On Metriplectic Dynamics: Joining Hamiltonian and Dissipative Dynamics

Philip J. Morrison

Department of Physics and Institute for Fusion Studies
The University of Texas at Austin
`morrison@physics.utexas.edu`
`http://www.ph.utexas.edu/~morrison/`

CHATS2023 Marseille, France
May 31, 2023

Collaborators: G. Flierl, M. Furukawa, C. Bressan, O. Maj, M. Kraus, E. Sonnendrücker, ...

Geometry of metriplectic 4-brackets: M. Updike

Dynamics – Theories – Models

Goal:

Predict the future or explain the past $\Rightarrow$

$$\dot{z} = V(z)\,, \qquad z \in \mathcal{Z}\,, \text{ Phase Space}$$

A dynamical system. Maps, ODEs, PDEs, etc.

Whence vector field V?

• Fundamental parent theory (microscopic, N interacting gravitating or charged particles, BBGKY hierarchy, Vlasov-Maxwell system, ...). Identify small parameters, rigorous asymptotics $\rightarrow$ Reduced Computable Model V.

• Phenomena based modeling using known properties, constraints, etc. used to intuit $\rightarrow$
Reduced Computable Model V. $\leftarrow$ structure can be useful.

Types of Vector Fields, V

Natural Split:

$$V(z) = V_H + V_D$$

- Hamiltonian vector fields, V_H: conservative, properties, etc.
- Dissipative vector fields, V_D: not conservative, relaxation, etc.

General Hamiltonian Form:

$$V_H = J\,\frac{\partial H}{\partial z} \qquad \text{or} \qquad V_H = \mathcal{J}\frac{\delta H}{\delta \psi}$$

where $J(z)$ is Poisson tensor/operator and H is the Hamiltonian. Basic product decomposition.

General Dissipation:

$$V_D = ?\ldots \qquad \rightarrow \qquad V_D = G\,\frac{\partial F}{\partial z}$$

Why investigate? General properties of theory. Useful for computation.

Overview

I. Review Hamiltonian systems via noncanonical Poisson brackets

II. Review previous bracket formalisms for dissipation

III. Encompassing metriplectic 4-bracket theory

I. Noncanonical Hamiltonian Dynamics

Hamilton's Canonical Equations

Phase Space with Canonical Coordinates: (q, p)

Hamiltonian function: $H(q, p)$ $\leftarrow$ the energy

Equations of Motion:

$$\boxed{\dot{p}_i = -\frac{\partial H}{\partial q^i}\,, \qquad \dot{q}^i = \frac{\partial H}{\partial p_i}\,,} \qquad i = 1, 2, \ldots N$$

Phase Space Coordinate Rewrite: $z = (q, p)\,, \quad \alpha, \beta = 1, 2, \ldots 2N$

$$\dot{z}^\alpha = J_c^{\alpha\beta}\frac{\partial H}{\partial z^\beta} = \{z^\alpha, H\}_c\,, \qquad (J_c^{\alpha\beta}) = \begin{pmatrix} 0_N & I_N \\ -I_N & 0_N \end{pmatrix},$$

$J_c :=$ Poisson tensor, Hamiltonian bi-vector, cosymplectic form

Noncanonical Hamiltonian Structure

Sophus Lie (1890) $\longrightarrow$ PJM (1980) $\longrightarrow$ Poisson Manifolds etc.

Noncanonical Coordinates:

$$\dot{z}^{\alpha} = \{z^{\alpha}, H\} = J^{\alpha\beta}(z)\frac{\partial H}{\partial z^{\beta}}$$

Noncanonical Poisson Bracket:

$$\{A, B\} = \frac{\partial A}{\partial z^{\alpha}} J^{\alpha\beta}(z)\frac{\partial B}{\partial z^{\beta}}$$

Poisson Bracket Properties:

antisymmetry $\longrightarrow$ $\{A, B\} = -\{B, A\}$

Jacobi identity $\longrightarrow$ $\{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\} = 0$

Leibniz $\longrightarrow$ $\{AC, B\} = A\{C, B\} + \{C, B\}A$

G. Darboux: $detJ \neq 0 \Longrightarrow J \rightarrow J_c$ Canonical Coordinates

Sophus Lie: $detJ = 0 \Longrightarrow$ Canonical Coordinates plus Casimirs
(Lie's distinguished functions!)

Flow on Poisson Manifold

Definition. A Poisson manifold $\mathcal{Z}$ is differentiable manifold with bracket

$$\{\, ,\, \} : C^\infty(\mathcal{Z}) \times C^\infty(\mathcal{Z}) \to C^\infty(\mathcal{Z})$$

st $C^\infty(\mathcal{Z})$ with $\{\, ,\, \}$ is a Lie algebra realization, i.e., is

i) bilinear,
ii) antisymmetric,
iii) Jacobi, and
iv) Leibniz, i.e., acts as a derivation.

Flows are integral curves of noncanonical Hamiltonian vector fields, JdH.

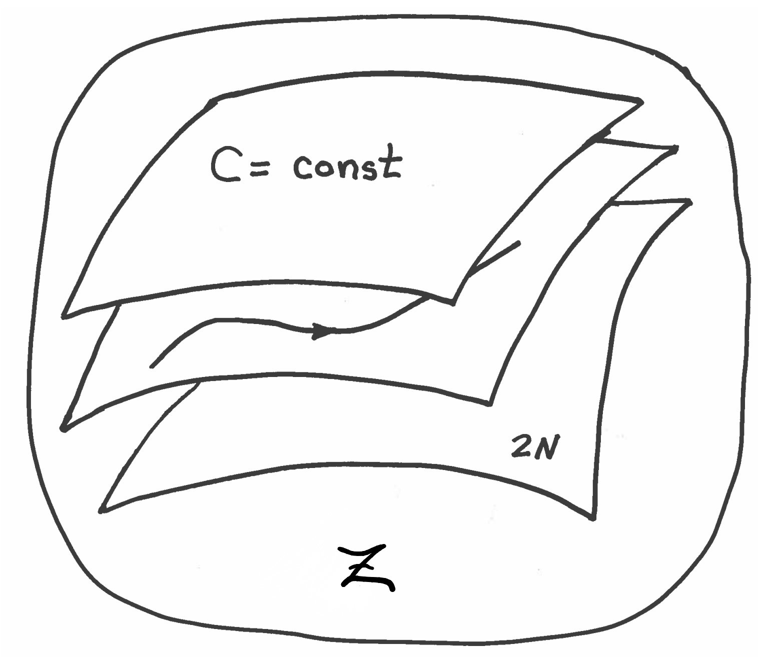
Because of degeneracy, $\exists$ functions C st $\{A, C\} = 0$ for all $A \in C^\infty(\mathcal{Z})$. Called Casimir invariants (Lie's distinguished functions!).

