Thermodynamically consistent 2-phase flow via metriplectic 4-bracket dynamics

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- Theory of thermodynamically consistent theories.
- An algorithm for constructing such theories.
- Use algorithm to construct consistent theories for 2-phase flow.

Old and New

Old:

- A. N. Kaufman and P. J. Morrison, “Algebraic Structure of the Plasma Quasilinear Equations,” Physics Letters A 88, 405–406 (1982).
- P. J. Morrison, “Bracket Formulation for Irreversible Classical Fields,” Physics Letters A **100**, 423–427 (1984).
- P. J. Morrison, “Some Observations Regarding Brackets and Dissipation,” arXiv:2403.14698v1 [mathph] 15 Mar 2024 (1984 CPAM report).
- P. J. Morrison, “A Paradigm for Joined Hamiltonian and Dissipative Systems,” Physica D **18**, 410–419 (1986).

New:

- **A. Zaidni**, P. J. Morrison, and S. Benjelloun,, “Thermodynamically Consistent Cahn-Hilliard-Navier-Stokes Equations using the Metriplectic Dynamics Formalism,” arXiv:2402.11116
- N. Sato and P. J. Morrison, “A Collision Operator for Describing Dissipation in Noncanonical Phase Space,” Fundamental Plasma Physics **10**, 100054 (18pp) (2024).
- P. J. Morrison and M. Updike, “Inclusive Curvature-Like Framework for Describing Dissipation: Metriplectic 4-Bracket Dynamics,” Physical Review E **109**, 045202 (22pp) (2024).

Thermodynamic Consistency – Examples

Navier-Stokes (**inconsistent):**

$$\partial_t \boldsymbol{v} = -\boldsymbol{v}\cdot\nabla\boldsymbol{v} - \frac{1}{\rho_0}\nabla p + \nu\nabla^2\boldsymbol{v}\,, \qquad \nabla\cdot\boldsymbol{v} = 0 \quad \Rightarrow \quad p[\boldsymbol{v}]$$

$$H = \int_\Omega \rho_0\,|\boldsymbol{v}|^2/2 \quad \text{and} \quad \dot{H} \le 0$$

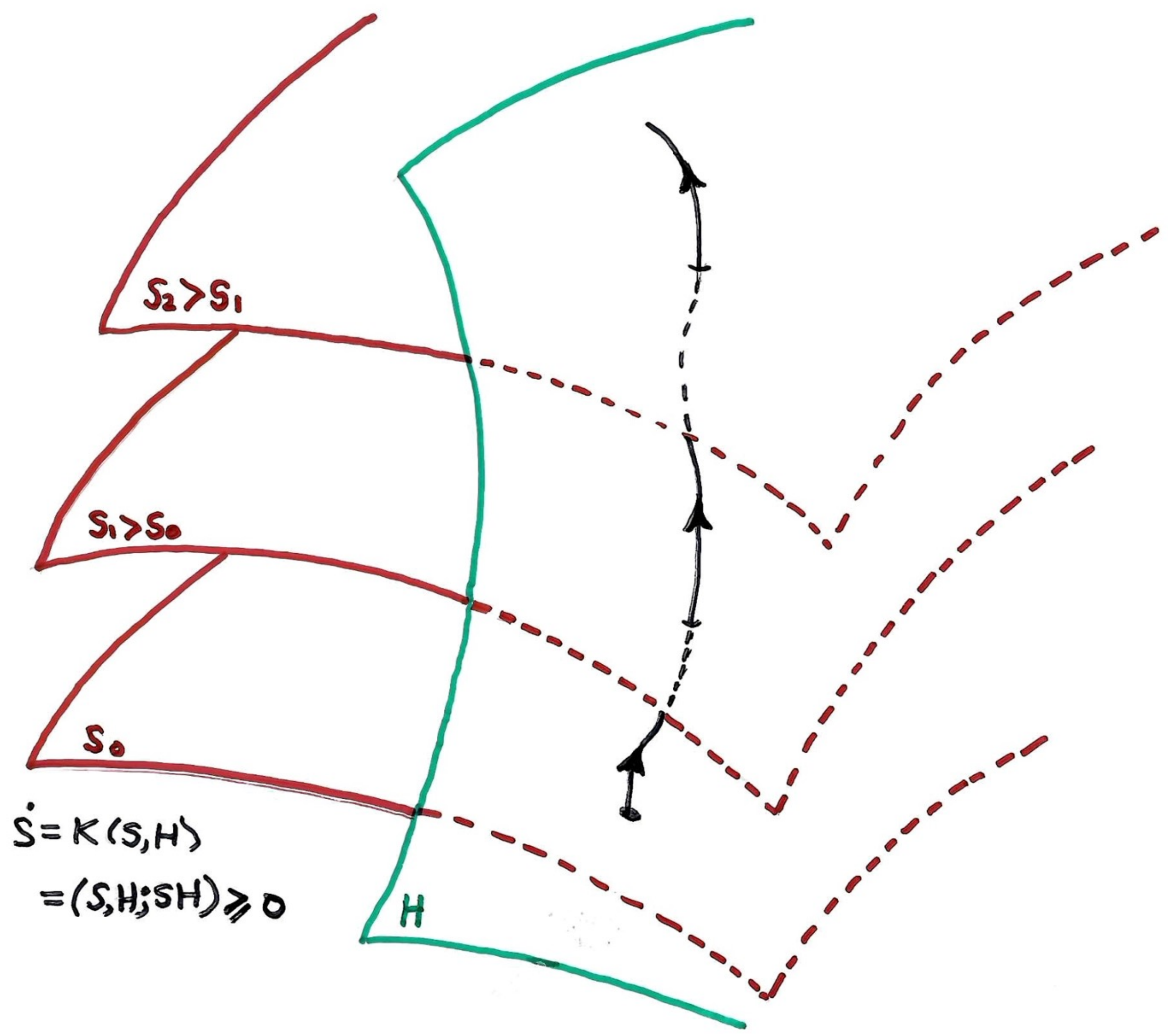
Thermodynamic Navier-Stokes (**consistent)** (Eckart 1940):

