On an inclusive curvature-like framework for describing dissipation: metriplectic 4-bracket dynamics

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Geometry of metriplectic 4-brackets: with Michael Updike

pjm & M. Updike, **arXiv:2306.06787v1** [math-ph] 11 Jun 2023.

Dynamics – Theories – Models

Goal:

Predict the future or explain the past $\Rightarrow$

$$\dot{z} = V(z)\,, \qquad z \in \mathcal{Z}\,, \text{ Phase Space}$$

A dynamical system. Maps, ODEs, PDEs, etc.

Whence vector field V?

• Fundamental parent theory (microscopic, N interacting gravitating or charged particles, BBGKY hierarchy, Vlasov-Maxwell system, ...). Identify small parameters, rigorous asymptotics $\rightarrow$ Reduced Computable Model V.

• Phenomena based modeling using known properties, constraints, etc. used to intuit $\rightarrow$
Reduced Computable Model V. $\leftarrow$ structure can be useful.

Types of Vector Fields, $V(z)$ (cont)

Only (?) Natural Split:

$$V(z) = V_H + V_D$$

• Hamiltonian vector fields, V_H: conservative, properties, etc.

• Dissipative vector fields, V_D: not conservative of something, relaxation/asymptotic stability, etc.

General Hamiltonian Form:

finite dim $\rightarrow$ $\quad V_H = J\,\dfrac{\partial H}{\partial z} \qquad$ or $\qquad V_H = \mathcal{J}\dfrac{\delta H}{\delta \psi}$ $\quad \leftarrow \infty$ dim

where $J(z)$ is Poisson tensor/operator and H is the Hamiltonian. Basic product decomposition.

General Dissipation:

$$V_D = ?... \qquad \rightarrow \qquad V_D = G\,\frac{\partial F}{\partial z}$$

Why investigate? General properties of theory. Build in thermodynamic consistency. Useful for computation.

Metriplectic Dynamics

(Metric $\cup$ Symplectic Flows)

- Natural split of vector fields

- Enforces thermodynamic consistency: $\dot{H} = 0$ the 1st Law and $\dot{S} \geq 0$ the 2nd Law.

- **Encompassing 4-bracket theory**: "curvature" as dissipation

Ideas of Casimirs are candidates for entropy, multibracket, curvature, etc. in PJM, *Bracket formulation for irreversible classical fields*, Phys. Lett. A **100**, 423–427 (1984).

Poisson Brackets – Flows on Poisson Manifolds

Definition. A Poisson manifold $\mathcal{Z}$ has bracket

$$\{\,,\,\}: C^\infty(\mathcal{Z}) \times C^\infty(\mathcal{Z}) \to C^\infty(\mathcal{Z})$$

st $C^\infty(\mathcal{Z})$ with $\{\,,\,\}$ is a Lie algebra realization, i.e., is

- bilinear,
- antisymmetric,
- Jacobi, and
- Leibniz, i.e., acts as a derivation $\Rightarrow$ vector field.

Geometrically $C^\infty(\mathcal{Z}) \equiv \Lambda^0(\mathcal{Z})$

$$\{f,g\} = J(\boldsymbol{d}f \wedge \boldsymbol{d}g) = \langle \boldsymbol{d}f, J\boldsymbol{d}g\rangle = J(\boldsymbol{d}f, \boldsymbol{d}g)\,.$$

Flows are integral curves of noncanonical Hamiltonian vector fields, JdH, i.e., $\dot{z} = J\partial H/\partial z$.

