Hamiltonian-Dirac Simulated Annealing: Application to Vortex States

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Goal

Use Hamiltonian structure of matter, vortex dynamics in particular, together w/ Dirac brackets to construct a numerical algorithm for calculating equilibrium states.

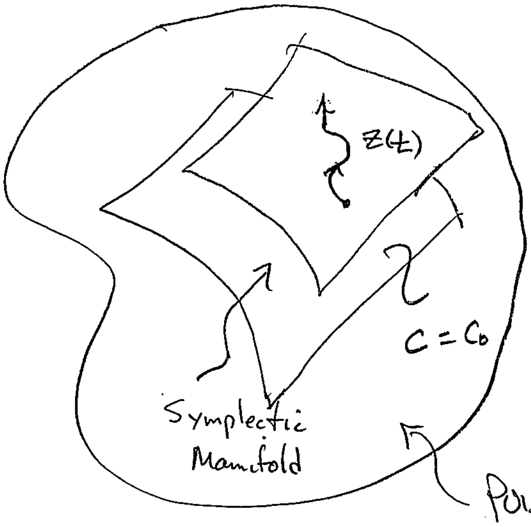
Noncanonical Hamiltonian Systems

Not Darboux & "too many" coords

$$\dot{z} = J\nabla H = \{z, H\} \qquad \{f,g\} = \frac{\partial f}{\partial z^i} J^{ij} \frac{\partial g}{\partial z^j} = \nabla f \cdot J \cdot \nabla g$$

$S = J\cdot\nabla J + \circlearrowleft = 0$ Jacobi Identity

J cosymplectic form dual to 2-form on leaf



equilibria

$J\nabla H = 0 \Rightarrow \nabla H \in \text{Null}(J)$

$\nabla H = -\lambda \nabla C$ spans Null(J)

C = Casimir Invariant

equilibria in frame

$J\cdot\nabla(H+L) = 0$

angular momenta

rotating $\Leftarrow$

EX. 2D Euler (PJM 1981 & P. Olver 1982)

$\xi = (x,y) \in D \subset \mathbb{R}^2, \mathbb{T}^2, \ldots$

$q(z,t)$: Vorticity-like field

$$\frac{\partial q}{\partial t} + [\psi, q] = 0 \; ; \qquad q = \mathcal{L}\psi$$

elliptic operator

$$\boxed{H[q] = \frac{1}{2}\int q\psi \, d^2x}$$

$$\boxed{\{F,G\} = \int_D q\,[F_q, G_q]\, d\xi} \leftarrow \text{P.B.}$$

$$\{F,G\} = \int_D d\xi \, \frac{\delta F}{\delta q(\xi)} \mathcal{J} \frac{\delta G}{\delta q(\xi)}$$

$F_q = \frac{\delta F}{\delta q}$ = Variational Derivative

$\mathcal{J} = -[q, \cdot\,]$ cosymp. operator

$[f,g] = f_x g_y - f_y g_x \qquad C[q] = \int \mathcal{C}(q)\, dz$

$q(\xi(\xi_0,t)) = \mathring{q}(\xi_0)$

Rearrangement area preserving

$$\boxed{L[q] = -\frac{1}{2}\int_D (x^2+y^2)\, q \, d\xi}$$ ang. momentum

$\{L, H\} = 0$

Metric Systems

$\dot{z} = g \cdot \nabla S \doteq (z, S)$

$(f, g) = \nabla S \cdot g \cdot \nabla S$

Cahn - Hilliard
Ricci

$\dot{S} = \nabla S \cdot g \cdot \nabla S \geq 0$

Boltzmann's "H-theorem"
Lyapunov's Thm. w/ S as Lyapunov function

Double Brackets (Vallis, Carnevale, Young) (Shepherd) (1990)

$\dot{z} = ((z, F))$

$\dot{z}^i = \sum_j J^{ij} J^{jl} \frac{\partial F}{\partial z^l}$

$\overline{J}^2 = J^2$

Flow preserves Casimirs

$\dot{F} \geq 0$ vanishes @ $\nabla F \in Null(J^2) = Null(J)$

Metriplectic Generalization

χ a field (PJM 1985)

