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NONCANONICAL HAMILTONIAN DENSITY FORMULATION OF HYDRODYNAMICS AND IDEAL MHD

BY

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Noncanonical Hamiltonian Density Formulation of Hydrodynamics and Ideal MHD

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ABSTRACT

We present a new Hamiltonian density formulation of a perfect fluid with or without a magnetic field. Contrary to previous work the dynamical variables are the physical variables, ρ, $\underset{\sim}{v}$, $\underset{\sim}{B}$, and s, which form a noncanonical set. A Poisson bracket which satisfies the Jacobi identity is defined. This formulation is transformed to a Hamiltonian system where the dynamical variables are the spatial Fourier coefficients of the fluid variables.

Several advantages may be gained from expressing a set of equations in Hamiltonian form. In addition to their formal elegance, Hamiltonian systems possess Poincaré invariants that influence the dispersion of an ensemble of systems with clustered initial conditions. A manifestly Hamiltonian formulation of a given problem makes it easier to find those approximations that preserve the Hamiltonian character. Here we present such a formulation of hydrodynamics and magnetohydrodynamics.

Hamiltonian systems are most elegant when expressed in canonical coordinates. Hydrodynamics is most usefully expressed in Eulerian variables. These two desiderata conflict. In practice, the penalty paid for adopting noncanonical coordinates is not severe, so that branch of the dichotomy is pursued here.

Previously, the equations of hydrodynamics[1] and MHD,[2] in both Eulerian and Lagrangian form, have been shown to arise from a suitable Hamilton's principle. Such a Lagrangian density formulation is the natural starting place for derivation of a Hamiltonian density description.[3] Typically, the Euler-Lagrange equation is the fluid equation of motion; the remaining fluid equations have the role of constraints. A Hamiltonian density formulation obtained by Legendre transformation necessarily embodies this division of roles. Alternatively, Hamiltonian type equations have been given directly for a fluid[4] and for ideal MHD.[5] In these formulations, Clebsch or other non physical variables are necessary and entropy convection is not included. Our formulation departs from previous work in that all of the fluid equations are, in principle, placed on equal footing; further, the dynamical variables are the physical variables. The fluid equations, including entropy convection and (but not necessarily) the Maxwell induction equation, are obtained in Poisson bracket form; the Hamiltonian density is the energy density of the fluid. The physical variables are noncanonical; this results in alteration of the usual Poisson bracket. The use of noncanonical variables has proven to be fruitful for Hamiltonian systems[6] and a Poisson bracket similar to ours has been used to express the Korteweg-deVries equation as a Hamiltonian system.[7,8]

We wish to cast the following set of equations into Hamiltonian form:

$$\mathbf{v}_t = -\nabla(v^2/2) + \mathbf{v} \times (\nabla \times \mathbf{v}) - \rho^{-1} \nabla(\rho^2 U_\rho) + \rho^{-1}(\nabla \times \mathbf{B}) \times \mathbf{B} \tag{1}$$

$$\rho_t = -\nabla \cdot (\rho \mathbf{v}) \tag{2}$$

$$\mathbf{B}_t = \nabla \times (\mathbf{v} \times \mathbf{B}) \tag{3}$$

$$s_t = -\mathbf{v} \cdot \nabla s \quad . \tag{4}$$

Equation (1) is the hydrodynamic force balance equation for a fluid with density ρ and velocity $\mathbf{v}$, with the addition of the magnetic body force term $\mathbf{J} \times \mathbf{B}$. We have eliminated $\mathbf{J}$ by making use of Ampere's law: $\mathbf{J} = \nabla \times \mathbf{B}$. The internal energy per unit mass, $U(\rho, s)$ is a prescribed function of ρ and the entropy per unit mass[9] s. The intensive variables, pressure p and temperature T, are obtained from this function: $p = \rho^2 U_\rho$ and $T = U_s$. Equation (2) is a mass conservation. Equation (3) is the Maxwell induction equation with the electric field eliminated by Ohm's law: $\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B} = 0$. Here infinite conductivity is assumed. Equation (4) expresses entropy convection; heat flow is assumed to vanish. The equation $\underset{\sim}{\nabla} \cdot \underset{\sim}{B} = 0$ enters our formulation only as an initial condition.

