On the General Metriplectic Formalism for Describing Dissipation ~~and its Computational Uses~~

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Geometry of metriplectic 4-brackets: M. Updike

pjm & M. Updike, **arXiv:2306.06787v1** [math-ph] 11 Jun 2023.

Dynamics – Theories – Models

Goal:

Predict the future or explain the past $\Rightarrow$

$$\dot{z} = V(z)\,, \qquad\qquad z \in \mathcal{Z}\,, \ \text{Phase Space}$$

A dynamical system. Maps, ODEs, PDEs, etc.

Whence vector field V?

• Fundamental parent theory (microscopic, N interacting gravitating or charged particles, BBGKY hierarchy, Vlasov-Maxwell system, ...). Identify small parameters, rigorous asymptotics $\rightarrow$ Reduced Computable Model V.

• Phenomena based modeling using known properties, constraints, etc. used to intuit $\rightarrow$
Reduced Computable Model V. $\leftarrow$ structure can be useful.

Types of Vector Fields, $V(z)$

<u>ODEs:</u> 1-parameter group of trans. $t \to \pm\infty$. **Reversible?**

<u>PDEs etc.:</u> group or semigroup. diffusive $t \to \infty$. **Irreversible?**

<u>Hamiltonian ODE or PDE:</u> group $t \to \pm\infty$. **Reversible?**

<u>Time Reversal Symmetry:</u> canonical coords (q, p), equation same if $p \to -p$ and $t \to -t$. Example of discrete symmetry.

Not all Hamiltonian system have time reversal symmetry!

<u>Conservative:</u> Hamiltonian (autonomous), dissipative or non-dissipative, asymptotic stability?

Types of Vector Fields, $V(z)$ (cont)

Only (?) Natural Split:

$$V(z) = V_H + V_D$$

- Hamiltonian vector fields, V_H: conservative, properties, etc.
- Dissipative vector fields, V_D: not conservative of something, relaxation/asymptotic stability, etc.

General Hamiltonian Form:

finite dim $\rightarrow$ $\quad V_H = J\,\frac{\partial H}{\partial z} \qquad$ or $\qquad V_H = \mathcal{J}\frac{\delta H}{\delta \psi}$ $\quad \leftarrow \infty$ dim

where $J(z)$ is Poisson tensor/operator and H is the Hamiltonian. Basic product decomposition.

General Dissipation:

$$V_D = ?... \qquad \rightarrow \qquad V_D = G\,\frac{\partial F}{\partial z}$$

Why investigate? General properties of theory. Useful for computation.

Overview

I. Review Hamiltonian systems via noncanonical Poisson brackets

II. Review previous bracket formalisms for dissipation

III. Encompassing metriplectic 4-bracket theory

I. Noncanonical Hamiltonian Dynamics

Hamilton's Canonical Equations

Phase Space with Canonical Coordinates: (q, p)

Hamiltonian function: $H(q, p)$ $\leftarrow$ the energy

Equations of Motion:

$$\boxed{\dot{p}_i = -\frac{\partial H}{\partial q^i}\,, \qquad \dot{q}^i = \frac{\partial H}{\partial p_i}\,,} \qquad i = 1, 2, \ldots N$$

Phase Space Coordinate Rewrite: $z = (q, p)\,, \quad \alpha, \beta = 1, 2, \ldots 2N$

$$\dot{z}^\alpha = J_c^{\alpha\beta}\frac{\partial H}{\partial z^\beta} = \{z^\alpha, H\}_c\,, \qquad (J_c^{\alpha\beta}) = \begin{pmatrix} 0_N & I_N \\ -I_N & 0_N \end{pmatrix},$$

$J_c :=$ Poisson tensor, Hamiltonian bivector, cosymplectic form

Noncanonical Hamiltonian Structure

Sophus Lie (1890) $\longrightarrow$ PJM & Greene (1980, noncanonical) $\longrightarrow$ A. Weinstein (1983, Poisson Manifolds etc.)

