

Thermodynamically consistent 2-phase flow via metriplectic 4-bracket dynamics

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NMPP, Max Planck, Garching

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- Theory of thermodynamically consistent theories.
- An algorithm for constructing such theories.
- Use algorithm to construct consistent theories for 2-phase flow.

Old and New

Old:

- A. N. Kaufman and P. J. Morrison, “[Algebraic Structure of the Plasma Quasilinear Equations](#),” Physics Letters A 88, 405–406 (1982).
- P. J. Morrison, “[Bracket Formulation for Irreversible Classical Fields](#),” Physics Letters A **100**, 423–427 (1984).
- P. J. Morrison, “[Some Observations Regarding Brackets and Dissipation](#),” arXiv:2403.14698v1 [math-ph] 15 Mar 2024 (1984 CPAM report).
- P. J. Morrison, “[A Paradigm for Joined Hamiltonian and Dissipative Systems](#),” Physica D **18**, 410–419 (1986).

New:

- **A. Zaidni**, P. J. Morrison, and S. Benjelloun, “[Thermodynamically Consistent Cahn-Hilliard-Navier-Stokes Equations using the Metriplectic Dynamics Formalism](#),” arXiv:2402.11116
- N. Sato and P. J. Morrison, “[A Collision Operator for Describing Dissipation in Noncanonical Phase Space](#),” Fundamental Plasma Physics **10**, 100054 (18pp) (2024).
- P. J. Morrison and M. Updike, “[Inclusive Curvature-Like Framework for Describing Dissipation: Metriplectic 4-Bracket Dynamics](#),” Physical Review E **109**, 045202 (22pp) (2024).

Thermodynamic Consistency – Examples

Navier-Stokes (**inconsistent**):

$$\partial_t \mathbf{v} = -\mathbf{v} \cdot \nabla \mathbf{v} - \frac{1}{\rho_0} \nabla p + \nu \nabla^2 \mathbf{v}, \quad \nabla \cdot \mathbf{v} = 0 \quad \Rightarrow \quad p[\mathbf{v}]$$

$$H = \int_{\Omega} \rho_0 |\mathbf{v}|^2 / 2 \quad \text{and} \quad \dot{H} \leq 0$$

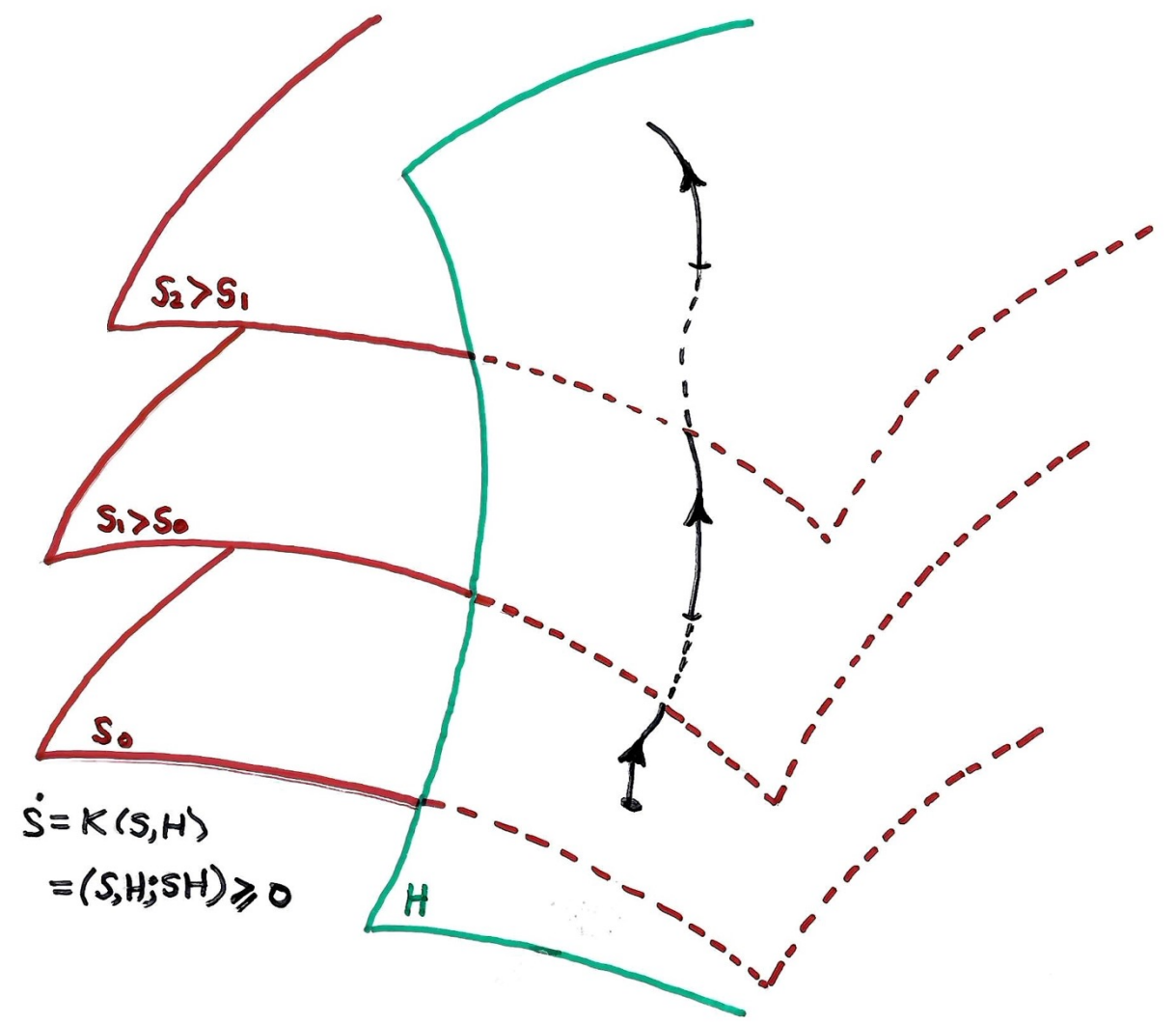
Thermodynamic Navier-Stokes (**consistent**) (Eckart 1940):