$$\partial_t \boldsymbol{v} = -\boldsymbol{v}\cdot\nabla\boldsymbol{v} - \frac{1}{\rho}\nabla p + \frac{1}{\rho}\nabla\cdot\mathcal{T} \qquad \leftarrow \ \mathcal{T} \text{ viscous stress tensor}$$

$$\partial_t \rho = -\nabla\cdot(\rho\boldsymbol{v})$$

$$\partial_t s = -\boldsymbol{v}\cdot\nabla s - \frac{1}{\rho T}\nabla\cdot\boldsymbol{q} + \frac{1}{\rho T}\mathcal{T}:\nabla\boldsymbol{v}\,, \qquad \boldsymbol{q} = -\kappa\nabla T \quad \leftarrow \ \mathcal{T} \text{ heat flux}$$

$$H = \int_\Omega \rho\,|\boldsymbol{v}|^2/2 + \rho U(\rho, s)\,, \quad \dot{H} = 0 \quad \text{and} \quad S = \int_\Omega \rho s \ \rightarrow \ \dot{S} \ge 0$$


$S_2 > S_1$
$S_1 > S_0$
S_0
H
$\dot{S} = K(S,H)$
$= (S,H;SH) \geq 0$

Cahn-Hilliard Equation (1958)

Equation of Motion:

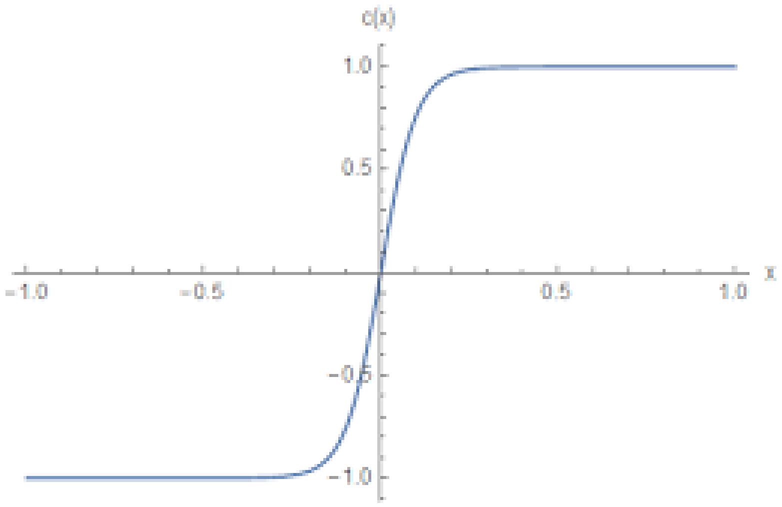
$$\frac{\partial c}{\partial t} = \nabla^2 \left(c^3 - c - \nabla^2 c \right) = \nabla^2 \frac{\delta \mathcal{F}}{\delta c},$$

for concentration c.