Noncanonical Coordinates:

$$\dot{z}^\alpha = \{z^\alpha, H\} = J^{\alpha\beta}(z)\frac{\partial H}{\partial z^\beta}$$

Noncanonical Poisson Bracket:

$$\{A, B\} = \frac{\partial A}{\partial z^\alpha} J^{\alpha\beta}(z)\frac{\partial B}{\partial z^\beta}$$

Poisson Bracket Properties:

antisymmetry $\longrightarrow$ $\{A, B\} = -\{B, A\}$

Jacobi identity $\longrightarrow$ $\{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\} = 0$

Leibniz $\longrightarrow$ $\{AC, B\} = A\{C, B\} + \{C, B\}A$

G. Darboux: $det J \neq 0 \Longrightarrow J \rightarrow J_c$ Canonical Coordinates

Sophus Lie: $det J = 0 \Longrightarrow$ Canonical Coordinates plus Casimirs (Lie's distinguished functions!)

Flow on Poisson Manifold

Definition. A Poisson manifold $\mathcal{Z}$ is differentiable manifold with bracket

$$\{\,,\,\} : C^\infty(\mathcal{Z}) \times C^\infty(\mathcal{Z}) \to C^\infty(\mathcal{Z})$$

st $C^\infty(\mathcal{Z})$ with $\{\,,\,\}$ is a Lie algebra realization, i.e., is

i) bilinear,
ii) antisymmetric,
iii) Jacobi, and
iv) Leibniz, i.e., acts as a derivation.

Flows are integral curves of noncanonical Hamiltonian vector fields, JdH.

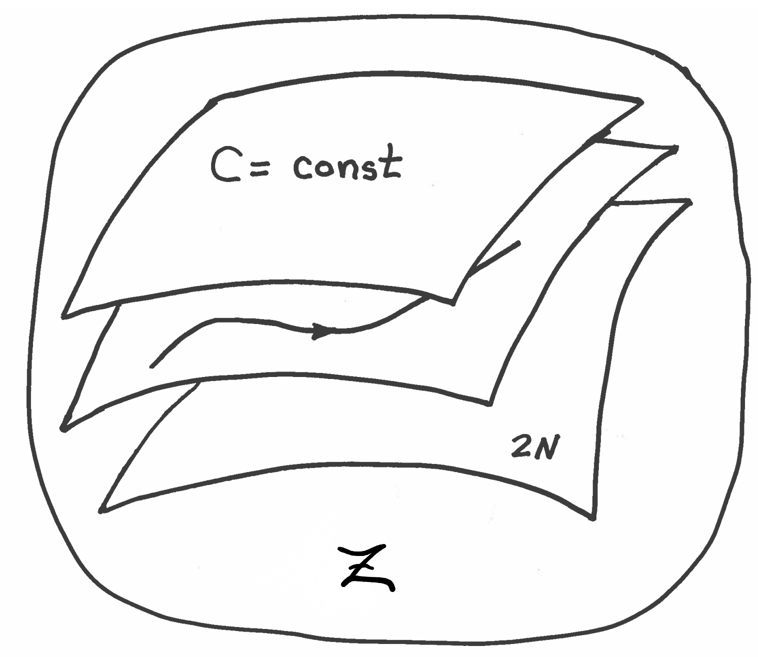
Because of degeneracy, $\exists$ functions C st $\{A, C\} = 0$ for all $A \in C^\infty(\mathcal{Z})$. Called Casimir invariants (Lie's distinguished functions!).

Poisson Manifold (phase space) $\mathcal{Z}$ Cartoon

Degeneracy in J $\Rightarrow$ Casimirs:

$$\{A, C\} = 0 \quad \forall\ A : \mathcal{Z} \to \mathbb{R}$$

Lie-Darboux Foliation by Casimir (symplectic) leaves:



Lie-Poisson Brackets

Lie-Poisson brackets are special kind of noncanonical Poisson bracket that are associated with any Lie algebra, say $\mathfrak{g}$.

Natural phase space $\mathfrak{g}^*$. For $f, g \in C^\infty(\mathfrak{g}^*)$ and $z \in \mathfrak{g}^*$.