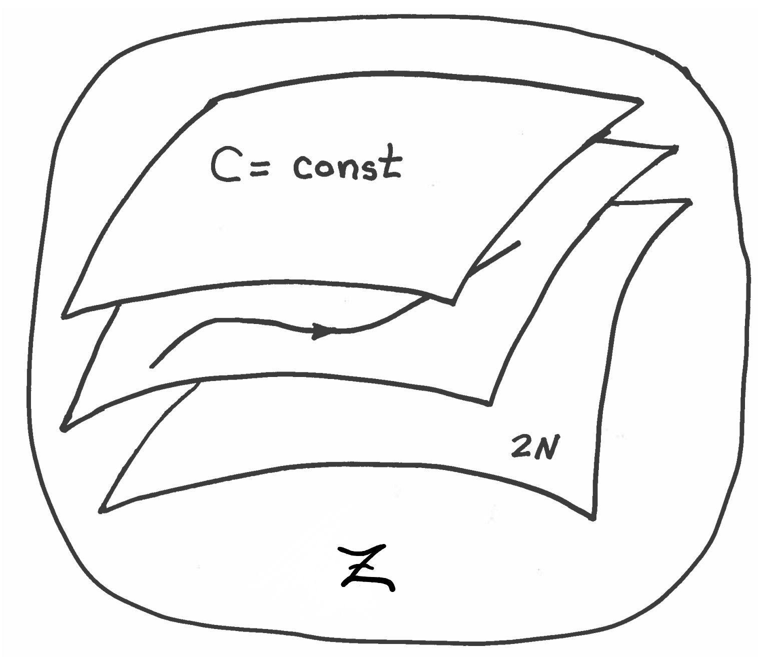
Because of degeneracy, $\exists$ functions C st $\{A,C\} = 0$ for all $A \in C^\infty(\mathcal{Z})$. Casimir invariants (Lie's distinguished functions!).

Poisson Manifold (phase space) $\mathcal{Z}$ Cartoon

Degeneracy in $J \Rightarrow$ Casimirs:

$$\{A, C\} = 0 \quad \forall\ A : \mathcal{Z} \to \mathbb{R}$$

Lie-Darboux Foliation by Casimir (symplectic) leaves:



Metriplectic 4-Bracket: $(f, k; g, n)$

Why a 4-Bracket

- Two slots for two fundamental functions: Hamiltonian, H, and Entropy (Casimir), S.

- Leaves two slots for bilinear bracket: one for observable one for generator

- Provides natural reductions to other bilinear brackets.

- The three slot brackets of pjm 1984 were not trilinear.

The Metriplectic 4-Bracket

4-bracket on 0-forms (functions):

$$(\,\cdot\,,\,\cdot\,;\,\cdot\,,\,\cdot\,)\colon \Lambda^0(\mathcal{Z}) \times \Lambda^0(\mathcal{Z}) \times \Lambda^0(\mathcal{Z}) \times \Lambda^0(\mathcal{Z}) \to \Lambda^0(\mathcal{Z})$$

For functions $f, k, g, n \in \Lambda^0(\mathcal{Z})$

$$(f,k;g,n) := R(\boldsymbol{d}f, \boldsymbol{d}k, \boldsymbol{d}g, \boldsymbol{d}n)\,,$$

In a coordinate patch the metriplectic 4-bracket has the form:

$$(f,k;g,n) = R^{ijkl}(z)\frac{\partial f}{\partial z^i}\frac{\partial k}{\partial z^j}\frac{\partial g}{\partial z^k}\frac{\partial n}{\partial z^l}\,. \qquad \leftarrow \text{quadravector?}$$

• A blend of ideas: Two important functions H and S, symmetries, curvature idea, multilinear brackets all in pjm 1984, 1986.

• Manifolds with both Poisson tensor J and compatible metric, g or connection.

Metriplectic 4-Bracket Properties

(i) linearity in all arguments, e.g,

$$(f+h,k;g,n)=(h,k;g,n)+(h,k;g,n)$$

(ii) algebraic identities/symmetries

$$\begin{aligned}(f,k;g,n) &= -(k,f;g,n)\\ (f,k;g,n) &= -(f,k;n,g)\\ (f,k;g,n) &= (g,n;f,k)\\ (f,k;g,n) &+ (f,g;n,k)+(f,n;k,g)=0 \qquad \leftarrow \text{not needed}\end{aligned}$$

(iii) derivation in all arguments, e.g.,

$$(fh,k;g,n)=f(h,k;g,n)+(f,k;g,n)h$$

which is manifest when written in coordinates. Here, as usual, fh denotes pointwise multiplication. Symmetries of algebraic curvature. Although $R^l{}_{ijk}$ or R_{lijk} but not R^{lijk}. **Metriplectic Minimum.**

Reduction to Metriplectic 2-Bracket

(PJM 1984, 1986)

Symmetric 2-bracket:

$$(f,g)_H = (f,H;g,H) = (g,f)_H$$

Dissipative dynamics:

$$\dot{z} = (z,S)_H = (z,H;S,H)$$

Energy conservation:

$$(f,H)_H = (H,f)_H = 0 \qquad \forall f\,.$$

Entropy dynamics:

$$\dot{S} = (S,S)_H = (S,H;S,H) \geq 0$$

Metriplectic 4-brackets $\rightarrow$ metriplectic 2-brackets of 1984, 1986!