"Free Energy":

$$\mathcal{F} = \int d^3x \left(\frac{c^4}{4} - \frac{c^2}{2} + |\nabla c|^2 \right) = H - \mathcal{T} S\,?$$

A phase separation (diffuse interface) solution:



Goal

Construct:

- Cahn-Hilliard $\cup$ Navier-Stokes = Cahn-Hilliard - Navier-Stokes (CHNS)
- Thermodynamically consisted with complete set of fluxes and affinities.

Vector Fields, $V(z)$

Natural Split:

$$V(z) = V_H + V_D$$

- Hamiltonian vector fields, V_H: conservative, properties, etc.
- Dissipative vector fields, V_D: not conservative of something, relaxation/asymptotic stability, etc.

General Hamiltonian Form:

$$\text{finite dim} \rightarrow \quad V_H = J\frac{\partial H}{\partial z} = \{z, H\} \qquad \text{or} \qquad V_H = \mathcal{J}\frac{\delta H}{\delta \psi} \quad \leftarrow \infty \text{ dim}$$

where $J(z)$ is Poisson tensor/operator, $\{f, g\}$ Poisson bracket, and H is the Hamiltonian.

General Dissipation:

$$V_D = ?... \qquad \rightarrow \qquad V_D = G\frac{\partial S}{\partial z}$$

Build in thermodynamic consistency: 1st law Hamiltonian $\dot{H} = 0$ and 2nd law entropy $\dot{S} \geq 0$.

Building Theories - Traditional

Identify configuration space:

- Coordinates $q \in \mathcal{Q}$.
- Identify kinetic and potential energies, T and V.
- Construct Lagrangian:

$$\mathcal{L} = T - V\,.$$

- Obtain Lagrange's equations of motion:

$$\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} = 0\,.$$

For both finite systems and field theories consider symmetries, etc.

New Metriplectic Algorithm

1. Identify dynamical variables defined on $\Omega \subset \mathbb{R}^N$; e.g. for CHNS

$$\Psi = \{\boldsymbol{v}, \rho, s, c\} \qquad \text{or} \qquad \Psi = \{\boldsymbol{m} = \rho\boldsymbol{v}, \rho, \sigma = \rho s, \tilde{c} = \rho c\}$$

2. Propose energy and entropy functionals, $H[\Psi]$ and $S[\Psi]$; for CHNS*

$$H^a = \int_\Omega \frac{\rho}{2}|\mathbf{v}|^2 + \rho u + \frac{\rho^a}{2}\lambda_u \Gamma^2(\nabla c) \quad \text{and} \quad S^a = \int_\Omega \rho s + \frac{\rho^a}{2}\lambda_s \Gamma^2(\nabla c)$$

3. Find Poisson bracket $\{F, G\}$ for which entropy S^a is a Casimir invariant, $\{F, S^a\} = 0\, \forall\, F$

4. Construct metriplectic 4-bracket $(F, K; G, N)$ via Kulkarni-Nomizu product to obtain EoMs:

$$\partial_t \Psi = \{\Psi, H\} + (\Psi, H; S, H)$$

Result automatically Thermodynamically consistent!

* Γ Euler homogenous deg 1 (Taylor 1992 weighted mean curvature surface effects); $\mathcal{F} = H - \mathcal{T}S \to$ CH.

Hamiltonian Review

Poisson Bracket: $\{f, g\}$

Hamilton's Canonical Equations

Phase Space with Canonical Coordinates: (q, p)

Hamiltonian function: $H(q, p)$ $\leftarrow$ the energy

Equations of Motion:

$$\boxed{\dot{p}_\alpha = -\frac{\partial H}{\partial q^\alpha}\,, \qquad \dot{q}^\alpha = \frac{\partial H}{\partial p_i}\,,} \qquad \alpha = 1, 2, \ldots N$$

Phase Space Coordinate Rewrite: $z = (q, p)\,, \quad i, j = 1, 2, \ldots 2N$

$$\dot{z}^i = J_c^{ij}\,\frac{\partial H}{\partial z^j} = \{z^i, H\}_c\,, \qquad (J_c^{ij}) = \begin{pmatrix} 0_N & I_N \\ -I_N & 0_N \end{pmatrix}\,,$$

$J_c :=$ Poisson tensor, Hamiltonian bi-vector, cosymplectic form

Noncanonical Hamiltonian Structure

Sophus Lie (1890) $\longrightarrow$ PJM (1980) $\longrightarrow$ Poisson Manifolds etc.

