0.375(-40) - 0.23913(-48) = -7.52174
\]

Back substitution:

\[
\begin{bmatrix}
-8 & 1 & -2 \\
0 & -5.75 & -1.5 \\
0 & 0 & 8.1087
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \begin{bmatrix}
-40 \\
-48 \\
-7.52174
\end{bmatrix}
\]
\[
x_3 = \frac{-7.52174}{8.1087} = -0.92761
\]
\[
x_2 = \frac{-48 + 1.5(-0.92761)}{-5.75} = 8.589812
\]
\[
x_1 = \frac{-40 + 2(-0.92761) - 1(8.589812)}{8} = 6.30563
\]

11.5 The problem can be written in matrix form as

\[
\begin{bmatrix}
0.866 & 0 & -0.5 & 0 & 0 & 0 & F_1 \\
0.5 & 0 & 0.866 & 0 & 0 & 0 & F_2 \\
-0.866 & -1 & 0 & -1 & 0 & 0 & F_3 \\
-0.5 & 0 & 0 & 0 & -1 & 0 & H \\
0 & 1 & 0.5 & 0 & 0 & 0 & V_2 \\
0 & 0 & -0.866 & 0 & 0 & -1 & V_3
\end{bmatrix} =
\begin{bmatrix}
F_{1,h} \\
F_{2,h} \\
F_{3,h} \\
F_{2,v} \\
F_{3,v} \\
V_3
\end{bmatrix}
\]

MATLAB can then be used to solve for the matrix inverse,

\[
\text{>> } A = [0.866 0 -0.5 0 0 0; \\
0.5 0.866 0 0 0 0; \\
-0.866 -1 0 -1 0 0; \\
-0.5 0 0 0 -1 0; \\
0 1 0.5 0 0 0; \\
0 0 -0.866 0 0 -1];
\]
\[
\text{>> } A^{-1} = \text{inv}(A)
\]

\[
A^{-1} =
\begin{bmatrix}
0.8660 & 0.5000 & 0 & 0 & 0 & 0 \\
0.2500 & -0.4330 & 0 & 0 & 1.0000 & 0 \\
-0.5000 & 0.8660 & 0 & 0 & 0 & 0 \\
-1.0000 & 0.0000 & -1.0000 & 0 & -1.0000 & 0 \\
-0.4330 & -0.2500 & 0 & -1.0000 & 0 & 0 \\
0.4330 & -0.7500 & 0 & 0 & 0 & -1.0000
\end{bmatrix}
\]

The forces in the members resulting from the two forces can be computed using the elements of the matrix inverse as in,

\[
F_1 = a_{12}^{-1}F_{2,v} + a_{13}^{-1}F_{3,h} = 0.5(-2000) + 0(-500) = -1000 + 0 = -1000
\]

\[
F_2 = a_{22}^{-1}F_{2,v} + a_{23}^{-1}F_{3,h} = -0.433(-2000) + 1(-500) = 866 - 500 = 366
\]

\[
F_3 = a_{32}^{-1}F_{2,v} + a_{33}^{-1}F_{3,h} = 0.866(-2000) + 0(-500) = -1732 + 0 = -1732
\]
12.3 The first iteration can be implemented as

\[ x_1 = \frac{27 - 2x_2 + x_3}{10} = \frac{27 - 2(0) + 0}{10} = -2.7 \]
\[ x_2 = \frac{-61.5 + 3x_1 - 2x_3}{-6} = \frac{-61.5 + 3(-2.7) - 2(0)}{-6} = 8.9 \]
\[ x_3 = \frac{-21.5 - x_1 - x_2}{5} = \frac{-21.5 - (-2.7) - 8.9}{5} = -6.62 \]

Second iteration:

\[ x_1 = \frac{27 - 2(8.9) - 6.62}{10} = 0.258 \]
\[ x_2 = \frac{-61.5 + 3(0.258) - 2(-6.62)}{-6} = 7.914333 \]
\[ x_3 = \frac{-21.5 - (0.258) - 7.914333}{5} = -5.934467 \]