Lie-Poisson bracket has the form

$$\begin{aligned} \{f, g\} &= \langle z, [\nabla f, \nabla g] \rangle \\ &= \frac{\partial f}{\partial z^i} c^{ij}{}_k \, z_k \frac{\partial g}{\partial z^j}\,, \qquad\qquad i, j, k = 1, 2, \ldots, \dim \mathfrak{g} \end{aligned}$$

Pairing $<\ ,\ >$: $\mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$, z^i coordinates for $\mathfrak{g}^*$, and $c^{ij}{}_k$ structure constants of $\mathfrak{g}$. Note $J^{ij} = c^{ij}{}_k \, z_k$.

Classical Field Theory for Classical Purposes

Dynamics of matter described by

- **Fluid models**
  - Euler's equations, Navier-Stokes, ...
- **Magnetofluid models**
  - MHD, XMHD (Hall, electron mass physics), 2-fluid, ...
- **Kinetic theories**
  - Vlasov-Maxwell, Landau-Lenard-Balescu, gyrokinetics, ...
- **Fluid-Kinetic hybrids**
  - MHD + hot particle kinetics, gyrokinetics, ...

Applications:

atmospheres, oceans, fluidics, natural and laboratory plasmas

Hamiltonian and Dissipative structures are <u>organizing principles</u>

Noncanonical MHD (pjm & Greene 1980)

Equations of Motion:

$$
\begin{aligned}
\text{Force} \quad & \rho\frac{\partial \boldsymbol{v}}{\partial t} = -\rho \boldsymbol{v}\cdot\nabla\boldsymbol{v} - \nabla p + \frac{1}{c}\boldsymbol{J}\times\boldsymbol{B} \\
\text{Density} \quad & \frac{\partial \rho}{\partial t} = -\nabla\cdot(\rho\boldsymbol{v}) \\
\text{Entropy} \quad & \frac{\partial s}{\partial t} = -\boldsymbol{v}\cdot\nabla s \\
\text{Ohm's Law} \quad & \boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B} = \eta\boldsymbol{J} = \eta\nabla\times\boldsymbol{B} \approx 0 \\
\text{Magnetic Field} \quad & \frac{\partial \boldsymbol{B}}{\partial t} = -\nabla\times\boldsymbol{E} = \nabla\times(\boldsymbol{v}\times\boldsymbol{B})
\end{aligned}
$$

Energy:

$$
H = \int_D d^3x \left(\frac{1}{2}\rho|\boldsymbol{v}|^2 + \rho U(\rho, s) + \frac{1}{2}|\boldsymbol{B}|^2\right)
$$

Thermodynamics:

$$
p = \rho^2\frac{\partial U}{\partial \rho} \qquad T = \frac{\partial U}{\partial s} \qquad \text{or} \quad p = \kappa\rho^\gamma
$$

Noncanonical Bracket:

$$
\begin{aligned}
\{F,G\} \;=\; -\int_D d^3x \Bigg(& \left[\frac{\delta F}{\delta \rho}\nabla\frac{\delta G}{\delta \boldsymbol{v}} - \frac{\delta G}{\delta \rho}\nabla\frac{\delta F}{\delta \boldsymbol{v}}\right] + \left[\frac{\delta F}{\delta \boldsymbol{v}} \cdot \left(\frac{\nabla \times \boldsymbol{v}}{\rho} \times \frac{\delta F}{\delta \boldsymbol{v}}\right)\right] \\
& + \frac{\nabla s}{\rho} \cdot \left[\frac{\delta F}{\delta \boldsymbol{v}} \cdot \nabla\frac{\delta G}{\delta s} - \frac{\delta G}{\delta \boldsymbol{v}} \cdot \nabla\frac{\delta F}{\delta s}\right] \\
& + \boldsymbol{B} \cdot \left[\frac{1}{\rho}\frac{\delta F}{\delta \boldsymbol{v}} \cdot \nabla\frac{\delta G}{\delta \boldsymbol{B}} - \frac{1}{\rho}\frac{\delta G}{\delta \boldsymbol{v}} \cdot \nabla\frac{\delta F}{\delta \boldsymbol{B}}\right] \\
& + \boldsymbol{B} \cdot \left[\nabla\left(\frac{1}{\rho}\frac{\delta F}{\delta \boldsymbol{v}}\right) \cdot \frac{\delta G}{\delta \boldsymbol{B}} - \nabla\left(\frac{1}{\rho}\frac{\delta G}{\delta \boldsymbol{v}}\right) \cdot \frac{\delta F}{\delta \boldsymbol{B}}\right] \Bigg) .
\end{aligned}
$$

