The metriplectic 4-bracket: a curvature-like framework for describing dissipation in joined Hamiltonian and dissipative fluid and plasma

Philip J. Morrison

Department of Physics
Institute for Fusion Studies, and ODEN Institute
The University of Texas at Austin
morrison@physics.utexas.edu
http://www.ph.utexas.edu/∼morrison/

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Collaborators: G. Flierl, M. Furukawa, C. Bressan, O. Maj, M. Kraus, E. Sonnendrücker, ...

Geometry of metriplectic 4-brackets: with Michael Updike

pjm & M. Updike, arXiv:2306.06787v1 [math-ph] 11 Jun 2023.

→ **Theory of thermodynamically consistent theories!**

Dynamics – Theories – Models

Goal:

Predict the future or explain the past $\Rightarrow$

$$\dot{z} = V(z)\,, \qquad\qquad z \in \mathcal{Z}\,, \text{ Phase Space}$$

A dynamical system. Maps, ODEs, PDEs, etc.

Whence vector field V?

• Fundamental parent theory (microscopic, N interacting gravitating or charged particles, BBGKY hierarchy, Vlasov-Maxwell system, ...). Identify small parameters, rigorous asymptotics $\rightarrow$ Reduced Computable Model V.

• Phenomena based modeling using known properties, constraints, etc. used to intuit $\rightarrow$
Reduced Computable Model V. $\leftarrow$ structure can be useful.

Types of Vector Fields, $V(z)$ (cont)

Only (?) Natural Split:

$$V(z) = V_H + V_D$$

- Hamiltonian vector fields, V_H: conservative, properties, etc.
- Dissipative vector fields, V_D: not conservative of something, relaxation/asymptotic stability, etc.

General Hamiltonian Form:

finite dim $\rightarrow$ $\quad V_H = J\,\frac{\partial H}{\partial z}$ or $V_H = \mathcal{J}\frac{\delta H}{\delta \psi}$ $\quad \leftarrow \infty$ dim

where $J(z)$ is Poisson tensor/operator and H is the Hamiltonian. Basic product decomposition.

General Dissipation:

$$V_D = ?... \qquad \rightarrow \qquad V_D = G\,\frac{\partial F}{\partial z}$$

Why investigate? General properties of theory. Build in thermodynamic consistency. Useful for computation.

Codifying Dissipation – Some History

Is there a framework for dissipation akin to the Hamiltonian formulation for nondissipative systems?

Rayleigh (1873): $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_\nu}\right) - \left(\frac{\partial \mathcal{L}}{\partial q_\nu}\right) + \left(\frac{\partial \mathcal{F}}{\partial \dot{q}_\nu}\right) = 0$

Linear dissipation e.g. of sound waves. *Theory of Sound*.

Cahn-Hilliard (1958): $\frac{\partial n}{\partial t} = \nabla^2 \frac{\delta F}{\delta n} = \nabla^2 \left(n^3 - n - \nabla^2 n\right)$

Phase separation, nonlinear diffusive dissipation, binary fluid ..

Other Gradient Flows: $\frac{\partial \psi}{\partial t} = \mathcal{G}\, \frac{\delta F}{\delta \psi}$

Otto, Ricci Flows, Poincarè conjecture on S^3, Perelman (2002)...

Metriplectic Dynamics

(Metric $\cup$ Symplectic Flows)

- Formalism for natural split of vector fields

- Enforces thermodynamic consistency: $\dot{H} = 0$ the 1st Law and $\dot{S} \geq 0$ the 2nd Law. Other invariants?

- **Encompassing 4-bracket theory**: "curvature" as dissipation

Ideas of Casimirs are candidates for entropy, multibracket, curvature, etc. in PJM, *Bracket formulation for irreversible classical fields*, Phys. Lett. A **100**, 423–427 (1984). Metriplectic in P. J. Morrison, Physica D 18, 410 (1986)

Poisson Brackets – Flows on Poisson Manifolds

Definition. A Poisson manifold $\mathcal{Z}$ has bracket

$$\{\, ,\,\}: C^\infty(\mathcal{Z}) \times C^\infty(\mathcal{Z}) \to C^\infty(\mathcal{Z})$$

st $C^\infty(\mathcal{Z})$ with $\{\, ,\,\}$ is a Lie algebra realization, i.e., is

- bilinear,
- antisymmetric,
- Jacobi, and
- Leibniz, i.e., acts as a derivation $\Rightarrow$ vector field.