Poisson Manifold (phase space) $\mathcal{Z}$ Cartoon

Degeneracy in J $\Rightarrow$ Casimirs:

$$\{A, C\} = 0 \quad \forall\ A : \mathcal{Z} \to \mathbb{R}$$

Lie-Darboux Foliation by Casimir (symplectic) leaves:



Lie-Poisson Brackets

Lie-Poisson brackets are special kind of noncanonical Poisson bracket that are associated with any Lie algebra, say $\mathfrak{g}$.

Natural phase space $\mathfrak{g}^*$. For $f, g \in C^\infty(\mathfrak{g}^*)$ and $z \in \mathfrak{g}^*$.

Lie-Poisson bracket has the form

$$\begin{aligned}\{f, g\} &= \langle z, [\nabla f, \nabla g]\rangle \\ &= \frac{\partial f}{\partial z^i} c^{ij}{}_k \, z_k \frac{\partial g}{\partial z^j}\,, \qquad\qquad i, j, k = 1, 2, \ldots, \dim \mathfrak{g}\end{aligned}$$

Pairing $<\ ,\ >$: $\mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$, z^i coordinates for $\mathfrak{g}^*$, and $c^{ij}{}_k$ structure constants of $\mathfrak{g}$. Note $J^{ij} = c^{ij}{}_k \, z_k$.

Classical Field Theory for Classical Purposes

Dynamics of matter described by

- **Fluid models**
  - Euler's equations, Navier-Stokes, ...
- **Magnetofluid models**
  - MHD, XMHD (Hall, electron mass physics), 2-fluid, ...
- **Kinetic theories**
  - Vlasov-Maxwell, Landau-Lenard-Balescu, gyrokinetics, ...
- **Fluid-Kinetic hybrids**
  - MHD + hot particle kinetics, gyrokinetics, ...

Applications:

atmospheres, oceans, fluidics, natural and laboratory plasmas

Hamiltonian and Dissipative structures are <u>organizing principles</u>

Free Rigid Body

Angular momenta (L^1, L^2, L^3), Lie-Poisson bracket with Lie algebra $\mathfrak{so}(3)$, $c^{ij}_{\ k} = -\epsilon_{ijk}$.

Hamiltonian:

$$H = \frac{(L^1)^2}{2I_1} + \frac{(L^2)^2}{2I_2} + \frac{(L^3)^2}{2I_3}$$

principal moments of inertia, I_i

Casimir

$$C = (L^1)^2 + (L^3)^2 + (L^3)^2\,,$$

Euler's equations:

$$\dot{L}^i = \{L^i, H\}$$

Noncanonical MHD (pjm & Greene 1980)

Equations of Motion:

Force	$\rho\frac{\partial \boldsymbol{v}}{\partial t} = -\rho\boldsymbol{v}\cdot\nabla\boldsymbol{v} - \nabla p + \frac{1}{c}\boldsymbol{J}\times\boldsymbol{B}$
Density	$\frac{\partial \rho}{\partial t} = -\nabla\cdot(\rho\boldsymbol{v})$
Entropy	$\frac{\partial s}{\partial t} = -\boldsymbol{v}\cdot\nabla s$
Ohm′s Law	$\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B} = \eta\boldsymbol{J} = \eta\nabla\times\boldsymbol{B}\approx 0$
Magnetic Field	$\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla\times\boldsymbol{E} = \nabla\times(\boldsymbol{v}\times\boldsymbol{B})$

Energy:

$$H = \int_D d^3x\,\left(\frac{1}{2}\rho|\boldsymbol{v}|^2 + \rho U(\rho,s) + \frac{1}{2}|\boldsymbol{B}|^2\right)$$

Thermodynamics:

$$p = \rho^2\frac{\partial U}{\partial \rho} \qquad T = \frac{\partial U}{\partial s} \qquad \text{or} \qquad p = \kappa\rho^\gamma$$

Noncanonical Bracket:

$$\begin{aligned}\{F,G\} \;=\; -\int_D d^3x \Bigg(& \left[\frac{\delta F}{\delta \rho}\nabla\frac{\delta G}{\delta \boldsymbol{v}} - \frac{\delta G}{\delta \rho}\nabla\frac{\delta F}{\delta \boldsymbol{v}}\right] + \left[\frac{\delta F}{\delta \boldsymbol{v}}\cdot\left(\frac{\nabla\times\boldsymbol{v}}{\rho}\times\frac{\delta F}{\delta \boldsymbol{v}}\right)\right] \\ & + \frac{\nabla s}{\rho}\cdot\left[\frac{\delta F}{\delta \boldsymbol{v}}\cdot\nabla\frac{\delta G}{\delta s} - \frac{\delta G}{\delta \boldsymbol{v}}\cdot\nabla\frac{\delta F}{\delta s}\right] \\ & + \boldsymbol{B}\cdot\left[\frac{1}{\rho}\frac{\delta F}{\delta \boldsymbol{v}}\cdot\nabla\frac{\delta G}{\delta \boldsymbol{B}} - \frac{1}{\rho}\frac{\delta G}{\delta \boldsymbol{v}}\cdot\nabla\frac{\delta F}{\delta \boldsymbol{B}}\right] \\ & + \boldsymbol{B}\cdot\left[\nabla\left(\frac{1}{\rho}\frac{\delta F}{\delta \boldsymbol{v}}\right)\cdot\frac{\delta G}{\delta \boldsymbol{B}} - \nabla\left(\frac{1}{\rho}\frac{\delta G}{\delta \boldsymbol{v}}\right)\cdot\frac{\delta F}{\delta \boldsymbol{B}}\right]\Bigg).\end{aligned}$$

Dynamics:

$$\frac{\partial\rho}{\partial t} = \{\rho, H\}\,, \quad \frac{\partial s}{\partial t} = \{s, H\}\,, \quad \frac{\partial \boldsymbol{v}}{\partial t} = \{\boldsymbol{v}, H\}\,, \text{ and } \frac{\partial \boldsymbol{B}}{\partial t} = \{\boldsymbol{B}, H\}\,.$$

Densities:

$$\boldsymbol{M} := \rho\boldsymbol{v} \qquad \sigma := \rho s \qquad\qquad \text{Lie – Poisson form}$$

Casimir Invariants:

Recall $\mathcal{J}\delta H/\delta\psi$, Casimirs determined by $\mathcal{J}$ for any H.

Casimir Invariants:

$$\{F, C\}^{MHD} = 0 \qquad \forall \text{ functionals } F.$$

Casimirs Invariant entropies:

$$C_S = \int d^3x \, \rho f(s) \,, \qquad \text{f arbitrary}$$

Casimirs Invariant helicities:

$$C_B = \int d^3x \, \boldsymbol{B} \cdot \boldsymbol{A} \,, \qquad C_V = \int d^3x \, \boldsymbol{B} \cdot \boldsymbol{v}$$

Helicities have topological content, linking etc.