The energy density of a fluid described by Eqs. (1)-(4) is $H = \rho v^2/2 + \rho U(\rho,s) + B^2/2$, where $\rho v^2/2$ is the kinetic energy density and the remaining two terms are the internal and magnetic energy densities. We take this as our Hamiltonian density and construct the Hamiltonian $\hat{H}\{\rho,s,\underset{\sim}{v},\underset{\sim}{B}\} = \int_V H(\rho,s,\underset{\sim}{v},\underset{\sim}{B})\, d\tau$ where the curley brackets are used to indicate that $\hat{H}$ is a functional of the enclosed functions. The integration is over a fixed spatial region V. We desire a Poisson bracket, such that Eqs. (1)-(4) can be represented in the form

$$\bar{\chi}^i_t = [\bar{\chi}^i, \hat{H}] \qquad i = 0,1,2\,.\,.\,.7\,, \tag{5}$$

where the $\bar{\chi}^i$ are suitable functional dynamical variables.

Before writing this bracket (Eq. 6 below), we briefly discuss the structure of our formulation. Quite generally consider the vector space $\mathcal{V}$, over the real numbers $\mathcal{R}$, whose elements are functionals of the form

$$\hat{F}\{\underset{\sim}{\chi}\} = \int_V F(\underset{\sim}{x},t;\, \underset{\sim}{\chi},\, \partial\underset{\sim}{\chi}/\partial x_\alpha,\, \partial^2\underset{\sim}{\chi}/\partial x_\alpha \partial x_\beta\, ,\,.\,.\,.)\, d\tau$$

where $\underset{\sim}{\chi}$ is an n-tuple of $C^\infty(V)$ functions $\chi^i(\underset{\sim}{x},t)$. (In particular, $\chi^o \equiv \rho$, $\chi^1 \equiv s$, $\chi^{2,3,4} \equiv \underset{\sim}{v}$, and $\chi^{5,6,7} \equiv \underset{\sim}{B}$.) The notation $\partial\underset{\sim}{\chi}/\partial x_\alpha$ is used to indicate that F depends on the derivatives of χ^i with respect to each of the three spatial variables x_α , $\alpha = 1,2,3$.

We assume F has a finite number of arguments and is a C^∞ function in each of them. Further, we assume that F and all its derivatives vanish on ∂V. The bracket we obtain is a bilinear function which maps $\mathcal{V} \times \mathcal{V}$ to $\mathcal{V}$. In addition, the bracket possesses the following two important properties:

(i) $[\hat{F},\hat{F}] = 0$ for every $\hat{F} \in \mathcal{V}$. For $\mathcal{V}$ over $\mathcal{R}$, this is equivalent to $[\hat{F},\hat{G}] = -[\hat{G},\hat{F}]$ for $\hat{F},\hat{G} \in \mathcal{V}$,

(ii) the Jacobi identity[10] $[\hat{E},[\hat{F},\hat{G}]] + [\hat{F},[\hat{G},\hat{E}]] + [\hat{G},[\hat{E},\hat{F}]] = 0$ for every $\hat{E},\hat{F},\hat{G} \in \mathcal{V}$.

A vector space together with a bracket which has the above properties defines a Lie algebra.[11]