Metriplectic 4-Bracket: Encompassing Definition of Dissipation

- Lots of geometry on Poisson manifolds with metric or connection. Emerges naturally.

- Entropy production and positive contravariant sectional curvature. For $\sigma, \eta \in \Lambda^1(\mathcal{Z})$, entropy production by

$$\dot{S} = K(\sigma, \eta) := (S, H; S, H)\,,$$

where the second equality follows if $\sigma = \boldsymbol{d}S$ and $\eta = \boldsymbol{d}H$.

Binary Brackets for Dissipation circa 1980 $\rightarrow$

- Symmetric Bilinear Brackets (pjm 1980 $-\ldots$ unpublished, 1984 reduced MHD)

- Antisymmetric Bracket (possibly degenerate) (Kaufman and pjm 1982)

- Metriplectic Dynamics (pjm 1984,1984, 1986, $\ldots$ Kaufman 1984 no degeneracy)

- GENERIC (Grmela 1984, with Oettinger 1997, . . .) Binary but **not** Symmetric and **not** Bilinear $\Leftrightarrow$ Metriplectic Dynamics!

- Double Brackets (Vallis, Carnevale, Young, Shepherd; Brockett, Bloch ... 1989)

4-Bracket Reduction to K-M Brackets

(Kaufman and Morrison 1982)

Done for plasma quasilinear theory.

Dynamics:

$$\dot{z} = [z, H]_S = (z, H; S, H)$$

Bracket Properties:

$$[f, g]_S = (f, g; S, H)$$

- bilinear
- antisymmetric, possibly degenerate
- energy conservation and entropy production

$$\dot{H} = [H, H]_S = 0 \quad \text{and} \quad \dot{S} = [S, H]_S \geq 0 \quad \Rightarrow \quad z \mapsto z_{eq}$$

4-Bracket Reduction to Double Brackets

(Vallis, Carnevale; Brockett, Bloch ... 1989)

Interchanging the role of H with a Casimir S:

$$(f, g)_S = (f, S; g, S)$$

Can show with assumptions (Koszul construction)

$$(C, g)_S = (C, S; g, S) = 0$$

for any Casimir C. Therefore $\dot{C} = 0$.

Beautiful geometry re Fernandes-Koszul connection!

4-Bracket Reduction GENERIC

(Grmela 1984, with Öttinger 1997)

- Bracket <u>not bilinear</u> and <u>not symmetric</u>

GENERIC Vector Field in terms of dissipation function $\Xi(z, z_*)$:

$$\dot{z}^i = Y_S^i = \left.\frac{\partial \Xi(z, z_*)}{\partial z_{*i}}\right|_{z_* = \partial S/\partial z} .$$

Special Case:

$$\Xi(z, z_*) = \frac{1}{2} \frac{\partial S}{\partial z^i} G^{ij}(z) \frac{\partial S}{\partial z^j} \quad \Rightarrow \quad Y_S^i = G^{ij}(z) \frac{\partial S}{\partial z^j} ,$$

- Exists a bracket and procedure for linearizing and symmetrizing

$\Rightarrow$

GENERIC (1997) $\equiv$ Metriplectic (1984,1986)!

General Constructions

- For any Riemannian manifold $\exists$ metriplectic 4-bracket. This means there is a wide class of them, but the bracket tensor does not need to come from Riemann tensor only satisfy the bracket properties.

Construction via Kulkarni-Nomizu Product

Given σ and μ, two symmetric rank-2 tensor fields operating on 1-forms $\boldsymbol{d}f, \boldsymbol{d}k$ and $\boldsymbol{d}g, \boldsymbol{d}n$, the K-N product is

$$\begin{aligned}\sigma \owedge \mu\, (\boldsymbol{d}f, \boldsymbol{d}k, \boldsymbol{d}g, \boldsymbol{d}n) \;&=\; \sigma(\boldsymbol{d}f, \boldsymbol{d}g)\, \mu(\boldsymbol{d}k, \boldsymbol{d}n)\\ &-\; \sigma(\boldsymbol{d}f, \boldsymbol{d}n)\, \mu(\boldsymbol{d}k, \boldsymbol{d}g)\\ &+\; \mu(\boldsymbol{d}f, \boldsymbol{d}g)\, \sigma(\boldsymbol{d}k, \boldsymbol{d}n)\\ &-\; \mu(\boldsymbol{d}f, \boldsymbol{d}n)\, \sigma(\boldsymbol{d}k, \boldsymbol{d}g)\,.\end{aligned}$$