Noncanonical Coordinates:

$$\dot{z}^i = \{z^i, H\} = J^{ij}(z)\frac{\partial H}{\partial z^j}$$

Noncanonical Poisson Bracket:

$$\{f,g\} = \frac{\partial f}{\partial z^i} J^{ij}(z) \frac{\partial g}{\partial z^j}$$

Poisson Bracket Properties:

antisymmetry $\longrightarrow$ $\{f,g\} = -\{g,f\}$

Jacobi identity $\longrightarrow$ $\{f,\{g,h\}\} + \{b,\{h,f\}\} + \{h,\{f,g\}\} = 0$

Leibniz $\longrightarrow$ $\{fh,g\} = f\{h,g\} + \{h,g\}f$

G. Darboux: $detJ \neq 0 \Longrightarrow J \rightarrow J_c$ Canonical Coordinates

Sophus Lie: $detJ = 0 \Longrightarrow$ Canonical Coordinates plus Casimirs (Lie's distinguished functions!)

Poisson Brackets – Flows on Poisson Manifolds

Definition. A Poisson manifold $\mathcal{Z}$ has bracket

$$\{\,,\,\} : C^\infty(\mathcal{Z}) \times C^\infty(\mathcal{Z}) \to C^\infty(\mathcal{Z})$$

st $C^\infty(\mathcal{Z})$ with $\{\,,\,\}$ is a Lie algebra realization, i.e., is

- bilinear,
- antisymmetric,
- Jacobi, and
- Leibniz, i.e., acts as a derivation $\Rightarrow$ vector field.

Geometrically $C^\infty(\mathcal{Z}) \equiv \Lambda^0(\mathcal{Z})$ and $\boldsymbol{d}$ exterior derivative.

$$\{f, g\} = \langle \boldsymbol{d}f, J\boldsymbol{d}g\rangle = J(\boldsymbol{d}f, \boldsymbol{d}g)\,.$$

J the Poisson tensor/operator. Flows are integral curves of noncanonical Hamiltonian vector fields, $J\boldsymbol{d}H$, i.e.,

$$\dot{z}^i = J^{ij}(z)\frac{\partial H(z)}{\partial z^j}\,, \qquad \mathcal{Z}'s \text{ coordinate patch } z = (z^1, \dots, z^N)$$

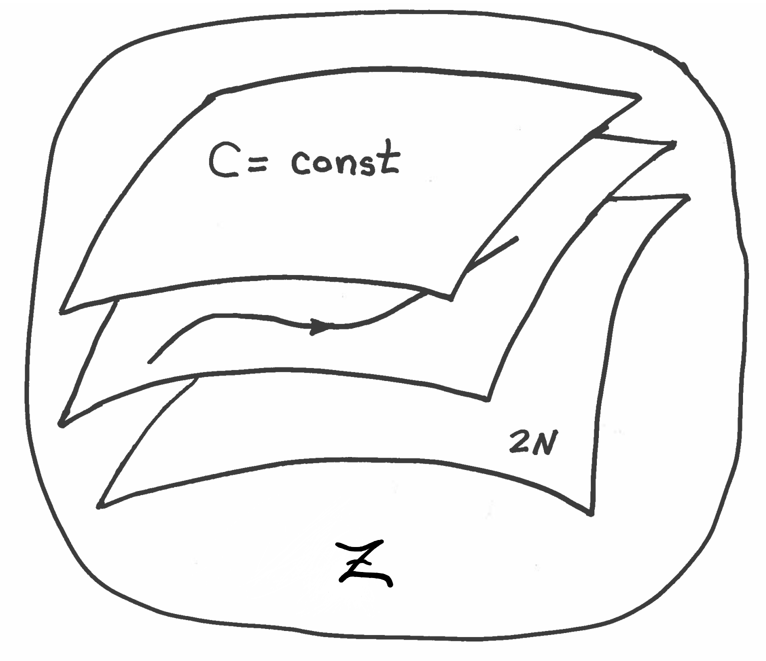
Because of degeneracy, $\exists$ functions C st $\{f, C\} = 0$ for all $f \in C^\infty(\mathcal{Z})$. Casimir invariants (Lie's distinguished functions!).