Maxwell-Vlasov Equations

Maxwell's Equations:

$$\frac{\partial \boldsymbol{B}}{\partial t} = -c\nabla \times \boldsymbol{E} \qquad \frac{\partial \boldsymbol{E}}{\partial t} = c\nabla \times \boldsymbol{B} - 4\pi \boldsymbol{J}_e$$

$$\nabla \cdot \boldsymbol{B} = 0 \qquad \nabla \cdot \boldsymbol{E} = 4\pi\rho_e$$

Coupling to Vlasov

$$\frac{\partial f_s}{\partial t} = -\boldsymbol{v} \cdot \nabla f_s - \frac{e_s}{m_s}\left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c} \times \boldsymbol{B}\right) \cdot \frac{\partial f_s}{\partial \boldsymbol{v}}$$

$$\rho_e(\boldsymbol{x}, t) = \sum_s e_s \int f_s(\boldsymbol{x}, \boldsymbol{v}, t)\, d^3v\,, \qquad \boldsymbol{J}_e(\boldsymbol{x}, t) = \sum_s e_s \int \boldsymbol{v}\, f_s(\boldsymbol{x}, \boldsymbol{v}, t)\, d^3v$$

$f_s(\boldsymbol{x}, \boldsymbol{v}, t)$ is a phase space density for particles of species s with charge and mass, e_s, m_s.

$$\psi = \Big(\boldsymbol{E}(\boldsymbol{x}, t),\, \boldsymbol{B}(\boldsymbol{x}, t),\, f_s(\boldsymbol{x}, \boldsymbol{v}, t)\Big)$$

Maxwell-Vlasov Hamiltonian Structure

Hamiltonian:

$$H=\sum_s \frac{m_s}{2}\int |\boldsymbol{v}|^2 f_s\, d^3x\, d^3v + \frac{1}{8\pi}\int (|\boldsymbol{E}|^2+|\boldsymbol{B}|^2)\, d^3x\,,$$

Bracket:

$$\begin{aligned}
\{F,G\} \;=\; & \sum_s \int \Big(\frac{1}{m_s} f_s \left(\nabla F_{f_s}\cdot\partial_{\boldsymbol{v}} G_{f_s} - \nabla G_{f_s}\cdot\partial_{\boldsymbol{v}} F_{f_s}\right) \\
& + \frac{e_s}{m_s^2 c} f_s\, \boldsymbol{B}\cdot\left(\partial_{\boldsymbol{v}} F_{f_s}\times\partial_{\boldsymbol{v}} G_{f_s}\right) \\
+ \;& \frac{4\pi e_s}{m_s} f_s \left(G_{\boldsymbol{E}}\cdot\partial_{\boldsymbol{v}} F_{f_s} - F_{\boldsymbol{E}}\cdot\partial_{\boldsymbol{v}} G_{f_s}\right)\Big) d^3x\, d^3v \\
+ \;& 4\pi c\int (F_{\boldsymbol{E}}\cdot\nabla\times G_{\boldsymbol{B}} - G_{\boldsymbol{E}}\cdot\nabla\times F_{\boldsymbol{B}})\, d^3x\,,
\end{aligned}$$

where $\partial_{\boldsymbol{v}} := \partial/\partial\boldsymbol{v}$, F_{f_s} means functional derivative of F with respect to f_s etc.

pjm 1980,1982; Marsden and Weinstein 1982

Maxwell-Vlasov Structure (cont)

Equations of Motion:

$$\frac{\partial f_s}{\partial t} = \{f_s, H\}\,, \quad \frac{\partial \boldsymbol{E}}{\partial t} = \{\boldsymbol{E}, H\}\,, \quad \frac{\partial \boldsymbol{B}}{\partial t} = \{\boldsymbol{B}, H\}\,.$$

Casimirs invariants:

$$\begin{aligned} \mathcal{C}_s^f[f_s] &= \int \mathcal{C}_s(f_s)\, d^3x d^3v \\ \mathcal{C}^E[\boldsymbol{E}, f_s] &= \int h^{\boldsymbol{E}}(x) \left(\nabla \cdot \boldsymbol{E} - 4\pi \sum_s e_s \int f_s\, d^3v \right) d^3x\,, \\ \mathcal{C}^B[\boldsymbol{B}] &= \int h^{\boldsymbol{B}}(x)\, \nabla \cdot \boldsymbol{B}\, d^3x\,, \end{aligned}$$

where $\mathcal{C}_s$, $h^{\boldsymbol{E}}$ and $h^{\boldsymbol{B}}$ are arbitrary functions of their arguments. These satisfy the degeneracy conditions

$$\{F, C\} = 0 \quad \forall F\,.$$

Summary

Poisson brackets defined by J, dynamics $\partial\psi/\partial t = \{\psi, H\}$:

$$\begin{aligned} J_{RB} &\rightarrow \text{Casimirs} \\ \mathcal{J}_{MHD} &\rightarrow \text{Casimirs} \\ \mathcal{J}_{M-V} &\rightarrow \text{Casimirs} \end{aligned}$$

Good theories in their ideal limit ($\nu, \eta, \cdots \rightarrow 0$) conserve energies, H, and have **Poisson brackets**. Bad theories do bad things: unaccounted energy, unphysical instabilities, etc.

Dissipation? Casimirs are candidates for entropies!

II. Dissipation Formalisms

Codifying Dissipation – Some History

Is there a framework for dissipation akin to the Hamiltonian formulation for nondissipative systems?

Rayleigh (1873): $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_\nu}\right) - \left(\frac{\partial \mathcal{L}}{\partial q_\nu}\right) + \left(\frac{\partial \mathcal{F}}{\partial \dot{q}_\nu}\right) = 0$
Linear dissipation e.g. of sound waves. *Theory of Sound*.

Cahn-Hilliard (1958): $\frac{\partial n}{\partial t} = \nabla^2 \frac{\delta F}{\delta n} = \nabla^2 \left(n^3 - n - \nabla^2 n\right)$
Phase separation, nonlinear diffusive dissipation, binary fluid ..

Other Gradient Flows: $\frac{\partial \psi}{\partial t} = \mathcal{G}\, \frac{\delta F}{\delta \psi}$
Otto, Ricci Flows, Poincarè conjecture on S^3, Perlman (2002)...

Bracket Dissipation 1980 $\rightarrow$

- Symmetric bilinear brackets (pjm 1982)
- Degenerate Antisymmetric Bracket (Kaufman and pjm 1982)
- Metriplectic Dynamics (pjm 1984,1986)
- Double Brackets (Vallis, Carnevale; Brockett, Bloch ... 1990)
- Generic (Grmela, Oettinger 1997) $\equiv$ Metriplectic Dynamics!

Brackets for Dissipation

Two ingredients: Bilinear Bracket + Generator

$$\dot{z} = \{z, H\} + (z, F)$$

where

$$(\, , \,) : C^{\infty}(\mathcal{Z}) \times C^{\infty}(\mathcal{Z}) \rightarrow C^{\infty}(\mathcal{Z})$$

What is F and what are the algebraic properties of $(\, , \,)$?

K-M Brackets 1982

Done for plasma quasilinear theory.