Dynamics:

$$
\frac{\partial \rho}{\partial t} = \{\rho, H\}, \quad \frac{\partial s}{\partial t} = \{s, H\}, \quad \frac{\partial \boldsymbol{v}}{\partial t} = \{\boldsymbol{v}, H\}, \text{ and } \frac{\partial \boldsymbol{B}}{\partial t} = \{\boldsymbol{B}, H\} .
$$

Densities:

$$
\boldsymbol{M} := \rho \boldsymbol{v} \qquad \sigma := \rho s \qquad \text{Lie – Poisson form}
$$

MHD Dynamics and Invariance

Dynamical (field) Variables:

$$\Psi := (\rho, \boldsymbol{v}, s, \boldsymbol{B})$$

Poisson Bracket:

$$\{F, G\} = \int_D d^3x \frac{\delta F}{\delta \Psi} \mathcal{J}(\Psi) \frac{\partial G}{\partial \Psi} .$$

$$\frac{\partial \Psi}{\partial t} = \{\Psi, H\} = \mathcal{J}(\Psi) \frac{\partial H}{\partial \Psi}$$

Poisson Operator $\mathcal{J}(\Psi)$: matrix differential operator

Algebra of (Galilean) Invariance:

$$P = \int_D d^3x \rho \boldsymbol{v} , \quad \boldsymbol{L} = \int_D d^3x \rho \, \boldsymbol{r} \times \boldsymbol{v} , \quad \text{etc.} \qquad \leftarrow \text{ 10 parameters}$$

Realization on functionals.

Casimir Invariants and the Kernel of $\mathcal{J}$:

Recall $\mathcal{J}\delta H/\delta\psi$, Casimirs determined by $\mathcal{J}$ for any H.

Casimir Invariants:

$$\{F, C\}^{MHD} = 0 \qquad \forall \text{ functionals } F.$$

Casimirs Invariant entropies:

$$C_S = \int d^3x\, \rho f(s)\,, \qquad \text{f arbitrary}$$

Casimirs Invariant helicities:

$$C_B = \int d^3x\, \boldsymbol{B}\cdot\boldsymbol{A}\,, \qquad C_V = \int d^3x\, \boldsymbol{B}\cdot\boldsymbol{v}$$

Helicities have topological content, linking etc.

II. Some Bracket Dissipation Formalisms

Binary Brackets for Dissipation circa 1980 $\rightarrow$

- Symmetric Bilinear Brackets (pjm 1980 $-\ldots$ unpublished, 1984 reduced MHD)

- Degenerate Antisymmetric Bracket (Kaufman and pjm 1982)

- Metriplectic Dynamics (pjm 1984,1984, 1986, $\ldots$ ANK 1984)

- Generic (Grmela 1984, with Oettinger 1997, $\ldots$) $\Leftrightarrow$ Metriplectic Dynamics! Binary but not Symmetric or Bilinear

- Double Brackets (Vallis, Carnevale; Brockett, Bloch ... 1989)

Brackets for Dissipation

Two ingredients: Binary or Bilinear Bracket + Generator

$$\dot{z} = \{z, H\} + (z, F)$$

where

$$(\;,\;) : C^{\infty}(\mathcal{Z}) \times C^{\infty}(\mathcal{Z}) \to C^{\infty}(\mathcal{Z})$$

What is F and what are the algebraic properties of $(\;,\;)$?