Poisson Manifold (phase space) $\mathcal{Z}$ Cartoon

Degeneracy in J $\Rightarrow$ Casimirs:

$$\{f, C\} = 0 \quad \forall\ f : \mathcal{Z} \to \mathbb{R}$$

Lie-Darboux Foliation by Casimir (symplectic) leaves:



3. Gibbs-Euler Poisson Bracket Dynamics

Hamiltonian:

$$H = \int_\Omega \frac{\rho|\mathbf{v}|^2}{2} + \rho u\,(\rho, s, c)\;, \qquad T = \frac{\partial u}{\partial s}\,, \qquad p = \rho^2 \frac{\partial u}{\partial \rho}\,, \quad \mu = \frac{\partial u}{\partial c}\,.$$

Poisson Bracket:

$$\{F, G\} = -\int_\Omega \mathbf{m} \cdot [F_{\boldsymbol{m}} \cdot \nabla G_{\boldsymbol{m}} - G_{\boldsymbol{m}} \cdot \nabla F_{\boldsymbol{m}}] + \rho\,[F_{\boldsymbol{m}} \cdot \nabla G_\rho - G_{\boldsymbol{m}} \cdot \nabla F_\rho]$$
$$+\,\sigma\,[F_{\boldsymbol{m}} \cdot \nabla G_\sigma - G_{\boldsymbol{m}} \cdot \nabla F_\sigma] + \tilde{c}\,[F_{\boldsymbol{m}} \cdot \nabla G_{\tilde{c}} - G_{\boldsymbol{m}} \cdot \nabla F_{\tilde{c}}]\,.$$

Equations of Motion:

$$\partial_t \mathbf{v} = \{\boldsymbol{v}, H\} = -\boldsymbol{v} \cdot \nabla \boldsymbol{v} - \nabla p/\rho\,, \qquad \partial_t \rho = \{\rho, H\} = -\nabla \cdot (\rho\,\boldsymbol{v})\,,$$
$$\partial_t \tilde{c} = \{\tilde{c}, H\} = -\nabla \cdot (\tilde{c}\,\boldsymbol{v})\,, \qquad \partial_t \sigma = \{\sigma, H\} = -\nabla \cdot (\sigma\,\boldsymbol{v})\,.$$

Casimir:

$$S = \int_\Omega \rho s \neq S^a\,!$$

Coordinate Change:

$$\rho s^a = \rho s + \frac{\rho^a}{2} \lambda_s \Gamma^2 (\nabla c)\,.$$

Note $F_m = \delta F / \delta \boldsymbol{m}$, etc., functional derivatives.

Metriplectic 4-Bracket: $(f, k; g, n)$

Why a 4-Bracket?

- Two slots for two fundamental functions: Hamiltonian, H, and Entropy (Casimir), S.

- There remains two slots for bilinear bracket: one for observable one for generator ($\mathcal{F}$?) s.t. $\dot{H} = 0$ and $\dot{S} \geq 0$.

- Provides natural reductions to other bilinear & binary brackets.

- The three slot brackets of pjm 1984 were not trilinear. Four needed to be multilinear.

The Metriplectic 4-Bracket

4-bracket on 0-forms (functions):

$$(\,\cdot\,,\,\cdot\,;\,\cdot\,,\,\cdot\,)\colon \Lambda^0(\mathcal{Z})\times\Lambda^0(\mathcal{Z})\times\Lambda^0(\mathcal{Z})\times\Lambda^0(\mathcal{Z})\to\Lambda^0(\mathcal{Z})$$

For functions $f,k,g,n\in\Lambda^0(\mathcal{Z})$

$$(f,k;g,n):=R(\boldsymbol{d}f,\boldsymbol{d}k,\boldsymbol{d}g,\boldsymbol{d}n)\,,$$

In a coordinate patch the metriplectic 4-bracket has the form:

$$(f,k;g,n)=R^{ijkl}(z)\frac{\partial f}{\partial z^i}\frac{\partial k}{\partial z^j}\frac{\partial g}{\partial z^k}\frac{\partial n}{\partial z^l}\,.$$ $\leftarrow$ quadravector?

- A blend of my previous ideas: Two important functions H and S, symmetries, curvature idea, multilinear brackets.

- Manifolds with both Poisson tensor, J^{ij}, and compatible quadravector R^{ijkl}, where S and H come from Hamiltonian part.