$$\partial_t \mathbf{v} = -\mathbf{v} \cdot \nabla \mathbf{v} - \frac{1}{\rho} \nabla p + \frac{1}{\rho} \nabla \cdot \mathcal{T} \quad \leftarrow \quad \mathcal{T} \text{ viscous stress tensor}$$

$$\partial_t \rho = -\nabla \cdot (\rho \mathbf{v})$$

$$\partial_t s = -\mathbf{v} \cdot \nabla s - \frac{1}{\rho T} \nabla \cdot \mathbf{q} + \frac{1}{\rho T} \mathcal{T} : \nabla \mathbf{v}, \quad \mathbf{q} = -\kappa \nabla T \quad \leftarrow \quad \mathcal{T} \text{ heat flux}$$

$$H = \int_{\Omega} \rho |\mathbf{v}|^2 / 2 + \rho U(\rho, s), \quad \dot{H} = 0 \quad \text{and} \quad S = \int_{\Omega} \rho s \rightarrow \dot{S} \geq 0$$



$$\dot{S} = K(S, H)$$

$$= (S, H; SH) \geq 0$$

Cahn-Hilliard Equation (1958)

Equation of Motion:

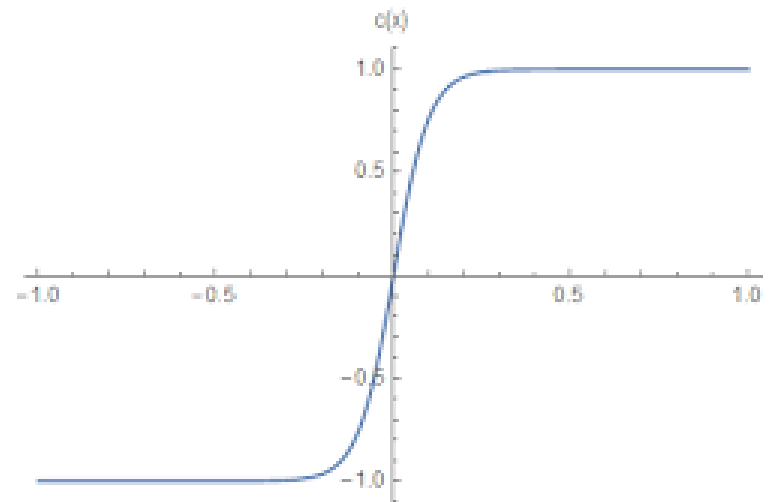
$$\frac{\partial c}{\partial t} = \nabla^2 (c^3 - c - \nabla^2 c) = \nabla^2 \frac{\delta \mathcal{F}}{\delta c},$$

for concentration c .