Geometrically $C^\infty(\mathcal{Z}) \equiv \Lambda^0(\mathcal{Z})$ and $\boldsymbol{d}$ exterior derivative.

$$\{f, g\} = J(\boldsymbol{d}f \wedge \boldsymbol{d}g) = \langle \boldsymbol{d}f, J\boldsymbol{d}g\rangle = J(\boldsymbol{d}f, \boldsymbol{d}g)\,.$$

J the Poisson tensor/operator. Flows are integral curves of non-canonical Hamiltonian vector fields, $J\boldsymbol{d}H$, i.e., $\dot{z}^i = J^{ij}\partial H/\partial z^j$.

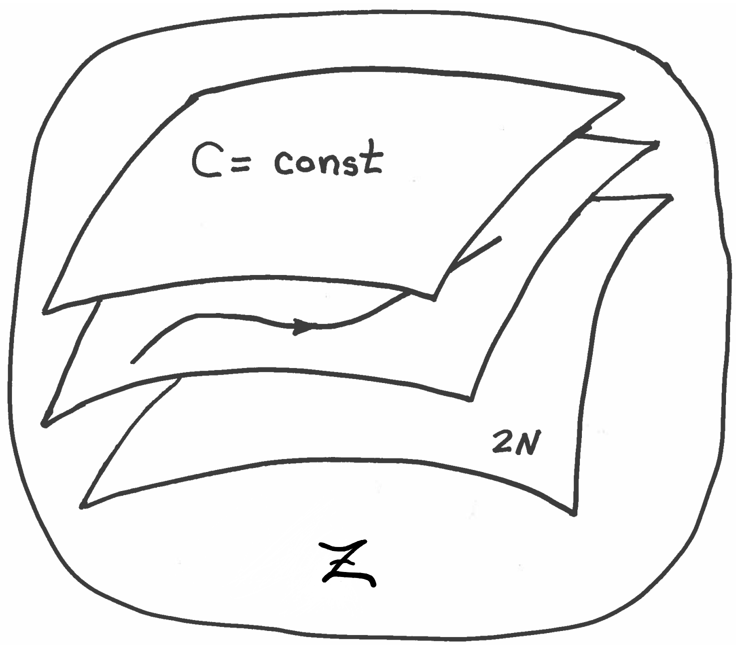
Because of degeneracy, $\exists$ functions C st $\{f, C\} = 0$ for all $f \in C^\infty(\mathcal{Z})$. Casimir invariants (Lie's distinguished functions!).

Poisson Manifold (phase space) $\mathcal{Z}$ Cartoon

Degeneracy in J $\Rightarrow$ Casimirs:

$$\{f, C\} = 0 \quad \forall\ f : \mathcal{Z} \rightarrow \mathbb{R}$$

Lie-Darboux Foliation by Casimir (symplectic) leaves:



Metriplectic 4-Bracket: $(f, k; g, n)$

Why a 4-Bracket?

• Two slots for two fundamental functions: Hamiltonian, H, and Entropy (Casimir), S.

• Leaves two slots for bilinear bracket: one for observable one for generator s.t. $\dot{H} = 0$ and $\dot{S} \geq 0$.

• Provides natural reductions to other bilinear brackets.

• The three slot brackets of pjm 1984 were not trilinear. Four needed to be multilinear.

The Metriplectic 4-Bracket

4-bracket on 0-forms (functions):

$$(\,\cdot\,,\,\cdot\,;\,\cdot\,,\,\cdot\,)\colon \Lambda^0(\mathcal{Z}) \times \Lambda^0(\mathcal{Z}) \times \Lambda^0(\mathcal{Z}) \times \Lambda^0(\mathcal{Z}) \rightarrow \Lambda^0(\mathcal{Z})$$

For functions $f, k, g, n \in \Lambda^0(\mathcal{Z})$

$$(f, k; g, n) := R(\boldsymbol{d}f, \boldsymbol{d}k, \boldsymbol{d}g, \boldsymbol{d}n)\,,$$

In a coordinate patch the metriplectic 4-bracket has the form:

$$(f, k; g, n) = R^{ijkl}(z)\frac{\partial f}{\partial z^i}\frac{\partial k}{\partial z^j}\frac{\partial g}{\partial z^k}\frac{\partial n}{\partial z^l}\,. \qquad \leftarrow \text{quadravector?}$$

- A blend of ideas: Two important functions H and S, symmetries, curvature idea, multilinear brackets all in pjm 1984, 1986.
- Manifolds with both Poisson tensor J and compatible metric, g or connection.