Dynamics:

$$\dot{z} = [z, H]_S$$

Properties:

- bilinear
- antisymmetric
- entropy production

$$\dot{S} = [S, H]_S \geq 0 \quad \Rightarrow \quad z \mapsto z_{eq}$$

Double Bracket 1989

Good Idea:

Vallis, Carnevale, and Young, Shepherd (1989,1990)

$$\frac{d\mathcal{F}}{dt} = \{\mathcal{F}, H\} + ((\mathcal{F}, H)) = ((\mathcal{F}, \mathcal{F})) \geq 0$$

where

$$((F, G)) = \int d^3x \, \frac{\delta F}{\delta \chi} \mathcal{J}^2 \frac{\delta G}{\delta \chi}$$

Lyapunov function, $\mathcal{F}$, yields asymptotic stability to rearranged equilibrium.

- Maximizing energy at fixed Casimir: Works fine sometimes, but limited to circular vortex states ....

Simulated Annealing

Use various bracket dynamics to effect extremization.

Many relaxation methods exist: gradient descent, etc.

Simulated annealing: an **artificial** dynamics that solves a variational principle with constraints for equilibria states.

Coordinates:

$$\dot{z}^i = ((z^i, H)) = J^{ik} g_{kl} J^{jl} \frac{\partial H}{\partial z^j}$$

symmetric, definite, and kernel of J.

$$\dot{C} = 0 \qquad \text{with} \qquad \dot{H} \leq 0$$

Simulated Annealing with Generalized (Noncanonical) Dirac Brackets

Dirac Bracket:

$$\{F, G\}_D = \{F, G\} + \frac{\{F, C_1\}\{C_2, G\}}{\{C_1, C_2\}} - \frac{\{F, C_2\}\{C_1, G\}}{\{C_1, C_2\}}$$

Preserves any two incipient constraints C_1 and C_2.

New Idea:

Do simulated Annealing with Generalized Dirac Bracket

$$((F, G))_D = \int d\mathbf{x} d\mathbf{x}' \{F, \zeta(\mathbf{x})\}_D \; \mathcal{G}(\mathbf{x}, \mathbf{x}') \; \{\zeta(\mathbf{x}'), G\}_D$$

Preserves any Casimirs of $\{F, G\}$ and Dirac constraints $C_{1,2}$

For successful implementation with contour dynamics see PJM (with Flierl) Phys. Plasmas **12** 058102 (2005).

Double Bracket SA for Reduced MHD

M. Furukawa, T. Watanabe, pjm, and K. Ichiguchi, *Calculation of Large-Aspect-Ratio Tokamak and Toroidally-Averaged Stellarator Equilibria of High-Beta Reduced Magnetohydrodynamics via Simulated Annealing,* Phys. Plasmas **25**, 082506 (2018).

High-beta reduced MHD (Strauss, 1977) given by

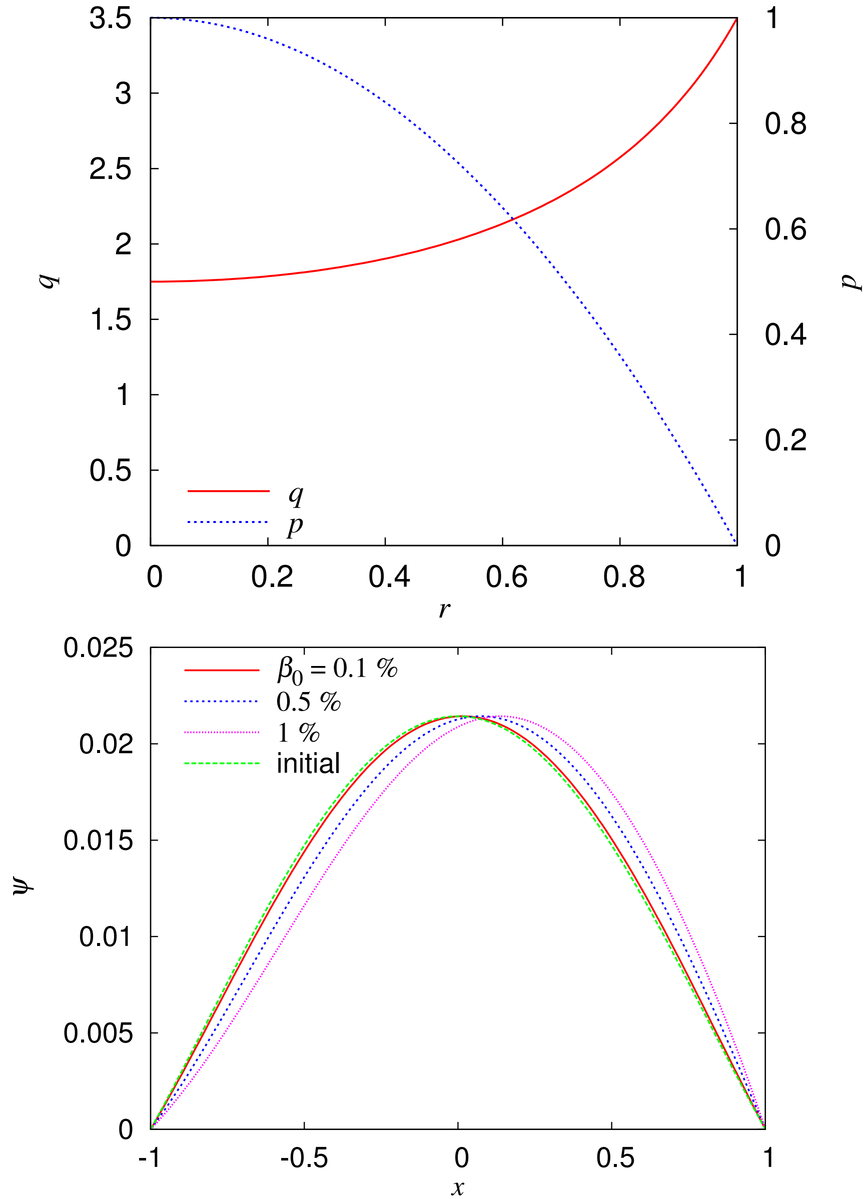
$$\begin{aligned}\frac{\partial U}{\partial t} &= [U,\varphi] + [\psi, J] - \epsilon\frac{\partial J}{\partial \zeta} + [P,h] \\ \frac{\partial \psi}{\partial t} &= [\psi,\varphi] - \epsilon\frac{\partial \varphi}{\partial \zeta} \\ \frac{\partial P}{\partial t} &= [P,\varphi]\end{aligned}$$

Extremization

$$\mathcal{F} = H + \sum_i C_i + \lambda^i P_i\,, \rightarrow \text{equilibria, maybe with flow}$$

Cs Casimirs and Ps dynamical invariants.

Sample Double Bracket SA equilibria



Nested Tori are level sets of ψ; q gives pitch of helical $\boldsymbol{B}$-lines.