K-M Brackets 1982

Done for plasma quasilinear theory.

Dynamics:

$$\dot{z} = [z, H]_S$$

Properties:

- bilinear
- antisymmetric, degenerate
- entropy production

$$\dot{S} = [S, H]_S \geq 0 \quad \Rightarrow \quad z \mapsto z_{eq}$$

Double Bracket 1989

Good Idea:

Vallis, Carnevale, and Young, Shepherd (1989,1990)

$$\frac{d\mathcal{F}}{dt} = \{\mathcal{F}, H\} + ((\mathcal{F}, H)) = ((\mathcal{F}, \mathcal{F})) \geq 0$$

where

$$((F, G)) = \int d^3x \, \frac{\delta F}{\delta \chi} \mathcal{J}^2 \frac{\delta G}{\delta \chi}$$

Lyapunov function, $\mathcal{F}$, yields asymptotic stability to rearranged equilibrium.

- Maximizing energy at fixed Casimir: Works fine sometimes, but limited to circular vortex states ....

Simulated Annealing

Use various bracket dynamics to effect extremization.

Many relaxation methods exist: gradient descent, etc.

Simulated annealing: an **artificial** dynamics that solves a variational principle with constraints for equilibria states.

Coordinates (pjm &Flierl 2011):

$$\dot{z}^i = ((z^i, H)) = J^{ik} g_{kl} J^{jl} \frac{\partial H}{\partial z^j}$$

symmetric, definite, and kernel of J.

$$\dot{C} = 0 \qquad \text{with} \qquad \dot{H} \leq 0$$

Metriplectic Dynamics pjm 1984, 1986

A dynamical model of thermodynamics that ‘captures’:.

- First Law: conservation of energy

- Second Law: entropy production

• Proposed as a general type of dynamical system in pjm 1984, 1986 and many examples satisfying axioms were given.

• Kaufman 1984 had all but degeneracy in $(\, , \,)$.

Metriplectic Dynamics – Entropy, Degeneracies, and 1st and 2nd Laws

- Casimirs of noncanonical PB $\{\,,\}$ are 'candidate' entropies. Election of particular $S \in \{\text{Casimirs}\} \Rightarrow$ thermal equilibrium (relaxed) state.

- Generator: $F = H + S$

- 1st Law: identify energy with Hamiltonian, H, then

$$\dot{H} = \{H, F\} + (H, F) = 0 + (H, H) + (H, S) = 0$$

  Foliate $\mathcal{Z}$ by level sets of H, with $(H, A) = 0 \ \forall\ A \in C^\infty(\mathcal{Z})$.

- 2nd Law: entropy production

$$\dot{S} = \{S, F\} + (S, F) = (S, S) \geq 0$$

  Lyapunov relaxation to the equilibrium state. Dynamics solves the equilibrium variational principle: $\delta F = \delta(H + S) = 0$.

Geometical Definition

A *metriplectic system* consists of a smooth manifold $\mathcal{Z}$, two smooth vector bundle maps $J, G : T^*\mathcal{Z} \to T\mathcal{Z}$ covering the identity, and two functions $H, S \in C^\infty(\mathcal{Z})$, the *Hamiltonian* and the *entropy* of the system, such that

(i) $\{f, g\} := \langle \mathbf{d}f, J(\mathbf{d}g)\rangle$ is a Poisson bracket; $J^* = -J$;

(ii) $(f, g) := \langle \mathbf{d}f, G(\mathbf{d}g)\rangle$ is a positive semidefinite symmetric bracket, i.e., $(\, , \,)$ is $\mathbb{R}$-bilinear and symmetric, so $G^* = G$, and $(f, f) \geq 0$ for every $F \in C^\infty(\mathcal{Z})$;

(iii) $\{S, f\} = 0$ and $(H, f) = 0$ for all $f \in C^\infty(\mathcal{Z})$

$$\iff J(\mathbf{d}S) = G(\mathbf{d}H) = 0.$$

Two examples of pjm 1984

Vlasov with Collisions

$$\frac{\partial f}{\partial t} = -v \cdot \nabla f - a \cdot \nabla_v f + \left.\frac{\partial f}{\partial t}\right)_c$$

where

$$\text{Collision term} \quad \rightarrow \quad \left.\frac{\partial f}{\partial t}\right)_c$$

could be, Landau, Lenard Balescu, etc.