Metriplectic 4-Bracket Properties

(i) $\mathbb{R}$-linearity in all arguments, e.g,

$$(f+h,k;g,n) = (f,k;g,n) + (h,k;g,n)$$

(ii) algebraic identities/symmetries

$$\begin{aligned}(f,k;g,n) &= -(k,f;g,n)\\ (f,k;g,n) &= -(f,k;n,g)\\ (f,k;g,n) &= (g,n;f,k)\end{aligned}$$

(iii) derivation in all arguments, e.g.,

$$(fh,k;g,n) = f(h,k;g,n) + (f,k;g,n)h$$

which is manifest when written in coordinates. Here, as usual, fh denotes pointwise multiplication. Symmetries of algebraic curvature without cyclic identity. Often see ${R^l}_{ijk}$ or R_{lijk} but not R^{lijk}! **Minimal Metriplectic.**

Existence – General Constructions

• For any Riemannian manifold $\exists$ metriplectic 4-bracket. This means there is a wide class of them, but the bracket tensor does not need to come from Riemann tensor only needs to satisfy the bracket properties.

• Methods of construction? We describe two: Kulkarni-Nomizu and Lie algebra based. Goal is to develop intuition like building Lagrangians.

Construction via Kulkarni-Nomizu Product

Given σ and μ, two symmetric rank-2 tensor fields operating on 1-forms (assumed exact) $\boldsymbol{d}f, \boldsymbol{d}k$ and $\boldsymbol{d}g, \boldsymbol{d}n$, the K-N product is

$$\begin{aligned}
\sigma \owedge \mu\,(\boldsymbol{d}f, \boldsymbol{d}k, \boldsymbol{d}g, \boldsymbol{d}n) \;&=\; \sigma(\boldsymbol{d}f, \boldsymbol{d}g)\,\mu(\boldsymbol{d}k, \boldsymbol{d}n) \\
&-\; \sigma(\boldsymbol{d}f, \boldsymbol{d}n)\,\mu(\boldsymbol{d}k, \boldsymbol{d}g) \\
&+\; \mu(\boldsymbol{d}f, \boldsymbol{d}g)\,\sigma(\boldsymbol{d}k, \boldsymbol{d}n) \\
&-\; \mu(\boldsymbol{d}f, \boldsymbol{d}n)\,\sigma(\boldsymbol{d}k, \boldsymbol{d}g)\,.
\end{aligned}$$

Metriplectic 4-bracket:

$$(f, k; g, n) = \sigma \owedge \mu(\boldsymbol{d}f, \boldsymbol{d}k, \boldsymbol{d}g, \boldsymbol{d}n)\,.$$

In coordinates:

$$R^{ijkl} = \sigma^{ik}\mu^{jl} - \sigma^{il}\mu^{jk} + \mu^{ik}\sigma^{jl} - \mu^{il}\sigma^{jk}\,.$$

Lie Algebras and Lie-Poisson Brackets

Lie Algebras: Denoted $\mathfrak{g}$, is a vector space (over $\mathbb{R}, \mathbb{C}$, for us $\mathbb{R}$) with binary, bilinear product $[\cdot,\cdot]\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$. In basis $\{e_i\}$, $[e_i, e_j] = c_{ij}^{\ \ k}\, e_k$. Structure constants $c_{ij}^{\ \ k}$. For example $\mathfrak{so}(3)$, which has $\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) + \boldsymbol{B}\times(\boldsymbol{C}\times\boldsymbol{A}) + \boldsymbol{C}\times(\boldsymbol{A}\times\boldsymbol{B}) \equiv 0$.

Lie-Poisson Brackets: special noncanonical Poisson brackets associated with any Lie algebra, $\mathfrak{g}$.

Natural phase space $\mathfrak{g}^*$. For $f, g \in C^\infty(\mathfrak{g}^*)$ and $z \in \mathfrak{g}^*$.

Lie-Poisson bracket has the form

$$\begin{aligned} \{f,g\} &= \langle z, [\nabla f, \nabla g]\rangle \\ &= \frac{\partial f}{\partial z^i} c^{ij}_{\ \ k}\, z_k \frac{\partial g}{\partial z^j}\,, \qquad\qquad i,j,k = 1,2,\ldots,\dim\mathfrak{g} \end{aligned}$$

Pairing $<\ ,\ >\colon \mathfrak{g}^*\times\mathfrak{g}\to\mathbb{R}$, z^i coordinates for $\mathfrak{g}^*$, and $c^{ij}_{\ \ k}$ structure constants of $\mathfrak{g}$. Note

$$J^{ij} = c^{ij}_{\ \ k}\, z_k\,.$$

Lie Algebra Based Metriplectic 4-Brackets

- For structure constants $c^{kl}{}_s$:

$$(f,k;g,n) = c^{ij}{}_r c^{kl}{}_s\, g^{rs} \frac{\partial f}{\partial z^i}\frac{\partial k}{\partial z^j}\frac{\partial g}{\partial z^k}\frac{\partial n}{\partial z^l}\,.$$

Lacks cyclic symmetry, but ∃ procedure to remove torsion (Bianchi identity) for any symmetric 'metric' g^{rs}. Dynamics does not see torsion, but manifold does.