“Free Energy”:

$$\mathcal{F} = \int d^3x \left(\frac{c^4}{4} - \frac{c^2}{2} + |\nabla c|^2 \right) \stackrel{?}{=} H - TS.$$

A phase separation (diffuse interface) solution:



Goal

Construct:

- Cahn-Hilliard \cup Navier-Stokes = Cahn-Hilliard - Navier-Stokes (CHNS)
- Thermodynamically consisted with complete set of fluxes and affinities.

Vector Fields, $V(z)$

Natural Split:

$$V(z) = V_H + V_D$$

- Hamiltonian vector fields, V_H : conservative, properties, etc.
- Dissipative vector fields, V_D : not conservative of something, relaxation/asymptotic stability, etc.

General Hamiltonian Form:

$$\text{finite dim} \rightarrow V_H = J \frac{\partial H}{\partial z} = \{z, H\} \quad \text{or} \quad V_H = \mathcal{J} \frac{\delta H}{\delta \psi} \leftarrow \infty \text{ dim}$$

where $J(z)$ is Poisson tensor/operator, $\{f, g\}$ Poisson bracket, and H is the Hamiltonian.

General Dissipation:

$$V_D = ? \dots \rightarrow V_D = G \frac{\partial S}{\partial z}$$

Build in thermodynamic consistency: 1st law Hamiltonian $\dot{H} = 0$ and 2nd law entropy $\dot{S} \geq 0$.

Building Theories - Traditional

Identify configuration space:

- Coordinates $q \in \mathcal{Q}$.
- Identify kinetic and potential energies, T and V .
- Construct Lagrangian:

$$\mathcal{L} = T - V .$$

- Obtain Lagrange's equations of motion:

$$\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} = 0 .$$

For both finite systems and field theories consider symmetries, etc.

New Metriplectic Algorithm - 4 Steps

1. Identify dynamical variables defined on $\Omega \subset \mathbb{R}^3$; e.g. for CHNS

$$\Psi = \{\mathbf{v}, \rho, s, c\} \quad \text{or} \quad \Psi = \{\mathbf{m} = \rho\mathbf{v}, \rho, \sigma = \rho s, \tilde{c} = \rho c\}$$

2. Propose energy and entropy functionals, $H[\Psi]$ and $S[\Psi]$; for CHNS*

$$H^a = \int_{\Omega} \frac{\rho}{2} |\mathbf{v}|^2 + \rho u + \frac{\rho^a}{2} \lambda_u \Gamma^2(\nabla c) \quad \text{and} \quad S^a = \int_{\Omega} \rho s + \frac{\rho^a}{2} \lambda_s \Gamma^2(\nabla c)$$

3. Find Poisson bracket $\{F, G\}$ for which entropy S^a is a Casimir invariant, $\{F, S^a\} = 0 \forall F$

4. Construct metriplectic 4-bracket $(F, K; G, N)$ via Kulkarni-Nomizu product to obtain EoMs:

$$\partial_t \Psi = \{\Psi, H\} + (\Psi, H; S, H)$$

Result automatically Thermodynamically consistent!

* Here $a \in \{0, 1\}$ is a parameter; Γ Euler homogenous deg 1 (Taylor 1992 weighted mean curvature surface effects); when $\Gamma^2(\nabla c) = |\nabla c|^2$, cf. $\mathcal{F} = H - \mathcal{T}S$ of C-H.