Conserves, mass, momentum, energy,

$$\frac{dH}{dt} = \frac{d}{dt}\int \frac{1}{2} m v^2 f + \text{interaction} = 0$$

and makes entropy

$$\frac{dS}{dt} = -\frac{d}{dt}\int f \ln(f) \geq 0$$

Landau Collision Operator

Metriplectic bracket:

$$(A,B) \;=\; \int dz \int dz' \left[\frac{\partial}{\partial v_i}\frac{\delta A}{\delta f(z)} - \frac{\partial}{\partial v'_i}\frac{\delta A}{\delta f(z')}\right] T_{ij}(z,z') \times \left[\frac{\partial}{\partial v_j}\frac{\delta B}{\delta f(z)} - \frac{\partial}{\partial v'_j}\frac{\delta B}{\delta f(z')}\right]$$

$$T_{ij}(z,z') = w_{ij}(z,z') f(z) f(z')/2$$

Conservation and Lyapunov:

$w_{ij}(z,z') = w_{ji}(z,z') \quad w_{ij}(z,z') = w_{ij}(z',z) \quad g_i w_{ij} = 0$ with $g_i = v_i - v'_i$

Landau kernel:

$$w_{ij}^{(L)} = (\delta_{ij} - g_i g_j/g^2)\delta(\mathbf{x}-\mathbf{x}')/g$$

Entropy:

$$S[f] = \int dz\, f \ln(f)$$

Ideal fluid with viscous heating and thermal conductivity.

$$\frac{\partial v_i}{\partial t} = \{v_i, \mathcal{H}\} \qquad (18)$$

$$\frac{\partial \rho}{\partial t} = \{\rho, \mathcal{H}\} \qquad (19)$$

$$\frac{\partial s}{\partial t} = \{s, \mathcal{H}\} \qquad (20)$$

where the GPB, {,}, is given by

$$\{F,G\} = -\int\left(\frac{\delta F}{\delta \rho}\vec{\nabla}\cdot\frac{\delta G}{\delta \vec{v}} + \frac{\delta F}{\delta \vec{v}}\cdot\vec{\nabla}\frac{\delta G}{\delta \rho} + \frac{\delta F}{\delta \vec{v}}\cdot\left[\frac{(\vec{\nabla}\times\vec{v})}{\rho}\times\frac{\delta G}{\delta \vec{v}}\right] + \frac{\vec{\nabla}s}{\rho}\cdot\left[\frac{\delta F}{\delta s}\frac{\delta G}{\delta \vec{v}} - \frac{\delta F}{\delta \vec{v}}\frac{\delta G}{\delta s}\right]\right) d^3x \ . \qquad (21)$$

Upon inserting the quantities shown on the right hand side of Eqs. (18)-(20), into Eq. (21) and performing the indicated operations one obtains, as noted, the invicid adiabatic limit of Eqs. (10)-(12).

The Casimirs for the bracket given by Eq. (21) are the total mass $M = \int \rho \, d^3x$ and a generalized entropy functional $\mathcal{S}_f = \int \rho f(s) \, d^3x$, where f is an arbitrary function of s. The latter quantity is added to the energy [Eq. (17)] to produce the generalized free energy of Eq. (4): $\mathcal{G} = \mathcal{H} + \mathcal{S}_f$.