Double Bracket SA for Stability

M. Furukawa and P. J. Morrison, *Stability analysis via simulated annealing and accelerated relaxation* , Phys. Plasmas, 2022.

Since SA searches for an energy extremum, it can also be used for stability analysis when initiated from a state where a perturbation is added to an equilibrium. Three steps:

1) choose **any** equilibrium of unknown stability

2) perturb the equilibrium with dynamically accessible (leaf) perturbation

3) perform double bracket SA

If it finds the equilibrium, then is is an energy extremum and must be stable

Sample Double Bracket SA unstable equilibria

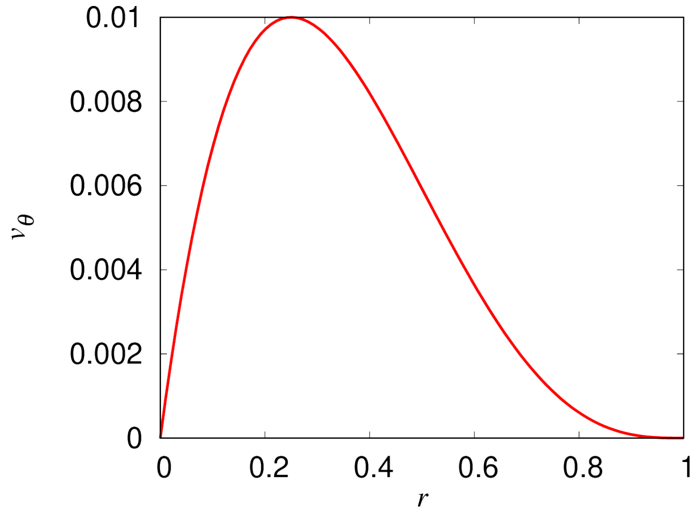
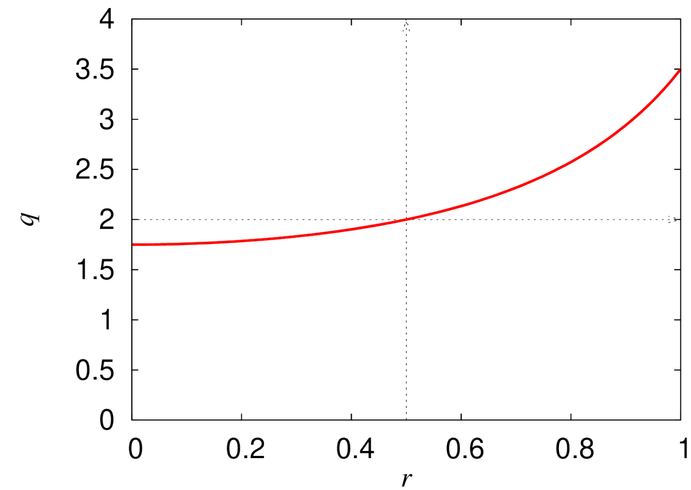
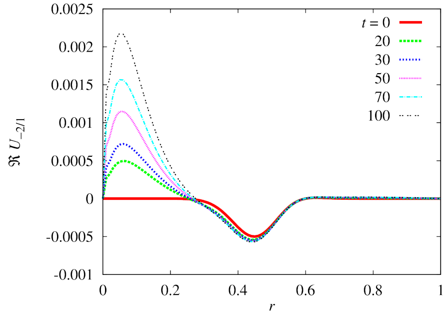
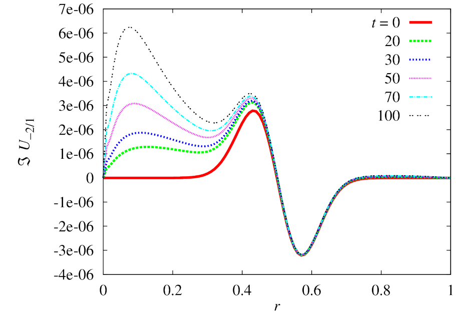


FIG. 12: Poloidal rotation velocity v_θ profile.

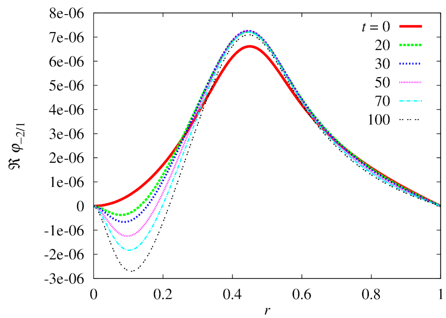




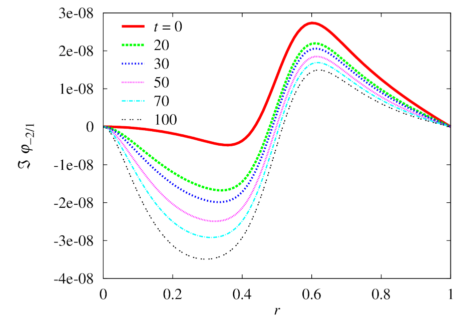
(a) Radial profile of $\Re\, U_{-2,1}$.



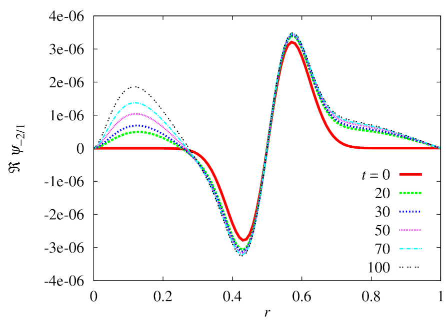
(b) Radial profile of $\Im\, U_{-2,1}$.



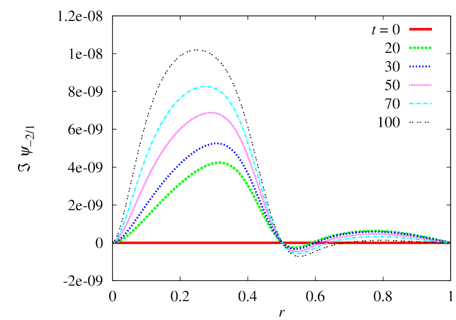
(c) Radial profile of $\Re\, \varphi_{-2,1}$.



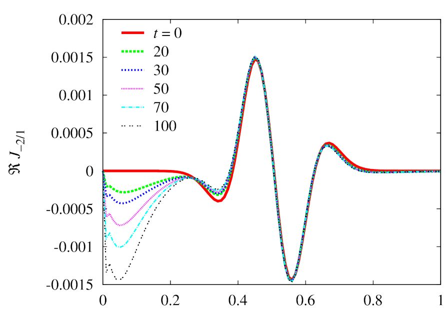
(d) Radial profile of $\Im\, \varphi_{-2,1}$.