Hamiltonian Review

Poisson Bracket: $\{f, g\}$

Hamilton's Canonical Equations

Phase Space with Canonical Coordinates: (q, p)

Hamiltonian function: $H(q, p)$ ← the energy

Equations of Motion:

$$\dot{p}_\alpha = -\frac{\partial H}{\partial q^\alpha}, \quad \dot{q}^\alpha = \frac{\partial H}{\partial p_\alpha}, \quad \alpha = 1, 2, \dots, N$$

Phase Space Coordinate Rewrite: $z = (q, p)$, $i, j = 1, 2, \dots, 2N$

$$\dot{z}^i = J_c^{ij} \frac{\partial H}{\partial z^j} = \{z^i, H\}_c, \quad J_c = \begin{pmatrix} 0_N & I_N \\ -I_N & 0_N \end{pmatrix},$$

$J_c :=$ Poisson tensor, Hamiltonian bi-vector, cosymplectic form

Noncanonical Hamiltonian Structure

Sophus Lie (1890) \longrightarrow PJM (1980) \longrightarrow Poisson Manifolds etc.

Noncanonical Coordinates:

$$\dot{z}^a = \{z^a, H\} = J^{ab}(z) \frac{\partial H}{\partial z^b}, \quad a, b = 1, 2, \dots, M$$

Noncanonical Poisson Bracket:

$$\{f, g\} = \frac{\partial f}{\partial z^a} J^{ab}(z) \frac{\partial g}{\partial z^b}, \quad J(z) \neq J_c$$

Poisson Bracket Properties:

antisymmetry $\longrightarrow \{f, g\} = -\{g, f\}$

Jacobi identity $\longrightarrow \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$

Leibniz $\longrightarrow \{fh, g\} = f\{h, g\} + \{h, g\}f$

G. Darboux: $\det J \neq 0 \implies J \rightarrow J_c$ Canonical Coordinates

Sophus Lie: $\det J = 0 \implies$ Canonical Coordinates plus Casimirs (Lie's distinguished functions!)

Poisson Brackets – Flows on Poisson Manifolds

Definition. A Poisson manifold \mathcal{Z} has bracket

$$\{, \} : C^\infty(\mathcal{Z}) \times C^\infty(\mathcal{Z}) \rightarrow C^\infty(\mathcal{Z})$$

st $C^\infty(\mathcal{Z})$ with $\{, \}$ is a Lie algebra realization, i.e., is

- bilinear,
- antisymmetric,
- Jacobi, and
- Leibniz, i.e., acts as a derivation \Rightarrow vector field.

Geometrically $C^\infty(\mathcal{Z}) \equiv \Lambda^0(\mathcal{Z})$ and d exterior derivative.

$$\{f, g\} = \langle df, Jdg \rangle = J(df, dg).$$

J the Poisson tensor/operator. Flows are integral curves of noncanonical Hamiltonian vector fields, JdH , i.e.,

$$\dot{z}^a = J^{ab}(z) \frac{\partial H(z)}{\partial z^b}, \quad \mathcal{Z}'s \text{ coordinate patch } z = (z^1, \dots, z^M)$$

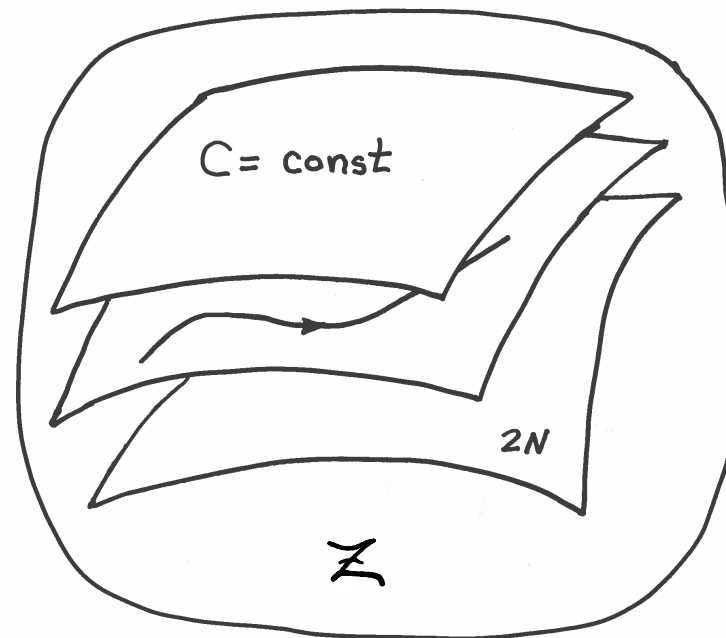
Because of degeneracy, \exists functions C st $\{f, C\} = 0$ for all $f \in C^\infty(\mathcal{Z})$. Casimir invariants.