In order to obtain the dissipative terms, we introduce the following symmetric bracket:

$$(F,G) = \frac{1}{\lambda}\int\left\{\frac{1}{\rho}\frac{\delta F}{\delta v_i}\frac{\partial}{\partial x_k}\left[\frac{\sigma_{ik}}{\rho}\frac{\delta G}{\delta s}\right] + \frac{1}{\rho}\frac{\delta G}{\delta v_i}\frac{\partial}{\partial x_k}\left[\frac{\sigma_{ik}}{\rho}\frac{\delta F}{\delta s}\right] + \frac{\sigma_{ik}}{T}\frac{\partial v_i}{\partial x_k}\left[\frac{1}{\rho^2}\frac{\delta F}{\delta s}\frac{\delta G}{\delta s}\right] + T^2\kappa\frac{\partial}{\partial x_k}\left[\frac{1}{\rho T}\frac{\delta F}{\delta s}\right]\frac{\partial}{\partial x_k}\left[\frac{1}{\rho T}\frac{\delta G}{\delta s}\right] + T\,\Lambda_{ikmn}\frac{\partial}{\partial x_m}\left[\frac{1}{\rho}\frac{\delta F}{\delta v_n}\right]\frac{\partial}{\partial x_k}\left[\frac{1}{\rho}\frac{\delta G}{\delta v_i}\right]\right\} d^3x \ , \qquad (23)$$

FIG. 10. This . ...

III. Metriplectic 4-Brackets for Dissipation

The Metriplectic 4-Bracket

4-bracket on 0-forms (functions):

$$(\,\cdot\,,\,\cdot\,;\,\cdot\,,\,\cdot\,)\colon \Lambda^0(\mathcal{Z})\times\Lambda^0(\mathcal{Z})\times\Lambda^0(\mathcal{Z})\times\Lambda^0(\mathcal{Z})\to\Lambda^0(\mathcal{Z})$$

For functions f, k, g, and n

$$(f,k;g,n) := R(\boldsymbol{d}f,\boldsymbol{d}k,\boldsymbol{d}g,\boldsymbol{d}n)\,,$$

In a coordinate patch the metriplectic 4-bracket has the form:

$$(f,k;g,n) = R^{ijkl}(z)\frac{\partial f}{\partial z^i}\frac{\partial k}{\partial z^j}\frac{\partial g}{\partial z^k}\frac{\partial n}{\partial z^l}\,. \qquad \leftarrow \text{quadravector?}$$

- A blend of ideas: Two important functions H and S, symmetries, curvature idea, multilinear brackets all in pjm 1984, 1986.
- Manifolds with both Poisson tensor J and compatible metric, g or connection.

Metriplectic 4-Bracket Properties

(i) linearity in all arguments, e.g,

$$(f+h,k;g,n)=(h,k;g,n)+(h,k;g,n)$$

(ii) algebraic identities/symmetries

$$\begin{aligned}(f,k;g,n) &= -(k,f;g,n)\\ (f,k;g,n) &= -(f,k;n,g)\\ (f,k;g,n) &= (g,n;f,k)\\ (f,k;g,n) &+ (f,g;n,k)+(f,n;k,g)=0 \qquad \leftarrow \text{not needed}\end{aligned}$$

(iii) derivation in all arguments, e.g.,

$$(fh,k;g,n)=f(h,k;g,n)+(f,k;g,n)h$$

which is manifest when written in coordinates. Here, as usual, fh denotes pointwise multiplication. Symmetries of algebraic curvature. Although $R^l{}_{ijk}$ or R_{lijk} but not R^{lijk}. **Metriplectic Minimum.**

Reduction to Metriplectic 2-Bracket

Symmetric 2-bracket:

$$(f,g)_H = (f,H;g,H) = (g,f)_H$$

Dissipative dynamics:

$$\dot{z} = (z,S)_H\,,$$

Energy conservation:

$$(f,H)_H = (H,f)_H = 0 \qquad \forall\, f\,.$$

Entropy dynamics:

$$\dot{S} = (S,S)_H = (S,H;S,H) \geq 0$$

Metriplectic 4-brackets $\rightarrow$ metriplectic 2-brackets of 1984, 1986!

Reduction to K-M

Kaufman & pjm, Phys. Lett. A 88, 405 (1982).