Now we introduce the following bracket[12]

$$[\hat{F},\hat{G}] = -\int_V \Bigg\{ \left[\frac{\delta\hat{F}}{\delta\rho}\,\underset{\sim}{\nabla}\cdot\frac{\delta\hat{G}}{\delta\underset{\sim}{v}} + \frac{\delta\hat{F}}{\delta\underset{\sim}{v}}\cdot\underset{\sim}{\nabla}\,\frac{\delta\hat{G}}{\delta\rho}\right] + \left[\frac{\delta\hat{F}}{\delta\underset{\sim}{v}}\cdot\left(\frac{(\underset{\sim}{\nabla}\times\underset{\sim}{v})}{\rho}\times\frac{\delta\hat{G}}{\delta\underset{\sim}{v}}\right)\right]$$

$$+ \left[\rho^{-1}\underset{\sim}{\nabla}s\cdot\left(\frac{\delta\hat{F}}{\delta s}\,\frac{\delta\hat{G}}{\delta\underset{\sim}{v}} - \frac{\delta\hat{G}}{\delta s}\,\frac{\delta\hat{F}}{\delta\underset{\sim}{v}}\right)\right] + \left[\rho^{-1}\,\frac{\delta\hat{F}}{\delta\underset{\sim}{v}}\cdot\left(\underset{\sim}{B}\times\left(\underset{\sim}{\nabla}\times\frac{\delta\hat{G}}{\delta\underset{\sim}{B}}\right)\right)\right.$$

$$\left. + \frac{\delta\hat{F}}{\delta\underset{\sim}{B}}\cdot\left(\underset{\sim}{\nabla}\times\left(\underset{\sim}{B}\times\rho^{-1}\,\frac{\delta\hat{G}}{\delta\underset{\sim}{v}}\right)\right)\right]\Bigg\}\,d\tau \equiv \int_V \frac{\delta\hat{F}}{\delta\chi^i}\,O^{ij}\,\frac{\delta\hat{G}}{\delta\chi^j}\,d\tau. \qquad (6)$$

Here the notation $\delta\hat{F}/\delta\chi^i$ means the functional derivative with respect to χ^i. Suppose each χ^i contains an additional parameter dependence $\chi^i(\underset{\sim}{x},\alpha_i,t)$. We define the functional derivative by[8]

$$\frac{\partial\hat{F}}{\partial\alpha_i} = \int_V \frac{\delta\hat{F}}{\delta\chi^i}\,\frac{\partial\chi^i}{\partial\alpha_i}\,d\tau \qquad \text{(not summed)}. \qquad (7)$$

This functional derivative has the role, in finite dimensional Hamiltonian systems, of the derivatives with respect to phase coordinates $\partial F/\partial q_i$, $\partial F/\partial p_i$. In finite degree of freedom systems the Poisson bracket is written

$$[F,G] = \frac{\partial F}{\partial z^i} J^{ij} \frac{\partial G}{\partial z^j}$$

where the z^i are the phase space coordinates, $z^i \in \{q_1, \ldots q_N, p_1, \ldots p_N\}$. In canonical coordinates the symplectic structure J^{ij}, is

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

where I is the unit $N \times N$ matrix. In a noncanonical system this matrix may be full and depend on the dynamical variables. Clearly, this is the case for our bracket, Eq. (6). The symplectic structure here is the operator O^{ij} which, in addition to depending on the dynamical variables, contains derivatives.

Now we complete the description of our formulation and demonstrate the relationship between this bracket and Eqs. (1)-(4). We define a set $\mathcal{D} \subset \mathcal{V}$ whose elements are of the form

$$\bar{\chi}^i\{\chi^i\} = \int_V f_i(\underset{\sim}{x})\, \chi^i(\underset{\sim}{x},t)\, d\tau \qquad i = 0,1,2\ldots.7 \text{ (not summed)}$$

where $\chi^i \in C^\infty(V)$ and the f_i are arbitrary functions[13] of $\underset{\sim}{x}$ alone, which vanish on ∂V. $\mathcal{D}$ is thus the set of dynamical variables. Substituting $\bar{\chi}^o$ and $\hat{H}$ into Eq. (5) yields

$$\frac{\partial \bar{\chi}^o}{\partial t} - [\bar{\chi}^o, \hat{H}] = \int_V f_o(\underset{\sim}{x}) \left(\frac{\partial \rho}{\partial t} + \underset{\sim}{\nabla}\cdot(\rho \underset{\sim}{v})\right) d\tau = 0 \qquad (8)$$

Since $f_o(\underset{\sim}{x})$ is an arbitrary function, by the Du Bois-Reymond[14] lemma, Eq. (8) implies Eq. (2). Eqs. (1,3,4) follow, from the remaining dynamical variables of the set $\mathcal{D}$, in a similar manner.