Poisson Manifold (phase space) \mathcal{Z} Cartoon

Degeneracy in $J \Rightarrow$ Casimirs:

$$\{f, C\} = 0 \quad \forall f : \mathcal{Z} \rightarrow \mathbb{R}$$

Lie-Darboux Foliation by Casimir (symplectic) leaves:



3. Gibbs-Euler Poisson Bracket Dynamics

Hamiltonian:

$$H = \int_{\Omega} \frac{\rho |\mathbf{v}|^2}{2} + \rho u(\rho, s, c), \quad T = \frac{\partial u}{\partial s}, \quad p = \rho^2 \frac{\partial u}{\partial \rho}, \quad \mu = \frac{\partial u}{\partial c}.$$

Poisson Bracket:

$$\begin{aligned} \{F, G\} = & - \int_{\Omega} \mathbf{m} \cdot [F_{\mathbf{m}} \cdot \nabla G_{\mathbf{m}} - G_{\mathbf{m}} \cdot \nabla F_{\mathbf{m}}] + \rho [F_{\mathbf{m}} \cdot \nabla G_{\rho} - G_{\mathbf{m}} \cdot \nabla F_{\rho}] \\ & + \sigma [F_{\mathbf{m}} \cdot \nabla G_{\sigma} - G_{\mathbf{m}} \cdot \nabla F_{\sigma}] + \tilde{c} [F_{\mathbf{m}} \cdot \nabla G_{\tilde{c}} - G_{\mathbf{m}} \cdot \nabla F_{\tilde{c}}]. \end{aligned}$$

Equations of Motion:

$$\begin{aligned} \partial_t \mathbf{v} = \{\mathbf{v}, H\} &= -\mathbf{v} \cdot \nabla \mathbf{v} - \nabla p / \rho, & \partial_t \rho = \{\rho, H\} &= -\nabla \cdot (\rho \mathbf{v}), \\ \partial_t \tilde{c} = \{\tilde{c}, H\} &= -\nabla \cdot (\tilde{c} \mathbf{v}), & \partial_t \sigma = \{\sigma, H\} &= -\nabla \cdot (\sigma \mathbf{v}) \end{aligned}$$

Casimir:

$$S = \int_{\Omega} \rho s \neq S^a !$$

Coordinate Change:

$$\rho s^a = \rho s + \frac{\rho^a}{2} \lambda_s \Gamma^2(\nabla c), \quad \mathbf{m}, \rho, c \text{ unchanged.}$$

Note $F_{\mathbf{m}} = \delta F / \delta \mathbf{m}$, etc., functional derivatives.

Metriplectic 4-Bracket: $(f, k; g, n)$

Why a 4-Bracket?

- Two slots for two fundamental functions: Hamiltonian, H , and Entropy (Casimir), S .
- There remains two slots for bilinear bracket: one for observable one for generator, $\mathcal{F} = H - \mathcal{T}S$, s.t. $\dot{H} = 0$ and $\dot{S} \geq 0$.
- Provides natural reductions to other bilinear & binary brackets.
- The three slot brackets of pjm 1984 were not trilinear. Four needed to be multilinear.

The Metriplectic 4-Bracket

4-bracket on 0-forms (functions):

$$(\cdot, \cdot; \cdot, \cdot): \Lambda^0(\mathcal{Z}) \times \Lambda^0(\mathcal{Z}) \times \Lambda^0(\mathcal{Z}) \times \Lambda^0(\mathcal{Z}) \rightarrow \Lambda^0(\mathcal{Z})$$

For functions $f, k, g, n \in \Lambda^0(\mathcal{Z})$

$$(f, k; g, n) := R(df, dk, dg, dn),$$

In a coordinate patch the metriplectic 4-bracket has the form:

$$(f, k; g, n) = R^{ijkl}(z) \frac{\partial f}{\partial z^i} \frac{\partial k}{\partial z^j} \frac{\partial g}{\partial z^k} \frac{\partial n}{\partial z^l}. \quad \leftarrow \text{quadravector?}$$

- A blend of my previous ideas: Two important functions H and S , symmetries, curvature idea, multilinear brackets.
- Manifolds with both Poisson tensor, J^{ij} , and compatible quadravector R^{ijkl} , where S and H come from Hamiltonian part.

