I

Hamiltonian & Action Principle (HAP) Forms

Trieste Aug 14, 2009

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Whence HAPs?

Relativistic 2N-Body prob. + Maxwell's eqs

ultimate plasma description

⟹ Vlasov

Ideal MHD

Gyrokinetics

BBGKY

⋮

why HAPs?

structure of physics

approximation

unambiguous energy

{ imposes structure

singles out class of models

What about dissipation? where is it above?

add later get ideal part right

No fake dissipation *

Reduced Fluid Models (RFM) today

computable n-field eqs.
w/ important physics

2D Euler
HM
Reduced MHD
⋮
sophisticated
reconnection w/ e^- inertia
Gyro physics

Whence RFMs?

approx. $\epsilon \ll 1$ rational rigor

modeling practical mutilation ex post facto

today build action special for cm

Pedigree

Lagrange (1788), Clebsch (1859)

Kirchoff (1876), ... Eckart, Newcomb

Lin, Arnold (1960), Zakharov (1970's)

PJM (1979) ...

Approaches ⇒ Confusion

Lagrangian (material) variables

Eulerian (spatial) variables

Clebsch (potential) variables

Variable Changes

functional chain rule

rediscovery

My Refs

AIP conf. proc. 88 (1982)

RMP 70, 467 (1998)

PoP 12 058102 (2005)

Hamiltonian's Principle

the procedure

* config space $q^i(t)$
* kinetic potential $\quad i = 1, 2, \ldots N$ ← # dof

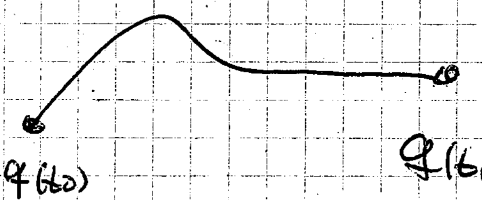
$$L = T - V$$

* action

$$S[q] = \int_{t_0}^{t_1} L(q, \dot{q}, t)\, dt$$

insert $q(t) \Rightarrow S = \#$

Paths



$\delta q(t_0) = \delta q(t_1) = 0$

extremal path

$$\delta S = \frac{d}{d\epsilon} S[q + \epsilon \delta q] \Big|_{\epsilon=0} = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} \right) \delta q^i \, dt$$

$$= \left\langle \frac{\delta S}{\delta q^i}, \delta q^i \right\rangle \qquad \text{Vary \& isolate}$$

cf $f(x, y, z)$

$\delta f = \nabla f \cdot \delta x \quad \Rightarrow \quad \boxed{\dfrac{\delta S}{\delta q^i} = 0} \quad \Rightarrow$ extremal

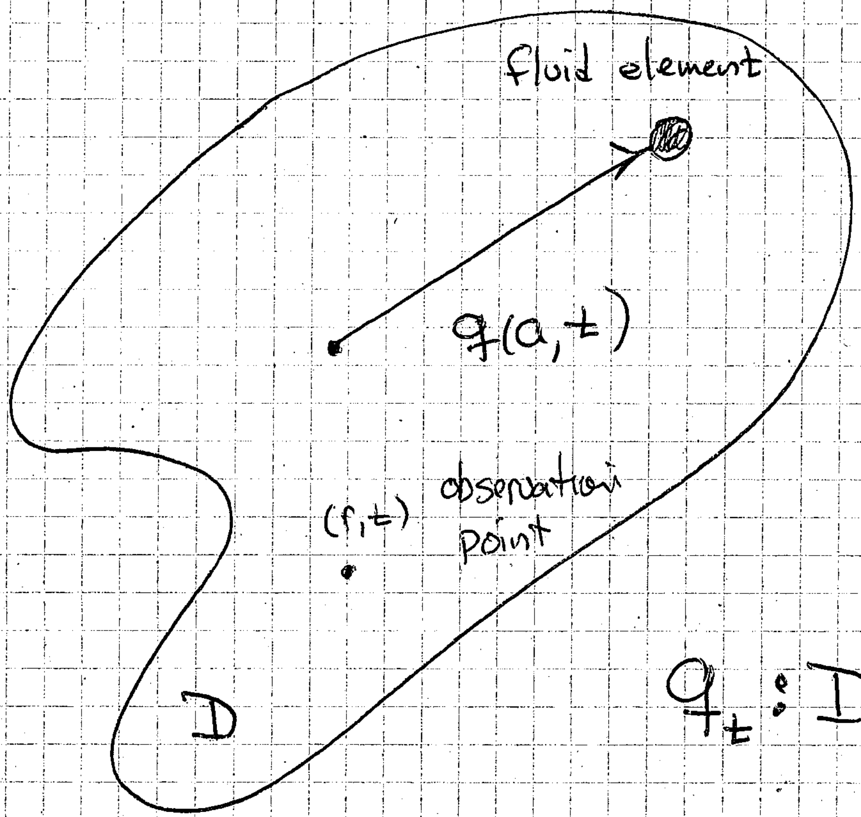
Functional Derivative → $\qquad$ Lag's eqs.