Several features of the bracket defined in Eq. (6), deserve comment. First, the density, ρ , appears in the denominator of several terms. This makes it awkward to evaluate the bracket exactly when polynomial or Fourier representations are used for the dynamical variables. This is easily rectified through a nonlinear transformation described below, and the resulting bracket, in terms of the new variables, has a pleasing form. Second, gradients appear throughout the bracket. This is reminiscent of the bracket used in Hamiltonian theories of the Korteweg-deVries equation[7,8]

$$[F,G] = \int dx \, \frac{\delta F}{\delta u} \left(\frac{\partial}{\partial x} \frac{\partial G}{\partial u} \right) .$$

Two methods have been used to reduce the Kdv bracket to canonical form. Gardner[8] used a Fourier transform to convert the derivatives to numbers, and then scaled the coefficients to achieve canonical form. Similarly motivated, we also consider Fourier transforms below. In another approach to the Korteweg-deVries equation, Zakharov and Faddeev[7] used a spectral transform to achieve canonical form. This method may be applicable here.

Another feasible attack on the derivatives in our Poisson bracket is to express velocities as appropriate derivatives of a new set of variables. Then, when the bracket is transformed into these new variables, the derivatives will effectively cancel out.

Indeed, the canonical variables used by Davydov[4] to express hydrodynamics and Zakharov and Kuznetzov[5] to express MHD (ignoring entropy) are of this type. This approach will not be further pursued here.

Our new set of Eulerian variables, which yields an improved Poisson bracket is $\{\rho,\sigma,\underset{\sim}{M},\underset{\sim}{B}\}$ where $\sigma = \rho s$ and $\underset{\sim}{M} = \rho\underset{\sim}{v}$; σ is the specific entropy and $\underset{\sim}{M}$ is the momentum density. Substitution of these variables into Eqs. (1)-(4) results in eight conservation equations. The pressure is now determined by $p = \rho^2(\tilde{U}_\rho + \sigma\rho^{-1}\tilde{U}_\sigma)$ where $\tilde{U}(\rho,\sigma) = U(\rho,s)$. As a result of the the transformation

$$\frac{\delta}{\delta\rho}\bigg|_{\underset{\sim}{v},s} = \frac{\delta}{\delta\rho}\bigg|_{\underset{\sim}{M},\sigma} + \rho^{-1}\underset{\sim}{M}\cdot\frac{\delta}{\delta\underset{\sim}{M}} + \sigma\rho^{-1}\frac{\delta}{\delta\sigma} ,$$

together with similar transformations for the remaining variables, Eq. (6) becomes

$$\begin{aligned}[\hat{F},\hat{G}] = -\int_V d\tau \Bigg\{ &\rho\left(\frac{\delta\hat{F}}{\delta\underset{\sim}{M}}\cdot\underset{\sim}{\nabla}\frac{\delta\hat{G}}{\delta\rho} - \frac{\delta\hat{G}}{\delta\underset{\sim}{M}}\cdot\underset{\sim}{\nabla}\frac{\delta\hat{F}}{\delta\rho}\right) + \underset{\sim}{M}\cdot\left(\frac{\delta\hat{F}}{\delta\underset{\sim}{M}}\cdot\underset{\sim}{\nabla}\frac{\delta\hat{G}}{\delta\underset{\sim}{M}} - \frac{\delta\hat{G}}{\delta\underset{\sim}{M}}\cdot\underset{\sim}{\nabla}\frac{\delta\hat{F}}{\delta\underset{\sim}{M}}\right) \\ &+ \sigma\left(\frac{\delta\hat{F}}{\delta\underset{\sim}{M}}\cdot\underset{\sim}{\nabla}\frac{\delta\hat{G}}{\delta\sigma} - \frac{\delta\hat{G}}{\delta\underset{\sim}{M}}\cdot\underset{\sim}{\nabla}\frac{\delta\hat{F}}{\delta\sigma}\right) + \underset{\sim}{B}\cdot\Bigg[\left(\frac{\delta\hat{F}}{\delta\underset{\sim}{M}}\cdot\underset{\sim}{\nabla}\frac{\delta\hat{G}}{\delta\underset{\sim}{B}} - \frac{\delta\hat{G}}{\delta\underset{\sim}{B}}\cdot\underset{\sim}{\nabla}\frac{\delta\hat{F}}{\delta\underset{\sim}{M}}\right) \\ &+ \left(\underset{\sim}{\nabla}\frac{\delta\hat{F}}{\delta\underset{\sim}{B}}\cdot\frac{\delta\hat{G}}{\delta\underset{\sim}{M}} - \underset{\sim}{\nabla}\frac{\delta\hat{G}}{\delta\underset{\sim}{B}}\cdot\frac{\delta\hat{F}}{\delta\underset{\sim}{M}}\right)\Bigg]\Bigg\} . \end{aligned} \tag{9}$$