Metriplectic 4-Bracket Properties

(i) linearity in all arguments, e.g,

$$(f+h,k;g,n) = (h,k;g,n) + (h,k;g,n)$$

(ii) algebraic identities/symmetries

$$\begin{aligned}
(f,k;g,n) &= -(k,f;g,n) \\
(f,k;g,n) &= -(f,k;n,g) \\
(f,k;g,n) &= (g,n;f,k) \\
(f,k;g,n) &+ (f,g;n,k) + (f,n;k,g) = 0 \qquad \leftarrow \text{not needed}
\end{aligned}$$

(iii) derivation in all arguments, e.g.,

$$(fh,k;g,n) = f(h,k;g,n) + (f,k;g,n)h$$

which is manifest when written in coordinates. Here, as usual, fh denotes pointwise multiplication. Symmetries of algebraic curvature. Although $R^{l}{}_{ijk}$ or R_{lijk} but not R^{lijk}. **Metriplectic Minimum.**

Reduction to Metriplectic 2-Bracket

(PJM 1984, 1986)

Symmetric 2-bracket:

$$(f,g)_H = (f,H;g,H) = (g,f)_H$$

Dissipative dynamics:

$$\dot{z} = (z,S)_H = (z,H;S,H)$$

Energy conservation:

$$(f,H)_H = (H,f)_H = 0 \qquad \forall f\,.$$

Entropy dynamics:

$$\dot{S} = (S,S)_H = (S,H;S,H) \geq 0$$

Metriplectic 4-brackets $\rightarrow$ metriplectic 2-brackets of 1984, 1986!

Metriplectic 4-Bracket: Encompassing Definition of Dissipation

• Lots of geometry on Poisson manifolds with metric or connection. Emerges naturally.

• If Riemannian, entropy production is positive contravariant sectional curvature. For $\sigma, \eta \in \Lambda^1(\mathcal{Z})$, entropy production by

$$\dot{S} = K(\sigma, \eta) := (S, H; S, H)\,,$$

where the second equality follows if $\sigma = \boldsymbol{d}S$ and $\eta = \boldsymbol{d}H$.

Binary Brackets for Dissipation circa 1980 $\rightarrow$

- Symmetric Bilinear Brackets (pjm 1980 $-\dots$ unpublished, 1984 reduced MHD)

- Antisymmetric Bracket (possibly degenerate) (Kaufman and pjm 1982)

- Metriplectic Dynamics (pjm 1984,1984, 1986, $\dots$ Kaufman 1984 no degeneracy)

- GENERIC (Grmela 1984, with Oettinger 1997, $\dots$) Binary but **not** Symmetric and **not** Bilinear $\Leftrightarrow$ Metriplectic Dynamics!

- Double Brackets (Vallis, Carnevale, Young, Shepherd; Brockett, Bloch ... 1989)

4-Bracket Reduction to K-M Brackets

(Kaufman and Morrison 1982)

Done for plasma quasilinear theory.

Dynamics:

$$\dot{z} = [z, H]_S = (z, H; S, H)$$

Bracket Properties:

$$[f, g]_S = (f, g; S, H)$$

- bilinear
- antisymmetric, possibly degenerate
- energy conservation and entropy production

$$\dot{H} = [H, H]_S = 0 \quad \text{and} \quad \dot{S} = [S, H]_S \geq 0 \quad \Rightarrow \quad z \mapsto z_{eq}$$

4-Bracket Reduction to Double Brackets

(Vallis, Carnevale; Brockett, Bloch ... 1989)

Interchanging the role of H with a Casimir S:

$$(f, g)_S = (f, S; g, S)$$

Can show with assumptions (Koszul construction)

$$(C, g)_S = (C, S; g, S) = 0$$

for any Casimir C. Therefore $\dot{C} = 0$.

Practical tool for equilibria computation $\rightarrow$ Beautiful geometry with Fernandes-Koszul connection!

4-Bracket Reduction GENERIC

(Grmela 1984, with Öttinger 1997)

- Bracket not bilinear and not symmetric

GENERIC Vector Field in terms of dissipation function $\Xi(z, z_*)$:

$$\dot{z}^i = Y_S^i = \left.\frac{\partial \Xi(z, z_*)}{\partial z_{*i}}\right|_{z_* = \partial S/\partial z} .$$

Special Case:

$$\Xi(z, z_*) = \frac{1}{2}\frac{\partial S}{\partial z^i} G^{ij}(z)\frac{\partial S}{\partial z^j} \quad \Rightarrow \quad Y_S^i = G^{ij}(z)\frac{\partial S}{\partial z^j} ,$$

- Exists a bracket and procedure for linearizing and symmetrizing

$\Rightarrow$

GENERIC (1997) $\equiv$ Metriplectic (1984,1986)!