$\frac{\partial \chi}{\partial t} = \{\chi, F\} + ((\chi, F))$

Ham. flow

metric flow

$$((F, G)) = \int d\xi' \int d\xi \, \frac{\delta F}{\delta \chi^i(\xi)} \mathcal{G}^{ij}(\xi, \xi') \frac{\delta G}{\delta \chi^j(\xi')}$$

$$\{F, G\} = \int d\xi' \int d\xi \, \frac{\delta F}{\delta \chi^i(\xi)} J^{ij}(\xi, \xi') \frac{\delta G}{\delta \chi^j(\xi')}$$

VCY: $\mathcal{G} = J^2$

Better: $J^{ik} K_{kl} J^{lj}$

smoothing

Dirac Brackets

Choose invariants $\{C_\alpha\}$

$\{C_\alpha, C_\beta\} = \mathcal{C}_{\alpha\beta}$

$\mathcal{C}^{\alpha\beta} = (\mathcal{C}_{\alpha\beta})^{-1}$

$$\boxed{\{F, G\}_D = \{F, G\} - \{F, C_\alpha\} \mathcal{C}^{\alpha\beta} \{C_\beta, G\}}$$

← Works for any "Good" bracket. Use Lie-Poisson

$\{C_\alpha, G\}_D = 0 \quad \forall G$

$(M, \{,\}_D)$ a Poisson Manifold

Our Double Bracket

$$((F,G)) = \alpha \int d\xi \int d\xi' \{F, q(\xi')\}_D \, K(\xi', \xi) \{q(\xi''), G\}_D$$

$$\boxed{\begin{array}{c} \alpha > 0 \\ \text{Max} \\ \alpha < 0 \\ \text{min} \end{array}}$$

$$\{F,G\}_D = \{F,G\} - \{F, C_\alpha\} C^{\alpha\beta} \{C_\beta, G\}$$

2D Euler

$\mathcal{J} K = \delta$ ← choose

$\{C_\alpha\} = \{ L, S, T, \ldots \}$

Ang. momentum

$S = \int d\xi \; xy \, q$

"strain moment"

$$\boxed{\mathcal{J} = H + C + P}$$

Generator

Dynamics

H:	$\frac{\partial O}{\partial t} = \{O, \mathcal{J}\}$	Hamiltonian
HD:	$\frac{\partial O}{\partial t} = \{O, \mathcal{J}\}_D$	Hamiltonian Dirac
SA:	$\frac{\partial O}{\partial t} = \sigma \{F, \mathcal{J}\} + \alpha ((O, \mathcal{J}))$	Sim. Annealing $\alpha \in \{\pm 1\}$ $\sigma \in \{0, 1\}$
DSA:	$\frac{\partial O}{\partial t} = \sigma \{O, \mathcal{J}\} + \alpha ((O, \mathcal{J}))_D$	Dirac Sim. Annealing

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Go to
PDF's & GIFS.

Fig.1 pdf — H-L diagram ; $\underline{h1}$

Fig 3 Gif — 2-fold $\boxed{H}$; els_1_m0 — IC $q = e^{-(r/r_0)^{10}}$

Fig 4b pdf — filaments ; el_20 — $r_0 = 1 + \epsilon \cos 2\theta$, $\epsilon = 0.4$

Fig 6. Gif — 2-fold $\boxed{SA\ \sigma = 0 \atop \alpha = 1}$; els_2_m0 — Kelvin max

Fig 7 pdf — H & C vs t for Fig 6 ; f6

Fig 8 Gif — 2-fold $\boxed{HD}$; els_3_m0 — can't go to max w/ L = const
$C_1 = -L$, $C_2 = S$ — Kirchoff-like

Fig 10 Gif — 2-fold $\boxed{DSA\ \sigma=0 \atop \alpha=1}$; els_4_m0 — finds ellipses w/ tines

Fig 11b pdf — H C vs t for Fig 10 ; els_4_m0_hc-d — C's const ~~yet~~ H↗

Fig. 12 Gif — 2-fold $\boxed{DSA\ \sigma=1 \atop \alpha=1}$; els_4_m1 — finds it w/ tines

Kelvin Sponge $\boxed{\text{Go to 6}}$

Fig 14 Gif — 2-fold $\boxed{SA\ \sigma=0 \atop \alpha=-1}$; els_2_p0 — Note expansion

Fig 15 pdf — H, L, S vs t ; els_2_p0_hcl — H decreasing, L decreasing, $C_2 = S \approx$ const

