Metriplectic Dynamics: The Geometrical Framework for Thermodynamically Consistent Dynamical Systems

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Metriplectic 4-Bracket:

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Theory of thermodynamically consistent theories (theory = dynamical system = $\mathfrak{X}(\mathcal{Z})$)

Dynamical thermodynamics (nonequilibrium thermodynamics) $\neg$ thermodynamics

$\frac{\partial}{\partial t}$ $\leftarrow$ yes! vs. $\frac{\partial}{\partial T}$ $\leftarrow$ no!

Finite dimensions $\exists$ rigor. **Infinite** dimensions $\exists$ wishful thinking.

Goal minimal geometric structure: Not too much, not too little!

Overview

I. Motivation

II. Metriplectic 4-Bracket

III. Unified Thermodynamic (UT) Algorithm and Examples

IV. Contact Hamiltonian and Metriplectic Systems

V. Final Comments

I. Motivation

Thermodynamic Consistency (TC)– Examples

Navier-Stokes is **inconsistent:**

$$\partial_t \boldsymbol{v} = -\boldsymbol{v} \cdot \nabla \boldsymbol{v} - \frac{1}{\rho_0} \nabla p + \nu \nabla^2 \boldsymbol{v}\,, \qquad \nabla \cdot \boldsymbol{v} = 0 \quad \Rightarrow \quad p[\boldsymbol{v}]$$

$H = \int_\Omega \rho_0\, |\boldsymbol{v}|^2/2$ and $\dot{H} \leq 0\,,$ $\nexists$ any thermodynamics!

Navier-Stokes-Fourier (NSF) is **consistent** (Eckart 1940):

$$\partial_t \boldsymbol{v} = -\boldsymbol{v} \cdot \nabla \boldsymbol{v} - \frac{1}{\rho} \nabla p + \frac{1}{\rho} \nabla \cdot \mathcal{T}$$

viscous stress tensor is $\mathcal{T}$

$$\partial_t \rho = -\nabla \cdot (\rho \boldsymbol{v})$$

$$\partial_t s = -\boldsymbol{v} \cdot \nabla s - \frac{1}{\rho T} \nabla \cdot \boldsymbol{q} + \frac{1}{\rho T} \mathcal{T} : \nabla \boldsymbol{v}$$

heat flux & viscous heating

$H = \int_\Omega \rho\, |\boldsymbol{v}|^2/2 + \rho u(\rho, s)\,,$ $\dot{H} = 0$ and $S = \int_\Omega \rho s \;\rightarrow\; \dot{S} \geq 0$

Example of **Thermodynamic Completion**, i.e. NS $\rightarrow$ NSF.

Thermodynamic Consistency

The realization in a **dynamical system** of the first and second laws of thermodynamics:

First Law is energy conservation:

$$\dot{H} = 0$$

Second Law is entropy production:

$$\dot{S} \geq 0$$

Good models **lift** thermodynamics to **dynamical systems.** They have two functions H, S.

Theories & Models as Dynamical Systems

Main Scientific Goal:

Predict the future or explain the past $\Rightarrow$

$$\dot{z} = V(z)\,, \qquad \text{dynamical variable } z \in \mathcal{Z} \text{ the Phase Space}$$

Ultimately a dynamical system. Vector fields on manifolds and Cauchy problem (IVP).

Examples: Maps, ODEs, PDEs, etc. finite-dimensional, infinite-dimensional (field theories)

Whence vector field V?

• Fundamental parent theory (microscopic, N interacting gravitating or charged particles, BBGKY hierarchy, Vlasov-Maxwell system, ...). Identify small parameters, limits, rigorous asymptotics, Hilbert's 6th $\rightarrow$ Reduced Computable Model for V.

• Phenomena based modeling using known properties, constraints, symmetries, etc. used to intuit $\rightarrow$ Reduced Computable Model V. $\leftarrow$ Here metriplectic structure can be useful.

Vector Field Splitting

$$V(z) = V_{nondissipative} + V_{dissipative}$$

How?

Vector Field Splitting

$$V(z) = V_{nondissipative} + V_{dissipative}$$

How?

$V_{nondissipative} \equiv$ Hamiltonian and $V_{dissipative} \equiv$?

What is Dissipation?