The error estimates can be computed as

\[ \varepsilon_{e,1} = \left| \frac{0.258 - 2.7}{0.258} \right| \times 100\% = 947\% \]
\[ \varepsilon_{e,2} = \left| \frac{7.914333 - 8.9}{7.914333} \right| \times 100\% = 12.45\% \]
\[ \varepsilon_{e,3} = \left| \frac{-5.934467 - (-6.62)}{-5.934467} \right| \times 100\% = 11.55\% \]

The remainder of the calculation proceeds until all the errors fall below the stopping criterion of 5%. The entire computation can be summarized as

<table>
<thead>
<tr>
<th>iteration</th>
<th>unknown</th>
<th>value</th>
<th>( \varepsilon_e )</th>
<th>maximum ( \varepsilon_e )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( x_1 )</td>
<td>2.7</td>
<td>100.00%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( x_2 )</td>
<td>8.9</td>
<td>100.00%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( x_3 )</td>
<td>-6.62</td>
<td>100.00%</td>
<td>100%</td>
</tr>
<tr>
<td>2</td>
<td>( x_1 )</td>
<td>0.258</td>
<td>946.51%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( x_2 )</td>
<td>7.914333</td>
<td>12.45%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( x_3 )</td>
<td>-5.934467</td>
<td>11.55%</td>
<td>947%</td>
</tr>
<tr>
<td>3</td>
<td>( x_1 )</td>
<td>0.523687</td>
<td>50.73%</td>
<td></td>
</tr>
</tbody>
</table>
\[
\begin{array}{ccc}
\tilde{x}_1 & 8.010001 & 1.19\% \\
\tilde{x}_2 & -6.00674 & 1.20\% \\
\hline
4 & x_1 & 0.497326 & 5.30\% \\
x_2 & 7.999091 & 0.14\% \\
x_3 & -5.99928 & 0.12\% \\
\hline
5 & x_1 & 0.500253 & 0.59\% \\
x_2 & 8.000112 & 0.01\% \\
x_3 & -6.00007 & 0.01\% \\
\end{array}
\]

Thus, after 5 iterations, the maximum error is 0.59% and we arrive at the result: \(x_1 = 0.500253\), \(x_2 = 8.000112\) and \(x_3 = -6.00007\).

13.5 Here is a MATLAB session that uses \texttt{eig} to determine the eigenvalues and the natural frequencies:

\begin{verbatim}
>> k=2;
>> kmw2=[2*k,-k,-k,-k,2*k,-k,-k,2*k];
>> [v,d]=eig(kmw2)

v =
0.5774    0.7634    0.2895
0.5774  -0.6325    0.5164
0.5774  -0.1310   -0.8059
d =
 0.0000  0.0000  0.0000
 0 6.0000  0.0000
 0  0 6.0000
\end{verbatim}

Therefore, the eigenvalues are 0, 6, and 6. Setting these eigenvalues equal to \(m\omega^2\), the three frequencies can be obtained.

\[m\omega_1^2 = 0 \Rightarrow \omega_1 = 0 \text{ (Hz) 1}\text{\textsuperscript{st} mode of oscillation}\]

\[m\omega_2^2 = 6 \Rightarrow \omega_2 = \sqrt{6} \text{ (Hz) 2}\text{\textsuperscript{nd} mode}\]

\[m\omega_3^2 = 6 \Rightarrow \omega_3 = \sqrt{6} \text{ (Hz) 3}\text{\textsuperscript{rd} mode}\]
The solution along with its second derivative can be substituted into the simultaneous ODEs. After simplification, the result is

\[
\begin{align*}
\left( \frac{1}{C_1} - L_2 \omega^2 \right) I_1 & - \frac{1}{C_1} I_2 & = 0 \\
- \frac{1}{C_1} I_1 & - \left( \frac{1}{C_1} + \frac{1}{C_2} - L_2 \omega^2 \right) I_2 & - \frac{1}{C_2} I_3 & = 0 \\
- \frac{1}{C_2} I_2 & - \left( \frac{1}{C_2} + \frac{1}{C_3} - L_4 \omega^2 \right) I_3 & = 0
\end{align*}
\]