Metriplectic 4-Bracket Properties

(i) \mathbb{R} -linearity in all arguments, e.g,

$$(f + h, k; g, n) = (f, k; g, n) + (h, k; g, n)$$

(ii) algebraic identities/symmetries

$$(f, k; g, n) = -(k, f; g, n)$$

$$(f, k; g, n) = -(f, k; n, g)$$

$$(f, k; g, n) = (g, n; f, k)$$

(iii) derivation in all arguments, e.g.,

$$(fh, k; g, n) = f(h, k; g, n) + (f, k; g, n)h$$

which is manifest when written in coordinates. Here, as usual, fh denotes pointwise multiplication. Symmetries of algebraic curvature without cyclic identity. Often see R^l_{ijk} or R_{lijk} but not R^{lijk} ! **Minimal Metriplectic.**

Existence – General Constructions

- For any Riemannian manifold \exists metriplectic 4-bracket. This means there is a wide class of them, but the bracket tensor does not need to come from Riemann tensor only needs to satisfy the bracket properties.
- Methods of construction? We describe two: Kulkarni-Nomizu and Lie algebra based. Goal is to develop intuition like building Lagrangians.

Construction via Kulkarni-Nomizu Product

Given σ and μ , two symmetric rank-2 tensor fields operating on 1-forms (assumed exact) df, dk and dg, dn , the K-N product is

$$\begin{aligned}\sigma \otimes \mu (df, dk, dg, dn) &= \sigma(df, dg) \mu(dk, dn) \\ &- \sigma(df, dn) \mu(dk, dg) \\ &+ \mu(df, dg) \sigma(dk, dn) \\ &- \mu(df, dn) \sigma(dk, dg).\end{aligned}$$

Metriplectic 4-bracket:

$$(f, k; g, n) = \sigma \otimes \mu (df, dk, dg, dn).$$

In coordinates:

$$R^{ijkl} = \sigma^{ik} \mu^{jl} - \sigma^{il} \mu^{jk} + \mu^{ik} \sigma^{jl} - \mu^{il} \sigma^{jk}.$$

Lie Algebras and Lie-Poisson Brackets

Lie Algebras: Denoted \mathfrak{g} , is a vector space (over \mathbb{R}, \mathbb{C} , for us \mathbb{R}) with binary, bilinear product $[\cdot, \cdot]: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$. In basis $\{e_i\}$, $[e_i, e_j] = c_{ij}^k e_k$. Structure constants c_{ij}^k . For example $\mathfrak{so}(3)$, which has $A \times (B \times C) + B \times (C \times A) + C \times (A \times B) \equiv 0$.

Lie-Poisson Brackets: special noncanonical Poisson brackets associated with any Lie algebra, \mathfrak{g} .

Natural phase space \mathfrak{g}^* . For $f, g \in C^\infty(\mathfrak{g}^*)$ and $z \in \mathfrak{g}^*$.

Lie-Poisson bracket has the form

$$\begin{aligned} \{f, g\} &= \langle z, [\nabla f, \nabla g] \rangle \\ &= \frac{\partial f}{\partial z^i} c_{ij}^k z_k \frac{\partial g}{\partial z^j}, \quad i, j, k = 1, 2, \dots, \dim \mathfrak{g} \end{aligned}$$

Pairing $\langle \cdot, \cdot \rangle: \mathfrak{g}^* \times \mathfrak{g} \rightarrow \mathbb{R}$, z^i coordinates for \mathfrak{g}^* , and c_{ij}^k structure constants of \mathfrak{g} . Note

$$J^{ij} = c_{ij}^k z_k.$$

Lie Algebra Based Metriplectic 4-Brackets

- For structure constants c^{kl}_s :

$$(f, k; g, n) = c^{ij}_r c^{kl}_s g^{rs} \frac{\partial f}{\partial z^i} \frac{\partial k}{\partial z^j} \frac{\partial g}{\partial z^k} \frac{\partial n}{\partial z^l}.$$

Lacks cyclic symmetry, but \exists procedure to remove torsion (Bianchi identity) for any symmetric 'metric' g^{rs} . Dynamics does not see torsion, but manifold does.