- For $g^{rs}_{CK} = c^{rl}{}_k\, c^{sk}{}_l$ the Cartan-Killing metric, torsion vanishes automatically. Completely determined by Lie algebra.

- Covariant connection $\nabla\colon \mathfrak{X}\times\mathfrak{X}\to\mathfrak{X}$. A contravariant connection $D\colon \Lambda^1(\mathcal{Z})\times\Lambda^1(\mathcal{Z})\to\Lambda^1(\mathcal{Z})$ satisfying Koszul identities, but Leibniz becomes $D_\alpha(f\gamma) = fD_\alpha\gamma + J(\alpha)[f]\gamma$ where $J(\alpha)[f] = \alpha_i J^{ij}\partial f/\partial z^j$ is a 0-form that replaces the term $\mathbf{X}(f)$ (Fernandes, 2000). Here $\alpha,\beta,\gamma\in\Lambda^1(\mathcal{Z})$, $f\in\Lambda^0(\mathcal{Z})$. Add a metric, build 4-bracket like curvature from connection.

4. K-N Metriplectic 4-Brackets for CHNS

K-N Form:

$$M(dF,dG) = F_{\sigma^a} G_{\sigma^a}\,,$$
$$\Sigma(dF,dG) = \nabla F_{\boldsymbol{m}} : \bar{\bar{\Lambda}}_1 : \nabla G_{\boldsymbol{m}} + \nabla F_{\sigma^a} \cdot \bar{\Lambda}_2 \cdot \nabla G_{\sigma^a} + \nabla \mathcal{L}^a_{\tilde{c}}(F) \cdot \bar{\Lambda}_3 \cdot \mathcal{L}^a_{\tilde{c}}(G)\,,$$

with pseudodifferential operator $\mathcal{L}^a_{\tilde{c}}(F) := \nabla\Big(F_{\tilde{c}} + \nabla \cdot (\rho^a \lambda_s \Gamma \boldsymbol{\xi} F_{\sigma^a})\,/\rho\Big).$

4-bracket:

$$\begin{aligned}(F,K;G,N)^a = \int_\Omega \frac{1}{T}\Big[&\, [K_{\sigma^a}\nabla F_{\mathbf{m}} - F_{\sigma^a}\nabla K_{\mathbf{m}}] : \bar{\bar{\Lambda}} : [N_{\sigma^a}\nabla G_{\mathbf{m}} - G_{\sigma^a}\nabla N_{\mathbf{m}}] \\ &+ \frac{1}{T}\Big[K_{\sigma^a}\nabla F_{\sigma^a} - F_{\sigma^a}\nabla K_{\sigma^a}\Big] \cdot \bar{\kappa} \cdot \Big[N_{\sigma^a}\nabla G_{\sigma^a} - G_{\sigma^a}\nabla N_{\sigma^a}\Big] \\ &+ \Big[K_{\sigma^a}\mathcal{L}^a_{\tilde{c}}(F) - F_{\sigma^a}\mathcal{L}^a_{\tilde{c}}(K)\Big] \cdot \bar{D} \cdot \Big[N_{\sigma^a}\mathcal{L}^a_{\tilde{c}}(G) - G_{\sigma^a}\mathcal{L}^a_{\tilde{c}}(N)\Big].\end{aligned}$$