Existence – General Constructions

- For any Riemannian manifold $\exists$ metriplectic 4-bracket. This means there is a wide class of them, but the bracket tensor does not need to come from Riemann tensor only needs to satisfy the bracket properties.

- Methods of construction?

Construction via Kulkarni-Nomizu Product

Given σ and μ, two symmetric rank-2 tensor fields operating on 1-forms $\boldsymbol{d}f, \boldsymbol{d}k$ and $\boldsymbol{d}g, \boldsymbol{d}n$, the K-N product is

$$\begin{aligned}
\sigma \owedge \mu\,(\boldsymbol{d}f, \boldsymbol{d}k, \boldsymbol{d}g, \boldsymbol{d}n) \;&=\; \sigma(\boldsymbol{d}f, \boldsymbol{d}g)\,\mu(\boldsymbol{d}k, \boldsymbol{d}n)\\
&-\; \sigma(\boldsymbol{d}f, \boldsymbol{d}n)\,\mu(\boldsymbol{d}k, \boldsymbol{d}g)\\
&+\; \mu(\boldsymbol{d}f, \boldsymbol{d}g)\,\sigma(\boldsymbol{d}k, \boldsymbol{d}n)\\
&-\; \mu(\boldsymbol{d}f, \boldsymbol{d}n)\,\sigma(\boldsymbol{d}k, \boldsymbol{d}g)\,.
\end{aligned}$$

Metriplectic 4-bracket:

$$(f, k; g, n) = \sigma \owedge \mu(\boldsymbol{d}f, \boldsymbol{d}k, \boldsymbol{d}g, \boldsymbol{d}n)\,.$$

In coordinates:

$$R^{ijkl} = \sigma^{ik}\mu^{jl} - \sigma^{il}\mu^{jk} + \mu^{ik}\sigma^{jl} - \mu^{il}\sigma^{jk}\,.$$

Lie-Algebra Based Metriplectic 4-Brackets

- For structure constants ${c^{kl}}_s$:

$$(f,k;g,n) = {c^{ij}}_r {c^{kl}}_s \, g^{rs} \frac{\partial f}{\partial z^i}\frac{\partial k}{\partial z^j}\frac{\partial g}{\partial z^k}\frac{\partial n}{\partial z^l}\,.$$

Lacks symmetry, but $\exists$ procedure to remove torsion (cyclic Bianchi identity) for any symmetric 'metric' g^{rs}. Dynamics does not see torsion, but manifold does.

- For $g^{rs}_{CK} = {c^{rl}}_k \, {c^{sk}}_l$ the Cartan-Killing metric, torsion vanishes automatically

Examples

- finite-dimensional
- 1+1 fluid theory
- 3+1 fluid theory
- kinetic theory

Free Rigid Body

Angular momenta (L^1, L^2, L^3), Lie-Poisson bracket with Lie algebra $\mathfrak{so}(3)$, $c^{ij}_{\ \ k} = -\epsilon_{ijk}$.

Hamiltonian:

$$H = \frac{(L^1)^2}{2I_1} + \frac{(L^2)^2}{2I_2} + \frac{(L^3)^2}{2I_3}$$

principal moments of inertia, I_i Casimir

$$C = ||L||^2 = (L^1)^2 + (L^3)^2 + (L^3)^2 = S\,,$$

Euler's equations:

$$\dot{L}^i = \{L^i, H\}$$

"Thermodynamics" $\to$ design a system s.t. $\dot{H} = 0$ and $\dot{S} \leq 0$.

“Thermodynamics” Free Rigid Body

Use K-N product. Choose $\sigma^{ij} = \mu^{ij} = g^{ij} \ \Rightarrow$

$$R^{ijkl} = K\left(g^{ik}g^{jl} - g^{il}g^{jk}\right)\,,$$

Riemannian *Space form* with constant sectional curvature K.

Assume Euclidean gives metriplectic 4-bracket:

$$(f,k;g,n) = K\left(\delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk}\right)\frac{\partial f}{\partial z^i}\frac{\partial k}{\partial z^j}\frac{\partial g}{\partial z^k}\frac{\partial n}{\partial z^l}\,,$$

Metriplectic 2-bracket:

$$(f,g)_H = (f,H;g,H)$$

Precisely bracket and dynamics of pjm 1986!

$$\dot{L}^i = \{L^i,H\} + (L^i,S)_H = \{L^i,H\} + (L^i,H;S,H)$$

1D fluid $u(x,t)$

Again use K-N product with operators Σ and M

$$(F,K;G,N) \;=\; \int_{\mathbb{R}} dx\, W\Big(\Sigma(F_u,G_u)M(K_u,N_u) \\ -\Sigma(F_u,N_u)M(K_u,G_u) + M(F_u,G_u)\Sigma(K_u,N_u) \\ -M(F_u,N_u)\Sigma(K_u,G_u)\Big)\,,$$