Metriplectic 4-bracket:

$$(f, k; g, n) = \sigma \owedge \mu(\boldsymbol{d}f, \boldsymbol{d}k, \boldsymbol{d}g, \boldsymbol{d}n)\,.$$

In coordinates:

$$R^{ijkl} = \sigma^{ik}\mu^{jl} - \sigma^{il}\mu^{jk} + \mu^{ik}\sigma^{jl} - \mu^{il}\sigma^{jk}\,.$$

Lie-Algebra Based Metriplectic 4-Brackets

- For structure constants $c^{kl}{}_s$:

$$(f,k;g,n) = c^{ij}{}_r c^{kl}{}_s\, g^{rs} \frac{\partial f}{\partial z^i}\frac{\partial k}{\partial z^j}\frac{\partial g}{\partial z^k}\frac{\partial n}{\partial z^l}\,.$$

Lacks symmetry, but $\exists$ procedure to remove torsion (cyclic Bianchi identity) for any symmetric 'metric' g^{rs}.

- For $g_{CK}^{rs} = c^{rl}{}_k\, c^{sk}{}_l$ the Cartan-Killing metric, torsion vanishes automatically

Final Comments

• See PJM & M. Updike, arXiv:2306.06787v1 [math-ph] 11 Jun 2023 for many examples, finite and infinite.

• Useful for thermodynamically consistent model building.

• Given that double brackets and metriplectic brackets have been used for computation of equilibria, metriplectic 4-bracket can be a new tool.

• New kind of structure to preserve.

References:

[1] P. J. Morrison and J. M. Greene, Phys. Rev. Lett. **45**, 790 (1980)
[2] P. J. Morrison, AIP Conf. Proc. **88**, 13 (1982)
[3] P. J. Morrison and R. D. Hazeltine, Phys. Fluids **27**, 886 (1984)
[4] A. N. Kaufman and P. J. Morrison, Phys. Lett. A **88**, 405 (1982)
[5] A. N. Kaufman, Phys. Lett. A 100, **419** (1984)
[6] P. J. Morrison, Phys. Lett. A **100**, 423 (1984)
[7] P. J. Morrison, Tech. Rep. PAM–**228**, Univ. Calif. Berkeley (1984)
[8] P. J. Morrison, Physica D **18**, 410 (1986)
[9] P. J. Morrison, J. Physics: Conf. Ser. **169**, 012006 (2009)
[10] M. Materassi and P. J. Morrison, J. Cybernetics & Physics **7**, 78 (2015)
[11] B. Coquinot and P. J. Morrison, J. Plasma Phys. **86**, 835860302 (2020)
[12] P. J. Morrison, Phys. Plasmas **24**, 055502 (2017)
[13] M. Kraus and E. Hirvijok, Phys. Plasmas **24**, 102311 (2017)
[14] C. Bressan, M. Kraus, P. J. Morrison, and O. Maj, J. Phys.: Conf. Series 1**125**, 012002 (2018)
[15] C. Bressan, Ph.D. thesis, Technical University of Munich (2022)
[16] R. W. Brockett, Proc. IEEE **27**, 799 (1988)
[17] G. Vallis, G. Carnevale, and W. Young, J. Fluid Mech. **207**, 133 (1989)
[18] G. R. Flierl and P. J. Morrison, Physica D **240**, 212 (2011)
[19] G. R. Flierl, P. J. Morrison, and R. V. Swaminathan, Fluids: Topical Collection "Geophysical Fluid Dynamics" **4**, 104 (2019)
[20] M. Furukawa and P. J. Morrison, Plasma Phys. Control. Fusion **59**, 054001 (2017)
[21] M. Furukawa, T. Watanabe, P. J. Morrison, and K. Ichiguchi, Phys. Plasmas **25**, 082506 (2018)
[22] M. Furukawa and P. J. Morrison, Phys. Plasmas **29**, 102504 (2022)
[23] P. J. Morrison and M. H. Updike, **arXiv:2306.06787v1** [math-ph] 11 Jun 2023.