Fig. 16 Gif — $\boxed{DSA\ \sigma=0 \atop \alpha=-1}$; els_4_p0

Fig. 21 Gif — 3-fold $\boxed{DSA\ \sigma=0 \atop \alpha=1}$; tri-db2 — 3 tines form
$\bar{X}$ shifts ⇒ more constraints

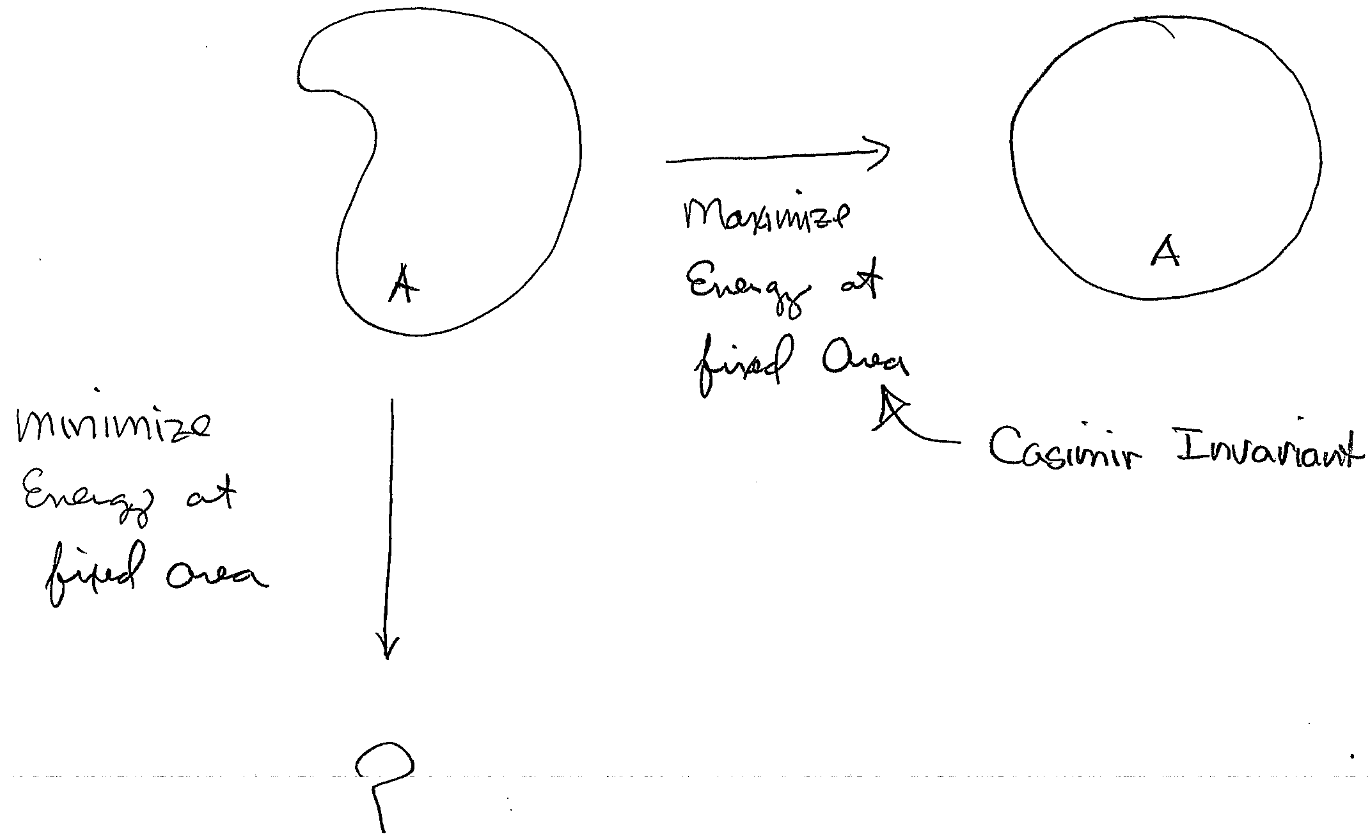
Fig. 27 Gif — Dipole $\boxed{DSA\ \sigma=0 \atop \alpha=1}$; dip_4_m0 — elongates ≈ Lamb Vortex

P_x, S

prevents translation

Kelvin Sponge $\quad H = \int_{\text{supp}} \frac{v^2}{2}\, d\xi$

Example: Vortex Patch



All runs done w/

- Domain 8 or 16 units
- 256 x 256 points
- Pseudospectral
- Integrals as sums
- Adams - Bashforth 2nd Order