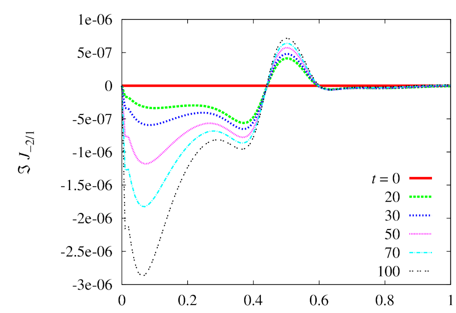
(e) Radial profile of $\Re\, \psi_{-2,1}$.



(f) Radial profile of $\Im\, \psi_{-2,1}$.



(g) Radial profile of $\Re\, J_{-2,1}$.



(h) Radial profile of $\Im\, J_{-2,1}$.

Metriplectic Dynamics 1984, 1986

A dynamical model of thermodynamics that ‘captures’:.

- First Law: conservation of energy

- Second Law: entropy production

Metriplectic Dynamics – Entropy, Degeneracies, and 1st and 2nd Laws

- Casimirs of noncanonical PB $\{\,,\,\}$ are 'candidate' entropies. Election of particular $S \in \{\text{Casimirs}\} \Rightarrow$ thermal equilibrium (relaxed) state.

- Generator: $F = H + S$

- 1st Law: identify energy with Hamiltonian, H, then

$$\dot{H} = \{H, F\} + (H, F) = 0 + (H, H) + (H, S) = 0$$

  Foliate $\mathcal{Z}$ by level sets of H, with $(H, A) = 0 \ \forall \ A \in C^\infty(\mathcal{Z})$.

- 2nd Law: entropy production

$$\dot{S} = \{S, F\} + (S, F) = (S, S) \geq 0$$

  Lyapunov relaxation to the equilibrium state. Dynamics solves the equilibrium variational principle: $\delta F = \delta(H + S) = 0$.

Geometical Definition

A *metriplectic system* consists of a smooth manifold $\mathcal{Z}$, two smooth vector bundle maps $J, G : T^*\mathcal{Z} \to T\mathcal{Z}$ covering the identity, and two functions $H, S \in C^\infty(\mathcal{Z})$, the *Hamiltonian* and the *entropy* of the system, such that

(i) $\{f, g\} := \langle \mathbf{d}f, J(\mathbf{d}g)\rangle$ is a Poisson bracket; $J^* = -J$;

(ii) $(f, g) := \langle \mathbf{d}f, G(\mathbf{d}g)\rangle$ is a positive semidefinite symmetric bracket, i.e., $(\,,\,)$ is $\mathbb{R}$-bilinear and symmetric, so $G^* = G$, and $(f, f) \geq 0$ for every $F \in C^\infty(\mathcal{Z})$;

(iii) $\{S, f\} = 0$ and $(H, f) = 0$ for all $f \in C^\infty(\mathcal{Z})$

$$\iff J(\mathbf{d}S) = G(\mathbf{d}H) = 0.$$

Two examples of pjm 1984

Vlasov with Collisions

$$\frac{\partial f}{\partial t} = -v \cdot \nabla f - a \cdot \nabla_v f + \left.\frac{\partial f}{\partial t}\right)_c$$

where

$$\text{Collision term} \quad \rightarrow \quad \left.\frac{\partial f}{\partial t}\right)_c$$

could be, Landau, Lenard Balescu, etc.

Conserves, mass, momentum, energy,

$$\frac{dH}{dt} = \frac{d}{dt}\int \frac{1}{2}mv^2 f + \text{interaction} = 0$$

and makes entropy

$$\frac{dS}{dt} = -\frac{d}{dt}\int f \ln(f) \geq 0$$

Landau Collision Operator

Metriplectic bracket:

$$(A,B) \;=\; \int dz \int dz' \left[\frac{\partial}{\partial v_i}\frac{\delta A}{\delta f(z)} - \frac{\partial}{\partial v_i'}\frac{\delta A}{\delta f(z')}\right] T_{ij}(z,z') \times \left[\frac{\partial}{\partial v_j}\frac{\delta B}{\delta f(z)} - \frac{\partial}{\partial v_j'}\frac{\delta B}{\delta f(z')}\right]$$

$$T_{ij}(z,z') = w_{ij}(z,z') f(z) f(z')/2$$

Conservation and Lyapunov:

$w_{ij}(z,z') = w_{ji}(z,z') \quad w_{ij}(z,z') = w_{ij}(z',z) \quad g_i w_{ij} = 0$ with $g_i = v_i - v_i'$

Landau kernel:

$$w_{ij}^{(L)} = (\delta_{ij} - g_i g_j / g^2)\delta(\mathbf{x}-\mathbf{x}')/g$$

Entropy:

$$S[f] = \int dz\, f \ln(f)$$

Ideal fluid with viscous heating and thermal conductivity.

$$\frac{\partial v_i}{\partial t} = \{v_i, \mathcal{H}\} \qquad (18)$$

$$\frac{\partial \rho}{\partial t} = \{\rho, \mathcal{H}\} \qquad (19)$$

$$\frac{\partial s}{\partial t} = \{s, \mathcal{H}\} \qquad (20)$$

where the GPB, $\{,\}$, is given by

$$\{F,G\} = -\int\Bigg(\frac{\delta F}{\delta \rho}\vec{\nabla}\cdot\frac{\delta G}{\delta \vec{v}} + \frac{\delta F}{\delta \vec{v}}\cdot\vec{\nabla}\frac{\delta G}{\delta \rho} + \frac{\delta F}{\delta \vec{v}}\cdot\left[\frac{(\vec{\nabla}\times\vec{v})}{\rho}\times\frac{\delta G}{\delta \vec{v}}\right] + \frac{\vec{\nabla}s}{\rho}\cdot\left[\frac{\delta F}{\delta s}\frac{\delta G}{\delta \vec{v}} - \frac{\delta F}{\delta \vec{v}}\frac{\delta G}{\delta s}\right]\Bigg)d^3x \,. \qquad (21)$$

Upon inserting the quantities shown on the right hand side of Eqs. (18)-(20), into Eq. (21) and performing the indicated operations one obtains, as noted, the invicid adiabatic limit of Eqs. (10)-(12).

The Casimirs for the bracket given by Eq. (21) are the total mass $M = \int \rho\, d^3x$ and a generalized entropy functional $\mathcal{S}_f = \int \rho f(s)\, d^3x$, where f is an arbitrary function of s. The latter quantity is added to the energy [Eq. (17)] to produce the generalized free energy of Eq. (4): $\mathcal{G} = \mathcal{H} + \mathcal{S}_f$.