Equations of Motion - Case $a = 1$

CHNS system for $a = 1$:

$$\begin{aligned}
\partial_t \boldsymbol{v} &= \{\boldsymbol{v}, H^1\}^1 + (\boldsymbol{v}, H^1; S^1, H^1)^1 \\
&= -\boldsymbol{v} \cdot \nabla \boldsymbol{v} - \frac{1}{\rho} \nabla \cdot \left[p\mathbf{I} + \lambda_f \rho \Gamma \boldsymbol{\xi} \otimes \nabla c \right] + \frac{1}{\rho} \nabla \cdot (\bar{\bar{\Lambda}} : \nabla \boldsymbol{v}) \,, \\
\partial_t \rho &= \{\rho, H^1\}^1 + (\rho, H^1; S^1, H^1)^1 = -\nabla \cdot (\rho \boldsymbol{v}) \,, \\
\partial_t \tilde{c} &= \{\tilde{c}, H^1\}^1 + (\tilde{c}, H^1; S^1, H^1)^1 = -\nabla \cdot (\tilde{c} \boldsymbol{v}) + \nabla \cdot (\bar{D} \cdot \nabla \mu_\Gamma^1) \,, \\
\partial_t \sigma_{\text{Total}}^1 &= \{\sigma_{\text{Total}}^1, H^1\}^1 + (\sigma_{\text{Total}}^1, H^1; S^1, H^1)^1 \\
&= -\nabla \cdot (\sigma_{\text{Total}}^1 \boldsymbol{v}) + \nabla \cdot \left(\frac{\bar{\kappa}}{T} \cdot \nabla T \right) + \frac{1}{T^2} \nabla T \cdot \bar{\kappa} \cdot \nabla T \\
&\quad + \frac{1}{T} \nabla \mathbf{v} : \bar{\bar{\Lambda}} : \nabla \mathbf{v} + \frac{1}{T} \nabla \mu_\Gamma^1 \cdot \bar{D} \cdot \nabla \mu_\Gamma^1 \,.
\end{aligned}$$

Special case does **not** agree with Guo and Lin, JFM (2015):
ours generalizes theirs and conserves energy, theirs does not!

Equations of Motion - Case $a = 0$

CHNS for $a = 0$:

$$
\begin{aligned}
\partial_t \boldsymbol{v} &= \{\boldsymbol{v}, H^0\}^0 + (\boldsymbol{v}, H^0; S^0, H^0)^0 \\
&= -\boldsymbol{v}\cdot\nabla\boldsymbol{v} - \frac{1}{\rho}\nabla\cdot\Big[\Big(p - \lambda_f \Gamma^2/2\Big)\mathbf{I} + \lambda_f \Gamma \boldsymbol{\xi}\otimes\nabla c\Big] + \frac{1}{\rho}\nabla\cdot(\bar{\bar{\Lambda}} : \nabla\boldsymbol{v})\,, \\
\partial_t \rho &= \{\rho, H^0\}^0 + (\rho, H^0; S^0, H^0)^0 = -\nabla\cdot(\rho\boldsymbol{v}) \\
\partial_t \tilde{c} &= \{\tilde{c}, H^0\}^0 + (\tilde{c}, H^0; S^0, H^0)^0 = -\nabla\cdot(\tilde{c}\boldsymbol{v}) + \nabla\cdot(\bar{D}\cdot\nabla\mu_\Gamma^0)\,, \\
\partial_t \sigma^0_{\mathrm{Total}} &= \{\sigma^0_{\mathrm{Total}}, H^0\}^0 + (\sigma^0_{\mathrm{Total}}, H^0; S^0, H^0)^0 \\
&= -\nabla\cdot(\sigma^0_{\mathrm{Total}}\boldsymbol{v}) + \nabla\cdot\left(\frac{\bar{\kappa}}{T}\cdot\nabla T\right) + \frac{1}{T^2}\nabla T\cdot\bar{\kappa}\cdot\nabla T \\
&\quad + \frac{1}{T}\nabla\boldsymbol{v} : \bar{\bar{\Lambda}} : \nabla\boldsymbol{v} + \frac{1}{T}\nabla\mu_\Gamma^0\cdot\bar{D}\cdot\nabla\mu_\Gamma^0\,.
\end{aligned}
$$

Special case agrees with Anderson et al. Physica D (2000)

Conclusion

Two thermodynamically consistent CHNS systems:

$$\dot{S}^a = (S^a, H^a; S^a, H^a)^a = K(S^a, H^a) \quad \leftarrow \text{sectional curvature}$$
$$= \int_\Omega \frac{1}{T}\left[\nabla \mathbf{v} : \bar{\bar{\Lambda}} : \nabla \mathbf{v} + \frac{1}{T}\nabla T \cdot \bar{\kappa} \cdot \nabla T + \nabla \mu_\Gamma^a \cdot \bar{D} \cdot \nabla \mu_\Gamma^a\right] \geq 0\,.$$