- Not all conservative systems are Hamiltonian
- Not all Hamiltonian systems are conservative
- Not all reversible systems are Hamiltonian
- All finite dynamical systems (autonomous ODEs) are reversible (1 parameter Lie group)
- Some infinite systems (PDEs) are reversible and some irreversible ($t \to -t$ Hadamard ill-posed)
- Not all Hamiltonian systems have time reversal symmetry
- Not all systems with time reversal symmetry are Hamiltonian
- $\exists$ systems with time reversible symmetry and asymptotic stability

Thermodynamically Consistent Dissipation:
Energy conserving systems with an increasing entropy that implies global asymptotic stability.

Such systems have a 'vector field' that naturally splits in Hamiltonian and dissipative parts. Hamiltonian is an unambiguous way to define nondissipative. The metriplectic 4-bracket is an unambiguous way to define dissipative. Together they $\Rightarrow$ thermodynamic consistency.

Toward a Thermodynamically Consistent Split

$$V(z) = V_H + V_D$$

Hamiltonian Form:

$$V_H = J \frac{\partial H}{\partial z}$$

where $J(z)$ is Poisson tensor/operator and H is the Hamiltonian. Basic product decomposition.

Dissipative Form:

$$V_D = \ldots ? \qquad \rightarrow \qquad V_D = G \frac{\partial F}{\partial z}$$

General degenerate '**metric** tensor' G of some kind for gradient system?

Frameworks for Dissipation – Some Works

Lagrangian/Action Based: Rayleigh (1873),: $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_\nu}\right) - \left(\frac{\partial \mathcal{L}}{\partial q_\nu}\right) + \left(\frac{\partial \mathcal{F}}{\partial \dot{q}_\nu}\right) = 0$
Linear dissipation e.g. of sound waves. *Theory of Sound.*
Gay-Balmaz & Yoshimura (2017) & Eldred (2020).

Gradient Based: Cahn-Hilliard (1958): $\frac{\partial n}{\partial t} = \nabla^2 \frac{\delta \mathcal{F}}{\delta n} = \nabla^2\left(n^3 - n - \nabla^2 n\right)$
Phase separation, nonlinear diffusive dissipation, binary fluid ..
Otto, Ricci Flows, Poincarè conjecture on S^3, Hamilton, Perelman (2002)... : $\frac{\partial \psi}{\partial t} = \mathcal{G}\,\frac{\delta \mathcal{F}}{\delta \psi}$

Bracket Based: Kaufman & pjm (1982, 1984), Grmela (1984), pjm (1986), Öttinger & Grmela (1997), ... : $\frac{\partial \psi}{\partial t} = (\psi, \mathcal{F})$ ← emerges here.
GENERIC (1997) ⇔ Metriplectic 2-bracket (1984)
Plasma models, kinetic theory, fluids.

New Metriplectic 4-Bracket Based: pjm, Updike, Zaidni (2024,2025):
An encompassing theory. pjm 2009, Kirchhoff & Maschke 2025

Contact structure vs. Metriplectic structure
Cartheodory, Gibbs, Souriau, Arnold, ...de León & Lainz, López-Gordón

Metriplectic Dynamics

<u>**Metri**</u>c (G) $\cup$ Sym<u>**plectic**</u> (J) Systems (pjm 1986) $\leftrightarrow V_D + V_H$

- Formalism for natural split of vector fields

- Enforces thermodynamic consistency: $\dot{H} = 0$ the 1st Law and $\dot{S} \geq 0$ the 2nd Law.

- Other invariants? E.g., collision operators preserve, mass, momentum, ... There exists some theory for building in, but won't discuss today.

- **Encompassing 4-bracket**: Entropy is a Casimir is & "curvature" is dissipation rate

Ideas of Casimirs are candidates for entropy, multibracket, curvature, etc. in pjm (1984). <u>Metriplectic</u> in pjm (1986). Recent work 2020 ... .

Metriplectic 4-Bracket Dynamics

Dynamical System (finite or infinite):

$$\dot{z} = \{z, H\} + (z, H; S, H)$$

Dynamics for any observable (functional of dynamical variables), z, is generated by multi-linear brackets, Poisson bracket + 4-bracket (2024), with Hamiltonian H and entropy = Casimir S.