Lagrangian Variables

(1788)

Naturally ~~HAP~~

Newton's 2nd



a = label st

$q(a,0) = a$

$$q_t : D \to D \subset \mathbb{R}^{2,3}$$

homeomorphism?

diffeomorphism?

Banach space?

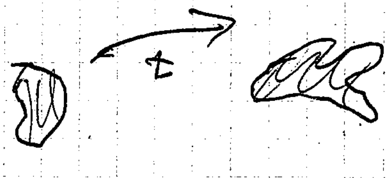
\$$10^6$

Deformation: $\frac{\partial q^i}{\partial a^j} =: q^i_{,j}$

$$J := \det(\partial q / \partial a)$$

$$J \neq 0 \Rightarrow a(q,t) \qquad q^i_{,k}\, a^k_{,j} = \delta^i_j$$

Diff. Forms



Vol $\quad d^3q = J\, d^3a$

area $\quad (d^2q)_i = J\, a^j_{,i}\, (d^2a)_j$

line $\quad (dq)_i = q^i_{,j}\, (da)_j$

$$J = \frac{1}{6} \epsilon_{kje}\, \epsilon^{imn}\, q^k_{,e}\, q^j_{,m}\, q^e_{,n}$$

Eulerian Variables "usual" description

Observe velocity a pt (r, t)

$\dot{q}(a,t)$ which one is there?

$$r = q(a,t) \Rightarrow a = q^{-1}(r,t)$$

Eul. vel. field

$$v(r,t) = \dot{q}(a,t)\Big|_{a=q^{-1}(r,t)} = \dot{q}\circ q^{-1}$$

Watching Fish.

Other properties

mass $\rho_0(a)$ st. $\rho_0(a)\,d^3a = dm$

entropy $S_0(a)$ entropy/mass

Vect. field $B_0^j(a)$

Geom. Constraints

$$S(r,t) = S_0(a)\Big|_{a=q^{-1}(r,t)}$$ entropy advection

$$\rho\, d^3x = \rho_0\, d^3a \Rightarrow \rho(r,t) = \frac{\rho_0(a)}{J}\Big|_{a=q^{-1}(r,t)}$$ mass

$$B\cdot d^2x = B_0\cdot d^2x \Rightarrow B^i = q^i{}_{,j}\frac{B_0^j}{J}\Big|_{a=q^{-1}(r,t)}$$ Flux

Lagrange-Euler Map

Densities

mass $$\rho(r,t) = \int_D d^3a\, \rho_0(a)\, \delta(r-q) = \frac{\rho_0}{J}\Big|_{a(r,t)}$$

$$\sigma = \rho s$$

entropy $$\sigma(r,t) = \int_D d^3a\, \sigma_0(a)\, \delta(r-q) = \frac{\sigma_0}{J}$$

momentum $$M_i(r,t) = \int_D d^3a\, \Pi_i(a,t)\, \delta(r-q) = \Pi/J$$

e.g. $\Pi = \rho_0 \dot{q}$

B-field $$B^i(r,t) = \int_D d^3a\, B_0^j(a)\, q^i_{,j}(a,t)\, \delta(r-q)$$

$$= \frac{q^i_{,j} B_0^j(a)}{J}$$

EX 1 show 2nd equal

Building an Action

(4 steps to enlightenment)
8? 3?

step 1 Select Domain

$q : D \to D$

$D \subset \mathbb{R}^{2,3}$ fluid

$D \subset \mathbb{R}^6$ kinetic

etc

Step 2 Select Eulerian Variables (observables)

Today MHD like $\{M, \rho, \sigma, B\}$

$\{U, \rho, s, B\}$

step 3 Eulerian Closure Principle

Pressure

energies must be "Eulerianizable"

property of media theories unlike QFT.