W a constant and $F_u = \delta F/\delta u$, etc.
Choose

$$M(F_u,G_u) = F_u G_u$$

$$\Sigma(F_u,G_u)(x) = \partial F_u(x)\mathcal{H}[G_u](x) + \partial G_u(x)\mathcal{H}[F_u](x)\,,$$

$\partial = \partial/\partial x$ and $\mathcal{H}$ the Hilbert transform $\Rightarrow$

$$(F,G)_H = (F,H;G,H) = \int_{\mathbb{R}} dx\, W\Big(\partial F_u \mathcal{H}[G_u] + \partial G_u \mathcal{H}[F_u]\Big)\,.$$

$$u_t = ...(u,S)_H = -2W\,\mathcal{H}[\partial u]\,.$$

Ott & Sudan 1969 fluid model of electron Landau damping (Hammett-Perkins 1990).

Thermodynamic Navier-Stokes: $\chi = \{\rho, \sigma = \rho s, \boldsymbol{M} = \rho \boldsymbol{v}\}$

K-N again:

$$M(F_\chi, G_\chi) = F_\sigma G_\sigma$$

$$\Sigma(F_\chi, G_\chi) = \widehat{\Lambda}_{ijkl}\, \partial_j F_{M_i} \partial_k G_{M_l} + a\, \nabla F_\sigma \cdot \nabla G_\sigma$$

$\partial_i := \partial / \partial x^i$ with general isotropic Cartesian tensor of order 4

$$\widehat{\Lambda}_{ikst} = \alpha \delta_{ik}\delta_{st} + \beta(\delta_{is}\delta_{kt} + \delta_{it}\delta_{ks}) + \ \gamma(\delta_{is}\delta_{kt} - \delta_{it}\delta_{ks})$$

Construct

$$(F, G)_H = (F, H; G, H) \quad \rightarrow \quad \chi_t = \{\chi, H\} + (\chi, S)_H \Rightarrow$$

using $S = \int d^3x\, \rho s$ and $H = \int d^3x \Big(\rho |\boldsymbol{v}|^2/2 + \rho U(\rho, s)\Big)$

$$\partial_t \boldsymbol{v} = -\boldsymbol{v} \cdot \nabla \boldsymbol{v} - \frac{1}{\rho} \nabla p + \frac{1}{\rho} \nabla \cdot \mathcal{T}$$

$$\partial_t \rho = -\nabla \cdot (\rho \boldsymbol{v})$$

$$\partial_t s = -\boldsymbol{v} \cdot \nabla s - \frac{1}{\rho T} \nabla \cdot \boldsymbol{q} + \frac{1}{\rho T} \mathcal{T} : \nabla \boldsymbol{v}\,, \qquad \boldsymbol{q} = -\kappa \nabla T$$

Collision Operator

Phase space $z = (\boldsymbol{x}, \boldsymbol{v})$, density $f(z,t)$

Define operator on $w \colon \mathbb{R}^6 \to \mathbb{R}$ (at fixed time)

$$P[w]_i = \frac{\partial w(z)}{\partial v_i} - \frac{\partial w(z')}{\partial v_i'}$$

$$\begin{aligned}(F, K; G, N) \;=\; & \int d^6 z \int d^6 z' \, \mathcal{G}(z, z') \\ & \times \, (\delta \owedge \delta)_{ijkl} \, P\left[F_f\right]_i P\left[K_f\right]_j P\left[G_f\right]_k P\left[N_f\right]_l \,,\end{aligned}$$

where simplest K-N

$$(\delta \owedge \delta)_{ijkl} = 2(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})\,.$$

with $S = -\int d^z f \ln f$

$$(f, H; SH) = ??$$

Landau-Lenard-Balescu collision operator!

Metriplectic 2-bracket $(f, g)_H$ in pjm 1984 again!

Final Comments

- See PJM & M. Updike, arXiv:2306.06787v1 [math-ph] 11 Jun 2023 for many more examples, finite and infinite.

- Useful for thermodynamically consistent model building, e.g., multiphase flow with many constitutive relation effects.

- Given that double brackets and metriplectic brackets have been used for computation of equilibria, metriplectic 4-bracket can be a new tool.

- New kind of structure to preserve: Symplectic, Poisson, FEEC, ... metriplectic 2-bracket, metriplectic 4-bracket?

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