Hamiltonian Review

Poisson Bracket: $\{f, g\}$

Hamilton's Canonical Equations

Phase Space with Canonical Coordinates: (q, p)

Hamiltonian function: $H(q, p)$ $\leftarrow$ the energy

Equations of Motion:

$$\boxed{\dot{p}_i = -\frac{\partial H}{\partial q^i} \quad \text{and} \quad \dot{q}^i = \frac{\partial H}{\partial p_i}} \qquad i = 1, 2, \ldots N$$

Phase Space Coordinate Rewrite: $z = (q, p)\,, \quad \alpha, \beta = 1, 2, \ldots 2N$

$$\dot{z}^\alpha = J_c^{\alpha\beta} \frac{\partial H}{\partial z^\beta} = \{z^\alpha, H\}_c \quad \text{where} \quad (J_c^{\alpha\beta}) = \begin{pmatrix} 0_N & I_N \\ -I_N & 0_N \end{pmatrix}$$

$J_c :=$ Poisson tensor, Hamiltonian bi-vector, cosymplectic form

Noncanonical Hamiltonian Structure

Sophus Lie (1890) $\longrightarrow$ PJM (noncanonical 1980) $\longrightarrow$ Weinstein (1982) Poisson Manifolds

Noncanonical Coordinates:

$$\dot{z}^\alpha = \{z^\alpha, H\} = J^{\alpha\beta}(z)\frac{\partial H}{\partial z^\beta}$$

Noncanonical Poisson Bracket:

$$\{f,g\} = \frac{\partial f}{\partial z^\alpha} J^{\alpha\beta}(z)\frac{\partial g}{\partial z^\beta}$$

Bilinear Poisson Bracket Properties:

antisymmetry $\rightarrow$ $\{f,g\} = -\{g,f\}$ $\rightarrow$ $J^{\alpha\beta} = -J^{\beta\alpha}$

Jacobi identity $\rightarrow$ $\{f,\{g,h\}\}+\{b,\{h,f\}\}+\{h,\{f,g\}\} = 0$ $\rightarrow$ Jacobiator $S^{\alpha\beta\gamma} = J^{\alpha\ell}\partial_\ell J^{\beta\gamma} + cyc \equiv 0$

Leibniz $\rightarrow$ $\{fh,g\} = f\{h,g\} + \{h,g\}f\,,$ fg pointwise

G. Darboux: $detJ \neq 0 \Longrightarrow J \rightarrow J_c$ Canonical Coordinates

Sophus Lie: $detJ = 0 \Longrightarrow$ Canonical Coordinates plus Casimirs $\leftarrow$ G. Sudarshan
(Lie's distinguished functions!)

Noncanonical Poisson Brackets – Flows on Poisson Manifolds

Definition. A Poisson manifold $\mathcal{Z}$ has bracket

$$\{\,,\}\colon C^\infty(\mathcal{Z})\times C^\infty(\mathcal{Z})\to C^\infty(\mathcal{Z})$$

st $C^\infty(\mathcal{Z})$ with $\{\,,\}$ is a Lie algebra realization, i.e., is

- $\mathbb{R}$-bilinear,
- antisymmetric,
- Jacobi identity
- Leibniz, i.e., acts as a derivation $\Rightarrow$ vector field.

Geometrically $C^\infty(\mathcal{Z})\equiv \Lambda^0(\mathcal{Z})$ and $\boldsymbol{d}$ exterior derivative.

$$\{f,g\}=\langle \boldsymbol{d}f, J\boldsymbol{d}g\rangle = J(\boldsymbol{d}f,\boldsymbol{d}g)=\frac{\partial f}{\partial z^\alpha}J^{\alpha\beta}\frac{\partial g}{\partial z^\beta}\,.$$

J the Poisson tensor/operator. Flows are integral curves of noncanonical Hamiltonian vector fields, $J\boldsymbol{d}H$, i.e.,

$$\dot z^\alpha = J^{\alpha\beta}(z)\frac{\partial H(z)}{\partial z^\beta}\,,\qquad \mathcal{Z}'s \text{ coordinate patch } z=(z^1,\dots,z^N)$$