Main Theorem:

* Action ECP $\Rightarrow$ Eq of motion Eulerianizable
* Action ECP $\Rightarrow$ $\exists$ noncanonical P.B.

$$\frac{1}{2}\int d^3a\, \rho_0\, \dot{q}^2 = \frac{1}{2}\int d^3x\, \rho\, U^2$$

Step 4 Symmetries

Galelian group, Lorentz/Poincare, . . .

MHD Action (Newcomb) IAEA (1962)

Kinetic $\quad T[q] = \frac{1}{2}\int d^3a\, \rho_0 \dot{q}^2 = \frac{1}{2}\int d^3x\, \rho v^2$

Internal $\quad V[q] = \int \rho_0\, d^3a\; V\left(\frac{\rho_0}{J}, \frac{\sigma_0}{J}, \frac{|q^i_{,j} B_0^j|}{J}\right)$

$$= \int \rho\, d^3x\; V(\rho, \sigma, |B|)$$

Thermodynamics (Lag. didn't know)

$U = \frac{\text{internal energy}}{\text{mass}}$ energy rep

U (vol, s)

$T = \left.\frac{\partial U}{\partial s}\right|_{vol}$ $\qquad p = -\left.\frac{\partial U}{\partial\, vol}\right|_s$ $\qquad vol \sim \frac{1}{\rho}$

$$\Updownarrow$$

$\therefore$ Write $\qquad p = \rho^2 \frac{\partial U}{\partial \rho}$

$U(\rho, s)$

Useful identities (exercise)

3D $\quad J\, a^i_{,k} = \frac{1}{2}\, \epsilon_{kjl}\, \epsilon^{imn}\, q^j_{,m}\, q^l_{,n}$

2D $\quad J\, a^i_{,k} = \epsilon_{kj}\, \epsilon^{il}\, q^j_{,l}$

$$\frac{1}{J} \frac{\partial J}{\partial q^i_{,j}} = a^j_{,i}$$

$$\frac{\partial}{\partial a^i} \left(J\, a^i_{,k} \right) = 0$$

$$J = \frac{1}{6}\, \epsilon_{kjl}\, \epsilon^{imn}\, q^k_{,i}\, q^j_{,m}\, q^l_{,n} \qquad 3D$$

$$J = \frac{1}{2}\, \epsilon_{kj}\, \epsilon^{il}\, q^k_{,i}\, q^j_{,l}$$

$$V[q] = \int d^3a\, \rho_0 \left[U\left(\frac{\rho_0}{\mathcal{J}}, S_0(a)\right) + \frac{(q^i_{,j} B_0^j)(q^i_{,k} B_0^k)}{2\mathcal{J}\rho_0} \right]$$

$$= \int d^3x \left[\rho\, U(\rho, s) + \frac{B^2}{2} \right]$$

I.E.   mag. energy

Action

$$S_{MHD}[q] = \int dt \int d^3a \left\{ \frac{\rho_0 \dot{q}^2}{2} - \rho_0 U - \frac{\rho_0}{2\rho} B^2 \right\}$$

$$\left.\frac{d}{d\epsilon} S[q + \epsilon \delta q]\right|_{\epsilon=0} = \int dt \int d^3a\, \frac{\delta S}{\delta q^i} \delta q^i = \left\langle \frac{\delta S}{\delta q}, \delta q \right\rangle$$

$$\frac{\delta S_{MHD}}{\delta q} = 0 \implies$$

$$\rho_0 \ddot{q}^i = \dots \qquad \text{complicated}$$

$\downarrow$ Eulerianize

$$\rho\left(\frac{\partial v}{\partial t} + v \cdot \nabla v\right) - B \cdot \nabla B + \nabla\left(p + \frac{B^2}{2}\right) = 0$$

Something New

Add term linear in $\dot{q}$ (ξ 2D)

$$G[q] = \int d^3a\, \Pi^* \cdot \dot{q} = \int_D d^3x\, M^* \cdot v$$

$$M^* = \frac{\Pi^*}{J}$$

Choose $\Pi^{*i} = J\, \epsilon^{ij} a^m{}_{,j} \frac{\partial L^*}{\partial a^m}$

$$L^* = L^*(\rho, \sigma, B, \nabla\rho, \nabla\sigma, \nabla B, \dots)$$

$$\Pi^* = -\frac{m}{2e} J\, \nabla\left(\frac{p}{B}\right) \times \hat{b} = \nabla \times L^*$$

$S_B = S_{MHD} + G$ above choice

$\frac{\delta S_B}{\delta q} = 0 \implies$

Braginskii MHD $\Big\}$ Gyroviscous tensor