Thus, we have formulated an eigenvalue problem. Further simplification results for the special case where the \(C\)'s and \(L\)'s are constant. For this situation, the system can be expressed in matrix form as

\[
\begin{bmatrix}
1 - \lambda & -1 & 0 \\
-1 & 2 - \lambda & -1 \\
0 & -1 & 2 - \lambda
\end{bmatrix}
\begin{bmatrix}
I_1 \\
I_2 \\
I_3
\end{bmatrix} = [0] \tag{1}
\]

where \(\lambda = \frac{L}{C} \omega^2\). MATLAB can be employed to determine values for the eigenvalues and eigenvectors.

\[
\begin{align*}
\text{>> a} & = [1 -1 0; -1 2 -1; 0 -1 2]; \\
\text{>> [v,d]} & = \text{eig(a)}
\end{align*}
\]

\[
\begin{align*}
v & = \\
& \begin{bmatrix}
-0.7370 & -0.5910 & 0.3280 \\
-0.5910 & 0.3280 & -0.7370 \\
-0.3280 & 0.7370 & 0.5910 \\
\end{bmatrix} \\
d & = \\
& \begin{bmatrix}
0.1981 & 0 & 0 \\
0 & 1.5550 & 0 \\
0 & 0 & 3.2470 \\
\end{bmatrix}
\end{align*}
\]

The matrix \(v\) consists of the system's three eigenvectors (arranged as columns), and \(d\) is a matrix with the corresponding eigenvalues on the diagonal. Thus, the package computes that the eigenvalues are \(\lambda = 0.1981, 1.555,\) and 3.247. These values in turn can be used to compute the natural circular frequencies of the system

\[
\omega = \begin{bmatrix}
0.4450 / \sqrt{LC} \\
1.2470 / \sqrt{LC} \\
1.8019 / \sqrt{LC}
\end{bmatrix}
\]

Aside from providing the natural frequencies, the eigenvalues can be substituted into Eq. 1 to gain further insight into the circuit's physical behavior. For example, substituting \(\lambda = 0.1981\) yields

\[
\begin{bmatrix}
0.8019 & -1 & 0 \\
-1 & 1.8019 & -1 \\
0 & -1 & 1.8019
\end{bmatrix}
\begin{bmatrix}
I_1 \\
I_2 \\
I_3
\end{bmatrix} = [0]
\]
Although this system does not have a unique solution, it will be satisfied if the currents are in fixed ratios, as in

\[ 0.8019i_i = i_x = 1.8019i_i \]  \hspace{1cm} (2) \]

Thus, as depicted in (a) in the figure below, they oscillate in the same direction with different magnitudes. Observe that if we assume that \( i_i = 0.737 \), we can use Eq. 2 to compute the other currents with the result

\[ \{i\} = \begin{bmatrix} 0.737 \\ 0.591 \\ 0.328 \end{bmatrix} \]

which is the first column of the \( \psi \) matrix calculated with MATLAB.

\[ \omega = \frac{0.4451}{\sqrt{LC}} \]

\[ \omega = \frac{1.2470}{\sqrt{LC}} \]

\[ \omega = \frac{1.8019}{\sqrt{LC}} \]

In a similar fashion, the second eigenvalue of \( \lambda = 1.555 \) can be substituted and the result evaluated to yield

\[ -1.8018i_i = i_x = 2.247i_i \]

As depicted in the above figure (b), the first loop oscillates in the opposite direction from the second and third. Finally, the third mode can be determined as

\[ -0.445i_i = i_x = -0.8718i_i \]

Consequently, as in the above figure (c), the first and third loops oscillate in the opposite direction from the second.