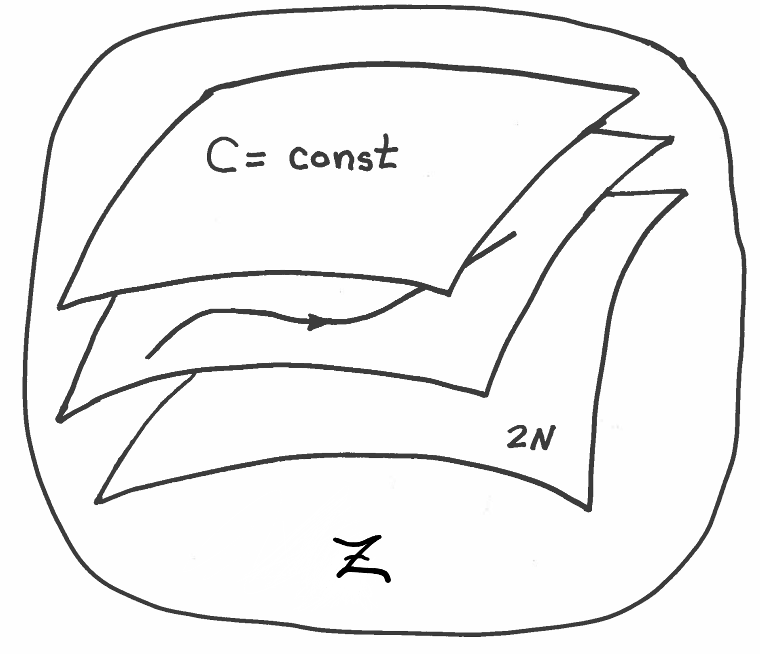
Because of degeneracy, $\exists$ functions C st $\{f,C\}=0$ for all $f\in C^\infty(\mathcal{Z})$, called Casimir invariants. **Casimirs are candidate entropies!**

Poisson Manifold (phase space) $\mathcal{Z}$ Cartoon

Degeneracy in J $\Rightarrow$ Casimirs:

$$\{f, C\} = 0 \quad \forall\, f : \mathcal{Z} \to \mathbb{R}$$

Lie-Darboux Foliation by Casimir (symplectic) leaves:



II. Metriplectic 4-Bracket: $(f, k; g, n)$

Why a 4-Bracket?

- One slot for dynamical variables (observables), z.

- Two slots for two fundamental functions: Hamiltonian, H, and Entropy (Casimir), S.

- There remains one slot for $\mathcal{F}$, free energy like generator $\mathcal{F} = H - TS$. Better argument: Needed to have multilinearity.

Comments:

- Provides natural reductions to other bilinear & binary brackets.

- The three slot brackets of pjm 1984 were not trilinear. Four needed to be multilinear.

The Metriplectic 4-Bracket

4-bracket on 0-forms (functions):

$$(\,\cdot\,,\,\cdot\,;\,\cdot\,,\,\cdot\,)\colon \Lambda^0(\mathcal{Z}) \times \Lambda^0(\mathcal{Z}) \times \Lambda^0(\mathcal{Z}) \times \Lambda^0(\mathcal{Z}) \to \Lambda^0(\mathcal{Z})$$

For functions $f, k, g, n \in \Lambda^0(\mathcal{Z})$ in a coordinate patch the 4-bracket has the form:

$$(f,k;g,n) = R^{ijkl}(z)\frac{\partial f}{\partial z^i}\frac{\partial k}{\partial z^j}\frac{\partial g}{\partial z^k}\frac{\partial n}{\partial z^l}\,.$$

$\leftarrow$ quadravector?

• Metriplectic manifolds have both Poisson tensor, J^{ij}, and compatible quadravector R^{ijkl}, where S (selected from set of Casimirs) and H comes from Hamiltonian part.

A blend of my previous early ideas 1980s: Two important functions H and S, symmetries, curvature idea, multi-brackets.

Metriplectic 4-Bracket Properties

(i) $\mathbb{R}$-linearity in all arguments, e.g, for $\lambda \in \mathbb{R}$

$$(f + \lambda h, k; g, n) = (f, k; g, n) + \lambda(h, k; g, n)$$

(ii) algebraic identities/symmetries

$$(f, k; g, n) = -(k, f; g, n)\,, \qquad (f, k; g, n) = -(f, k; n, g)\,, \qquad (f, k; g, n) = (g, n; f, k)$$

(iii) derivation in all arguments, e.g.,

$$(fh, k; g, n) = f(h, k; g, n) + (f, k; g, n)h$$

where as usual, fh denotes pointwise multiplication.