In order to obtain the dissipative terms, we introduce the following symmetric bracket:

$$(F,G) = \frac{1}{\lambda}\int\Bigg\{\frac{1}{\rho}\frac{\delta F}{\delta v_i}\frac{\partial}{\partial x_k}\left[\frac{\sigma_{ik}}{\rho}\frac{\delta G}{\delta s}\right] + \frac{1}{\rho}\frac{\delta G}{\delta v_i}\frac{\partial}{\partial x_k}\left[\frac{\sigma_{ik}}{\rho}\frac{\delta F}{\delta s}\right] + \frac{\sigma_{ik}}{T}\frac{\partial v_i}{\partial x_k}\left[\frac{1}{\rho^2}\frac{\delta F}{\delta s}\frac{\delta G}{\delta s}\right] + T^2\kappa\frac{\partial}{\partial x_k}\left[\frac{1}{\rho T}\frac{\delta F}{\delta s}\right]\frac{\partial}{\partial x_k}\left[\frac{1}{\rho T}\frac{\delta G}{\delta s}\right] + T\,\Lambda_{ikmn}\frac{\partial}{\partial x_m}\left[\frac{1}{\rho}\frac{\delta F}{\delta v_n}\right]\frac{\partial}{\partial x_k}\left[\frac{1}{\rho}\frac{\delta G}{\delta v_i}\right]\Bigg\}d^3x \,, \qquad (23)$$

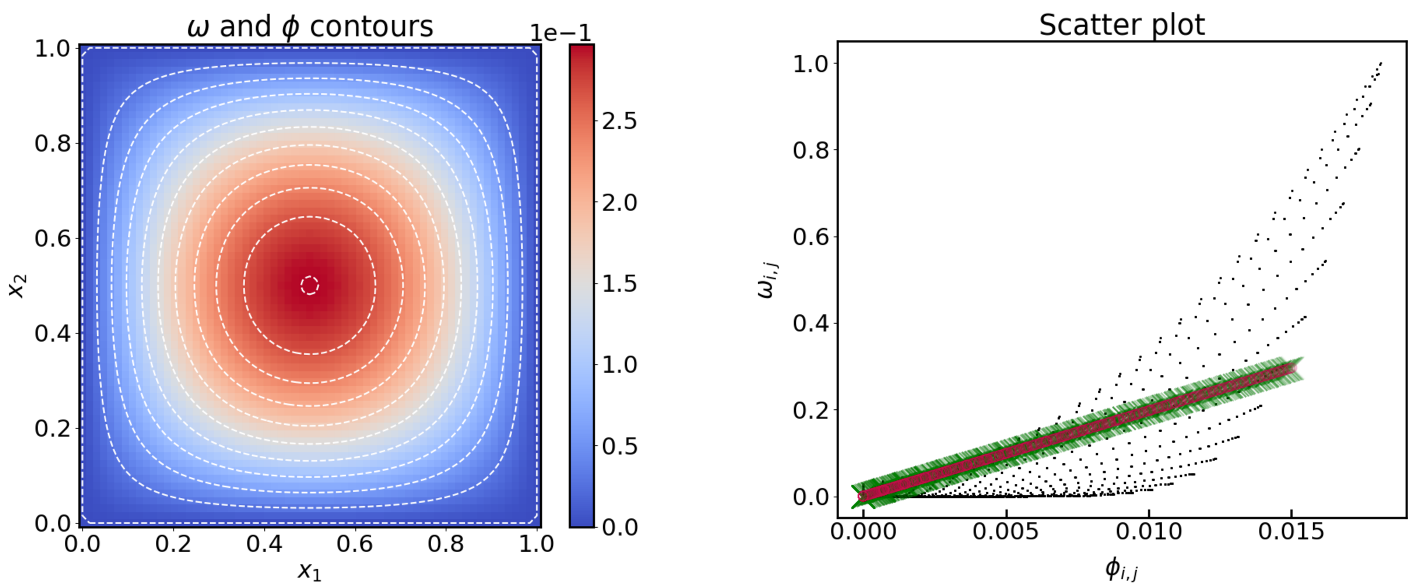
FIG. 10. This . ...

Metriplectic Simulated Annealing

Extremizes an entropy (Casimir) at fixed energy (Hamiltonian)

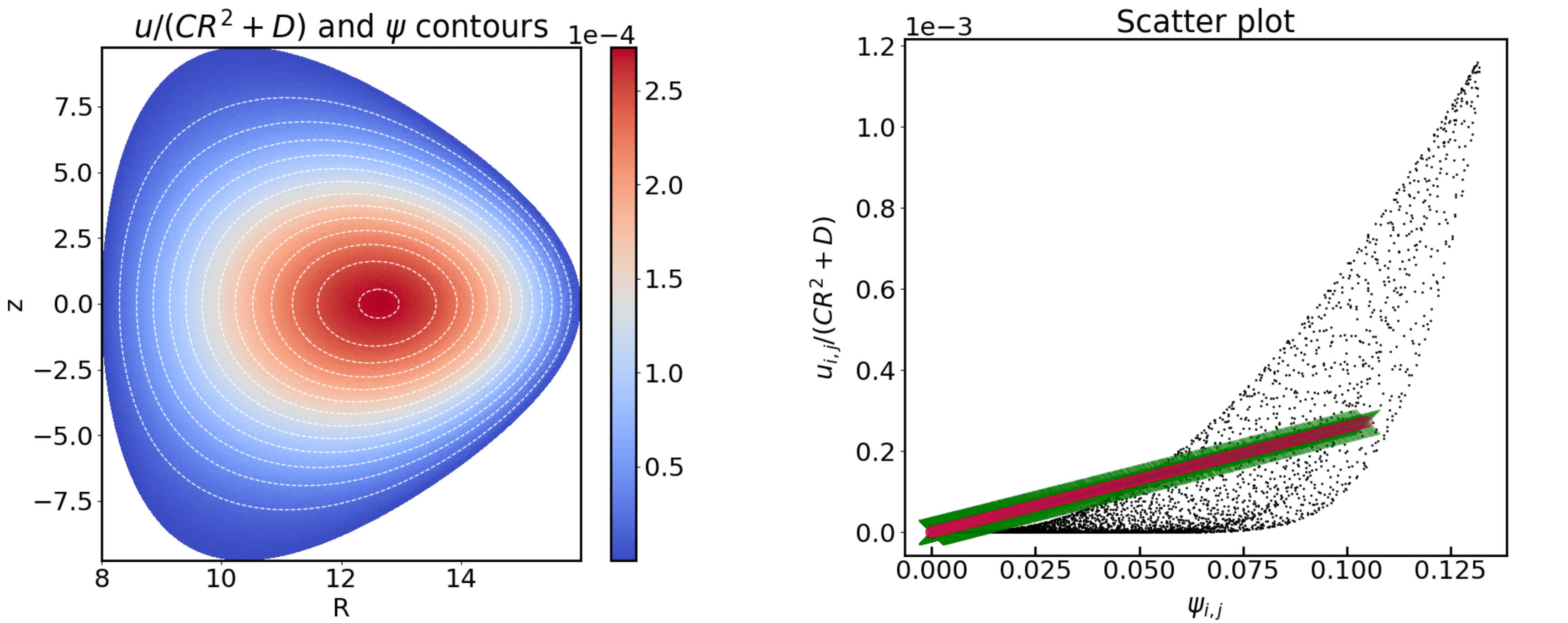
C. Bressen Ph.D. Thesis TUM, Garching 2022

Two cases: 2D Euler and Grad Shafranov MHD equilibria.