Notice that each term contains one Eulerian variable in the numerator; the terms in the denominator have been eliminated.

Now consider a transformation of the Hamiltonian coordinates from Eulerian variables to the coefficients of the Fourier transform of these variables. For convenience, we take V to be a unit cube and adopt periodic boundary conditions. Then

$$\rho = \sum_{\mathbf{k}} \rho_{\mathbf{k}}(t) \; \exp(2\pi \mathbf{k}\cdot\mathbf{x}) \tag{10}$$

where $\mathbf{k} \in Z\times Z\times Z$ ($Z \equiv$ integers). We observe from Eq. (7) that

$$\frac{\partial \hat{F}}{\partial \rho_{\mathbf{k}}} = \int_V \frac{\delta \hat{F}}{\delta \rho} \; \exp(2\pi i \mathbf{k}\cdot\mathbf{x}) \; d\tau \; . \tag{11}$$

Inverting Eq. (11), we obtain

$$\frac{\delta F}{\delta \rho} = \sum_{\mathbf{k}} \frac{\partial \hat{F}}{\partial \rho_{\mathbf{k}}} \; \exp(-2\pi i \mathbf{k}\cdot\mathbf{x}) \; . \tag{12}$$

Inserting Eqs. (10) and (12), and the analogous expressions for the other variables in our set, into Eq. (9) yields

$$[\hat{F},\hat{G}] = \sum_{\mathbf{k},\boldsymbol{\ell}} \frac{\partial \hat{F}}{\partial \mathbf{z}_{\mathbf{k}}} \cdot \underset{\approx}{O}_{\mathbf{k},\boldsymbol{\ell}} \cdot \frac{\partial \hat{G}}{\partial \mathbf{z}_{\boldsymbol{\ell}}} \tag{13}$$

where $\mathbf{z}_{\mathbf{k}}$ is the 8-tuple $(\rho_{\mathbf{k}}, \sigma_{\mathbf{k}}, \mathbf{M}_{\mathbf{k}}, \mathbf{B}_{\mathbf{k}})$, and the matrix $\underset{\approx}{O}_{\mathbf{k}\boldsymbol{\ell}}$ is