- For $g^{rs}_{CK} = c^{rl}_k c^{sk}_l$ the Cartan-Killing metric, torsion vanishes automatically. Completely determined by Lie algebra.

- Covariant connection $\nabla: \mathfrak{X} \times \mathfrak{X} \rightarrow \mathfrak{X}$. A contravariant connection $D: \Lambda^1(\mathcal{Z}) \times \Lambda^1(\mathcal{Z}) \rightarrow \Lambda^1(\mathcal{Z})$ satisfying Koszul identities, but Leibniz becomes $D_\alpha(f\gamma) = fD_\alpha\gamma + J(\alpha)[f]\gamma$ where $J(\alpha)[f] = \alpha_i J^{ij} \partial f / \partial z^j$ is a 0-form that replaces the term $\mathbf{X}(f)$ (Fernandes, 2000). Here $\alpha, \beta, \gamma \in \Lambda^1(\mathcal{Z})$, $f \in \Lambda^0(\mathcal{Z})$. Add a metric, build 4-bracket like curvature from connection.

4. K-N Metriplectic 4-Brackets for CHNS

K-N Form:

$$M(dF, dG) = F_{\sigma^a} G_{\sigma^a},$$

$$\Sigma(dF, dG) = \nabla F_{\mathbf{m}} : \bar{\Lambda}_1 : \nabla G_{\mathbf{m}} + \nabla F_{\sigma^a} \cdot \bar{\Lambda}_2 \cdot \nabla G_{\sigma^a} + \nabla \mathcal{L}_{\bar{c}}^a(F) \cdot \bar{\Lambda}_3 \cdot \mathcal{L}_{\bar{c}}^a(G),$$

with pseudodifferential operator $\mathcal{L}_{\bar{c}}^a F := \nabla (F_{\bar{c}} + \nabla \cdot (\rho^a \lambda_s \Gamma \xi F_{\sigma^a}) / \rho)$.

4-bracket:

$$\begin{aligned} (F, K; G, N)^a &= \int_{\Omega} \frac{1}{T} \left[[K_{\sigma^a} \nabla F_{\mathbf{m}} - F_{\sigma^a} \nabla K_{\mathbf{m}}] : \bar{\Lambda} : [N_{\sigma^a} \nabla G_{\mathbf{m}} - G_{\sigma^a} \nabla N_{\mathbf{m}}] \right. \\ &\quad + \frac{1}{T} [K_{\sigma^a} \nabla F_{\sigma^a} - F_{\sigma^a} \nabla K_{\sigma^a}] \cdot \bar{\kappa} \cdot [N_{\sigma^a} \nabla G_{\sigma^a} - G_{\sigma^a} \nabla N_{\sigma^a}] \\ &\quad \left. + [K_{\sigma^a} \mathcal{L}_{\bar{c}}^a(F) - F_{\sigma^a} \mathcal{L}_{\bar{c}}^a(K)] \cdot \bar{D} \cdot [N_{\sigma^a} \mathcal{L}_{\bar{c}}^a(G) - G_{\sigma^a} \mathcal{L}_{\bar{c}}^a(N)] \right]. \end{aligned}$$

Equations of Motion - Case $a = 1, \Gamma = |\nabla c|$

CHNS system for $a = 1$:

$$\begin{aligned}
 \partial_t \mathbf{v} &= \{\mathbf{v}, H^1\}^1 + (\mathbf{v}, H^1; S^1, H^1)^1 \\
 &= -\mathbf{v} \cdot \nabla \mathbf{v} - \frac{1}{\rho} \nabla \cdot [p\mathbf{I} + \lambda_f \rho \Gamma \boldsymbol{\xi} \otimes \nabla c] + \frac{1}{\rho} \nabla \cdot (\bar{\bar{\Lambda}} : \nabla \mathbf{v}), \\
 \partial_t \rho &= \{\rho, H^1\}^1 + (\rho, H^1; S^1, H^1)^1 = -\nabla \cdot (\rho \mathbf{v}), \\
 \partial_t \tilde{c} &= \{\tilde{c}, H^1\}^1 + (\tilde{c}, H^1; S^1, H^1)^1 = -\nabla \cdot (\tilde{c} \mathbf{v}) + \nabla \cdot (\bar{D} \cdot \nabla \mu_\Gamma^1), \\
 \partial_t \sigma_{\text{Total}}^1 &= \{\sigma_{\text{Total}}^1, H^1\}^1 + (\sigma_{\text{Total}}^1, H^1; S^1, H^1)^1 \\
 &= -\nabla \cdot (\sigma_{\text{Total}}^1 \mathbf{v}) + \nabla \cdot \left(\frac{\bar{\kappa}}{T} \cdot \nabla T \right) + \frac{1}{T^2} \nabla T \cdot \bar{\kappa} \cdot \nabla T \\
 &\quad + \frac{1}{T} \nabla \mathbf{v} : \bar{\bar{\Lambda}} : \nabla \mathbf{v} + \frac{1}{T} \nabla \mu_\Gamma^1 \cdot \bar{D} \cdot \nabla \mu_\Gamma^1.
 \end{aligned}$$

Special case has H and S same as Guo and Lin, JFM (2015). But EoMs do **not** agree!
 Ours generalizes theirs and conserves energy, theirs does **not**!

Equations of Motion - Case $a = 0, \Gamma$ general

CHNS for $a = 0$:

$$\begin{aligned}
 \partial_t \mathbf{v} &= \{\mathbf{v}, H^0\}^0 + (\mathbf{v}, H^0; S^0, H^0)^0 \\
 &= -\mathbf{v} \cdot \nabla \mathbf{v} - \frac{1}{\rho} \nabla \cdot \left[\left(p - \lambda_f \Gamma^2 / 2 \right) \mathbf{I} + \lambda_f \Gamma \boldsymbol{\xi} \otimes \nabla c \right] + \frac{1}{\rho} \nabla \cdot (\bar{\bar{\Lambda}} : \nabla \mathbf{v}), \\
 \partial_t \rho &= \{\rho, H^0\}^0 + (\rho, H^0; S^0, H^0)^0 = -\nabla \cdot (\rho \mathbf{v}) \\
 \partial_t \tilde{c} &= \{\tilde{c}, H^0\}^0 + (\tilde{c}, H^0; S^0, H^0)^0 = -\nabla \cdot (\tilde{c} \mathbf{v}) + \nabla \cdot (\bar{D} \cdot \nabla \mu_\Gamma^0), \\
 \partial_t \sigma_{\text{Total}}^0 &= \{\sigma_{\text{Total}}^0, H^0\}^0 + (\sigma_{\text{Total}}^0, H^0; S^0, H^0)^0 \\
 &= -\nabla \cdot (\sigma_{\text{Total}}^0 \mathbf{v}) + \nabla \cdot \left(\frac{\bar{\kappa}}{T} \cdot \nabla T \right) + \frac{1}{T^2} \nabla T \cdot \bar{\kappa} \cdot \nabla T \\
 &\quad + \frac{1}{T} \nabla \mathbf{v} : \bar{\bar{\Lambda}} : \nabla \mathbf{v} + \frac{1}{T} \nabla \mu_\Gamma^0 \cdot \bar{D} \cdot \nabla \mu_\Gamma^0.
 \end{aligned}$$

Special case agrees with Anderson et al. Physica D (2000).

Conclusions

- Produced a general thermodynamically consistent CHNS system.

$$\begin{aligned} \dot{S}^a &= (S^a, H^a; S^a, H^a)^a = K(S^a, H^a) \quad \leftarrow \text{sectional curvature} \\ &= \int_{\Omega} \frac{1}{T} \left[\nabla_{\mathbf{v}} : \bar{\bar{\Lambda}} : \nabla_{\mathbf{v}} + \frac{1}{T} \nabla T \cdot \bar{\kappa} \cdot \nabla T + \nabla \mu_{\Gamma}^a \cdot \bar{D} \cdot \nabla \mu_{\Gamma}^a \right] \geq 0. \end{aligned}$$

- General system reduces to two thermodynamically consistent CHNS systems: Anderson et al. **yes**, while Guo and Lin, **almost**.

Future Work?

- Apply algorithm to some plasma problem? Pellet injection, multi collisional species, comet tails, dusty plasmas, etc.?