K-M dynamics:

$$\dot{z}^i = [z^i, H]_S \, ,$$

K-M bracket emerges from any metriplectic 4-bracket:

$$[f, g]_S := (f, g; S, H)$$

Thus,

$$[f, g]_S = -[g, f]_S$$

and

$$\dot{H} = [H, H]_S = (H, H; S, H) = 0 \, ,$$

and

$$\dot{S} = [S, H]_S = (S, H; S, H) \geq 0$$

Reduction to Double Brackets

Interchanging the role of H with a Casimir S:

$$(f, g)_S = (f, S; g, S)$$

Can show with assumptions (Koszul construction)

$$(C, g)_S = (C, S; g, S) = 0$$

for any Casimir C. Therefore $\dot{C} = 0$.

Reduction to not bilinear and nonsymmetric Generic

- Exists a procedure for linearizing and symmetrizing.

Easy Construction: K-N Product

Given σ and μ, two symmetric rank-2 tensor fields operating on 1-forms $\boldsymbol{d}f, \boldsymbol{d}k$ and $\boldsymbol{d}g, \boldsymbol{d}n$, the Kulkarni-Nomizu (K-N) product is

$$\begin{aligned}
\sigma \owedge \mu \, (\boldsymbol{d}f, \boldsymbol{d}k, \boldsymbol{d}g, \boldsymbol{d}n) \;&=\; \sigma(\boldsymbol{d}f, \boldsymbol{d}g)\, \mu(\boldsymbol{d}k, \boldsymbol{d}n) \\
&-\; \sigma(\boldsymbol{d}f, \boldsymbol{d}n)\, \mu(\boldsymbol{d}k, \boldsymbol{d}g) \\
&+\; \mu(\boldsymbol{d}f, \boldsymbol{d}g)\, \sigma(\boldsymbol{d}k, \boldsymbol{d}n) \\
&-\; \mu(\boldsymbol{d}f, \boldsymbol{d}n)\, \sigma(\boldsymbol{d}k, \boldsymbol{d}g)\,.
\end{aligned}$$

Metriplectic 4-bracket:

$$(f, k; g, n) = \sigma \owedge \mu(\boldsymbol{d}f, \boldsymbol{d}k, \boldsymbol{d}g, \boldsymbol{d}n)\,.$$

In coordinates:

$$R^{ijkl} = \sigma^{ik}\mu^{jl} - \sigma^{il}\mu^{jk} + \mu^{ik}\sigma^{jl} - \mu^{il}\sigma^{jk}\,.$$

K-N Product $\rightarrow$ Landau Collision Operator

Metriplectic 4-bracket on functionals:

$$
\begin{aligned}
(F,K;G,N) &= \int\!\!\int d^6z\, d^6z'\, \mathcal{G}(z,z') \\
&\qquad \times(\Sigma \owedge M)(F_f,K_f,G_f,N_f)(z,z') \\
&= \int d^6z \int d^6z'\, \mathcal{G}(z,z') \\
&\quad \times\, (\delta \owedge \delta)^{ijkl}\, P\left[F_f\right]_i P\left[K_f\right]_j P\left[G_f\right]_k P\left[N_f\right]_l \,,
\end{aligned}
$$

where

$$
F_f := \frac{\delta F}{\delta f} \qquad \text{and} \qquad P[w]_i = \frac{\partial w(z)}{\partial v_i} - \frac{\partial w(z')}{\partial v_i'}
$$

$(f,H;g,H) = (f,g)_H$ becomes metriplectic 2-bracket (pjm 1984).

$(f,H;S,H) =$ Landau collision operator!

Metriplectic 4-Bracket: Encompassing Definition of Dissipation

- Lots of geometry on Poisson manifolds with metric or connection.

- Entropy production and positive contravariant sectional curvature. For $\sigma, \eta \in \Lambda^1(\mathcal{Z})$, entropy production by

$$K(\sigma, \eta) := (S, H; S, H)\,,$$

where the second equality follows if $\sigma = \boldsymbol{d}S$ and $\eta = \boldsymbol{d}H$.