$$\underset{\approx}{O}_{\mathbf{k}\boldsymbol{\ell}} = 2\pi i \left[\begin{array}{cc|c|ccc}
0 & 0 & -\rho_{\boldsymbol{\ell}+\mathbf{k}}\mathbf{k} & 0 & 0 & 0 \\
0 & 0 & -\sigma_{\boldsymbol{\ell}+\mathbf{k}}\mathbf{k} & 0 & 0 & 0 \\
\hline
\rho_{\boldsymbol{\ell}+\mathbf{k}}\boldsymbol{\ell} & \sigma_{\boldsymbol{\ell}+\mathbf{k}}\boldsymbol{\ell} & \boldsymbol{\ell}\,\mathbf{M}_{\boldsymbol{\ell}+\mathbf{k}} - \mathbf{M}_{\boldsymbol{\ell}+\mathbf{k}}\mathbf{k} & \multicolumn{3}{c}{\boldsymbol{\ell}\,\mathbf{B}_{\boldsymbol{\ell}+\mathbf{k}} - \boldsymbol{\ell}\cdot\mathbf{B}_{\boldsymbol{\ell}+\mathbf{k}}\underset{\approx}{I}'} \\
\hline
0 & 0 & & 0 & 0 & 0 \\
0 & 0 & -\mathbf{k}\,\mathbf{B}_{\boldsymbol{\ell}+\mathbf{k}} + \mathbf{k}\cdot\mathbf{B}_{\boldsymbol{\ell}+\mathbf{k}}\underset{\approx}{I}' & 0 & 0 & 0 \\
0 & 0 & & 0 & 0 & 0
\end{array} \right]$$

where $\underset{\approx}{I}'$ is the 3×3 unit matrix appropriate to the box in which it is contained. The matrix has the important property $\underset{\approx}{O}_{\underset{\sim}{k}\underset{\sim}{\ell}} = -\tilde{\underset{\approx}{O}}_{\underset{\sim}{\ell}\underset{\sim}{k}}$, where the tilde indicates transpose. Equation (13) can be written as follows:

$$[F,G] = \frac{\partial F}{\partial z^i} \, J^{ij} \, \frac{\partial G}{\partial z^j} \qquad i,j \in \{0,\pm1,\pm2,\ldots\} \tag{14}$$

where $z^i \in \{\rho_{\underset{\sim}{k}}, \sigma_{\underset{\sim}{k}}, \underset{\sim}{M}_{\underset{\sim}{k}}, \underset{\sim}{B}_{\underset{\sim}{k}} | \underset{\sim}{k} \in Z\times Z\times Z\}$. The matrix J has the property $J^{ij} = -J^{ji}$ and its elements can be obtained by a suitable map[15] of the indices of $\underset{\approx}{O}_{\underset{\sim}{k}\underset{\sim}{\ell}}$ onto Z. Clearly Eq. (14) is of the same form as finite Hamiltonian systems, but here J is of infinite order. Approximation techniques, along with the proof of the Jacobi identity, integral invariants and commutation relations, will be the subject of a future publication.

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[10]We have proved the Jacobi identity for functionals $\hat{F}, \hat{G}, \hat{E}$ with corresponding F,G,E, which are functions of $\underset{\sim}{x}$,t and $\underset{\sim}{\chi}$. This is sufficient for our purposes here. The more general case is of interest; e.g., the dynamical variable with the integrand $\underset{\sim}{\nabla} \times \underset{\sim}{v}$ yields, when substituted into Eq. (5), the vorticity equation.

[11] V. Arnold, Mathematical Methods of Classical Mechanics, (Springer Verlag, N.Y., 1978) p. 208; R. Bishop and R. Crittendon, Geometry of Manifolds, (Academic Press, N.Y., 1964) p. 26.

[12] The hydrodynamic bracket is obtained by setting $\underset{\sim}{B}=0$, isentropic hydrodynamics by further setting $s=0$, and irrotational isentropic hydrodynamics by finally setting $\underset{\sim}{\nabla} \times \underset{\sim}{v} = 0$. The Jacobi identity, as proved, is satisfied for these cases. (We use repeated index notation in the second equality here and henceforth, except where noted.)

[13] Conventionally, the functional derivative of a function with respect to itself is seen to be the Dirac delta function. For $f_i(\underset{\sim}{x}) \equiv \delta(\underset{\sim}{x}-\underset{\sim}{x}')$ this is obtained.

[14] D. Smith, Variational Methods in Optimization, (Prentice Hall, N.J., 1974), p. 357.

[15] A bijection $f:S\times Z\times Z\times Z \to Z$ where $S = \{0,1,2,3,4,5,6,7\}$ is needed for this purpose. A useful example will be given in a companion paper.

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