(a) **Color plot.** (b) **Scatter plot.**

Figure 6.7: **Relaxed state for the test case *euler-ilgr*.** The same as in Figure 6.2, but for the collision-like operator.



(a) **Color plot.**

(b) **Scatter plot.**

Figure 6.29: **Relaxed state for the *gs-imgc* test case.** The same as in Figure 6.23, but for the collision-like operator and the case of the Czarny domain discussed in Section A.4.2. With respect to Figure 6.27(b) for the diffusion-like operator, we see from (b) that the agreement between the relaxed state and the prediction of the variational principle is better.

III. Metriplectic 4-Brackets for Dissipation

The Metriplectic 4-Bracket

A blend of ideas: Two important functions H and S, symmetries, curvature idea, multilinear brackets all in pjm 1984, 1986. Manifolds with both Poisson tensor J and compatible metric, g.

4-bracket on 0-forms (functions):

$$(\,\cdot\,,\,\cdot\,;\,\cdot\,,\,\cdot\,)\colon \Lambda^0(\mathcal{Z})\times\Lambda^0(\mathcal{Z})\times\Lambda^0(\mathcal{Z})\times\Lambda^0(\mathcal{Z})\to\Lambda^0(\mathcal{Z})$$

For functions f, k, g, and n

$$(f,k;g,n) := R(\boldsymbol{d}f,\boldsymbol{d}k,\boldsymbol{d}g,\boldsymbol{d}n)\,,$$

In a coordinate patch the metriplectic 4-bracket has the form:

$$(f,k;g,n) = R^{ijkl}(z)\frac{\partial f}{\partial z^i}\frac{\partial k}{\partial z^j}\frac{\partial g}{\partial z^k}\frac{\partial n}{\partial z^l}\,.$$

Metriplectic 4-Bracket Properties

(i) linearity in all arguments, e.g,

$$(f+h,k;g,n)=(h,k;g,n)+(h,k;g,n)$$

(ii) algebraic identities/symmetries

$$\begin{aligned}
(f,k;g,n) &= -(k,f;g,n) \\
(f,k;g,n) &= -(f,k;n,g) \\
(f,k;g,n) &= (g,n;f,k) \\
(f,k;g,n) &+ (f,g;n,k)+(f,n;k,g)=0 \qquad \leftarrow \text{not needed}
\end{aligned}$$

(iii) derivation in all arguments, e.g.,

$$(fh,k;g,n)=f(h,k;g,n)+(f,k;g,n)h$$

which is manifest when written in coordinates. Here, as usual, fh denotes pointwise multiplication. Symmetries of algebraic curvature. Although $R^{l}{}_{ijk}$ or R_{lijk} but not R^{lijk}. **Metriplectic Minimum.**

Reduction to Metriplectic 2-Bracket

Symmetric 2-bracket:

$$(f,g)_H = (f,H;g,H) = (g,f)_H \tag{1}$$

Dissipative dynamics:

$$\dot{z} = (z,S)_H\,, \tag{2}$$

Energy conservation:

$$(f,H)_H = (H,f)_H = 0 \qquad \forall\, f\,. \tag{3}$$

Entropy dynamics:

$$\dot{S} = (S,S)_H = (S,H;S,H) \geq 0$$

Metriplectic 4-brackets $\rightarrow$ metriplectic 2-brackets of 1984, 1986!

Reduction to K-M

Kaufman & pjm, Phys. Lett. A 88, 405 (1982).

K-M dynamics:

$$\dot{z}^i = [z^i, H]_S \,,$$

K-M bracket emerges from any metriplectic 4-bracket:

$$[f, g]_S := (f, g; S, H)$$

Thus,

$$[f, g]_S = -[g, f]_S$$

and

$$\dot{H} = [H, H]_S = (H, H; S, H) = 0 \,,$$

and

$$\dot{S} = [S, H]_S = (S, H; S, H) \geq 0$$

Reduction to Double Brackets

Interchanging the role of H with a Casimir S:

$$(f, g)_S = (f, S; g, S)$$

Can show with assumptions (Kozul construction)

$$(C, g)_S = (C, S; g, S) = 0$$

for any Casimir C. Therefore $\dot{C} = 0$.

Easy Construction: K-N Product

Given σ and μ, two symmetric bivector fields operating on 1-forms $\boldsymbol{d}f, \boldsymbol{d}k$ and $\boldsymbol{d}g, \boldsymbol{d}n$, the Kulkarni-Nomizu (K-N) product is

$$\begin{aligned}
\sigma \owedge \mu\,(\boldsymbol{d}f, \boldsymbol{d}k, \boldsymbol{d}g, \boldsymbol{d}n) &= \sigma(\boldsymbol{d}f, \boldsymbol{d}g)\,\mu(\boldsymbol{d}k, \boldsymbol{d}n) \\
&- \sigma(\boldsymbol{d}f, \boldsymbol{d}n)\,\mu(\boldsymbol{d}k, \boldsymbol{d}g) \\
&+ \mu(\boldsymbol{d}f, \boldsymbol{d}g)\,\sigma(\boldsymbol{d}k, \boldsymbol{d}n) \\
&- \mu(\boldsymbol{d}f, \boldsymbol{d}n)\,\sigma(\boldsymbol{d}k, \boldsymbol{d}g)\,.
\end{aligned}$$

Metriplectic 4-bracket:

$$(f, k; g, n) = \sigma \owedge \mu(\boldsymbol{d}f, \boldsymbol{d}k, \boldsymbol{d}g, \boldsymbol{d}n)\,.$$

In coordinates:

$$R^{ijkl} = \sigma^{ik}\mu^{jl} - \sigma^{il}\mu^{jk} + \mu^{ik}\sigma^{jl} - \mu^{il}\sigma^{jk}\,.$$

K-N Product → Landau Collision Operator

Metriplectic 4-bracket on functionals:

$$
\begin{aligned}
(F, K; G, N) &= \int\!\!\int d^6z\, d^6z'\, \mathcal{G}(z, z') \\
&\qquad \times (\Sigma \bigcirc\!\!\!\!\wedge\, M)(F_f, K_f, G_f, N_f)(z, z') \\
&= \int d^6z \int d^6z'\, \mathcal{G}(z, z') \\
&\quad \times\, (\delta \bigcirc\!\!\!\!\wedge\, \delta)_{ijkl}\, P\left[F_f\right]_i P\left[K_f\right]_j P\left[G_f\right]_k P\left[N_f\right]_l \,,
\end{aligned}
$$

where

$$
F_f := \frac{\delta F}{\delta f} \qquad \text{and} \qquad P[w]_i = \frac{\partial w(z)}{\partial v_i} - \frac{\partial w(z')}{\partial v_i'}
$$

$(f, H; g, H) = (f, g)_H$ becomes metriplectic 2-bracket (pjm 1984).

$(f, H; S, H) =$ Landau collision operator!