Symmetries of algebraic curvature without torsion identity. **Minimal Metriplectic.**

Observation: Often see ${R^l}_{ijk}$ or R_{lijk} but not R^{lijk}! Never 4-bracket, i.e. action on 1-forms?

Properties – Existence – General Construction Methods

- Thermodynamic Consistency Built-in:

$$\dot{H} = \{H, H\} + (H, H; S, H) = 0 \qquad \text{and} \qquad \dot{S} = (S, H; S, H) \geq 0$$

Reduces to metriplectic 2-bracket (1984): $(F, G)_H = (F, H; G, H)$.

- For any Riemannian manifold $\exists$ metriplectic 4-bracket. This means there is a wide class of them, but the bracket tensor does not need to come from Riemann tensor only needs to satisfy the bracket properties.

- If Riemannian, **entropy production rate is** positive contravariant **sectional curvature**. For closed $\sigma, \eta \in \Lambda^1(\mathcal{Z})$, entropy production by

$$\dot{S} = K(\sigma, \eta) := (S, H; S, H) \geq 0\,,$$

where the second equality follows from $\sigma = \boldsymbol{d}S$ and $\eta = \boldsymbol{d}H$.

- Two methods of construction? **Kulkarni-Nomizu** (K-N) product and **Lie algebra** based. $K(\sigma, \eta) \geq 0$ automatic for K-N and easily made minimally degenerate!

Metriplectic 4-Bracket $\to$ 2-Bracket

Symmetric 2-bracket:

$$(f,g)_H = (f,H;g,H) = (g,f)_H$$

Dynamics:

$$\dot{z} = \{z,H\} + (z,S)_H = \{z,H\} + (z,H;S,H) = J\frac{\partial H}{\partial z} + G\frac{\partial S}{\partial z}$$

Degeneracies:

$$\partial S/\partial z \in \mathrm{ker}J \quad \text{and} \quad \partial H/\partial z \in \mathrm{ker}G$$

Energy conservation:

$$(g,H)_H = 0 \ \forall\, g \qquad \text{and} \qquad \{H,H\} = 0\,.$$

Entropy dynamics:

$$\dot{S} = \{S,S\} + (S,S)_H = (S,H;S,H) \geq 0$$

Methods of Construction

Construction via Kulkarni-Nomizu Product

Given σ and μ, two symmetric rank-2 tensor fields operating on 1-forms (assumed exact) $\boldsymbol{d}f, \boldsymbol{d}k$ and $\boldsymbol{d}g, \boldsymbol{d}n$, the K-N product is

$$\begin{aligned}\sigma \owedge \mu\,(\boldsymbol{d}f, \boldsymbol{d}k, \boldsymbol{d}g, \boldsymbol{d}n) &= \sigma(\boldsymbol{d}f, \boldsymbol{d}g)\,\mu(\boldsymbol{d}k, \boldsymbol{d}n) - \sigma(\boldsymbol{d}f, \boldsymbol{d}n)\,\mu(\boldsymbol{d}k, \boldsymbol{d}g) \\ &+ \mu(\boldsymbol{d}f, \boldsymbol{d}g)\,\sigma(\boldsymbol{d}k, \boldsymbol{d}n) - \mu(\boldsymbol{d}f, \boldsymbol{d}n)\,\sigma(\boldsymbol{d}k, \boldsymbol{d}g)\,.\end{aligned}$$

Metriplectic 4-bracket:

$$(f, k; g, n) = \sigma \owedge \mu(\boldsymbol{d}f, \boldsymbol{d}k, \boldsymbol{d}g, \boldsymbol{d}n)\,.$$

In coordinates:

$$R^{ijkl} = \sigma^{ik}\mu^{jl} - \sigma^{il}\mu^{jk} + \mu^{ik}\sigma^{jl} - \mu^{il}\sigma^{jk}\,.$$

If σ or μ defines inner product, then minimally degenerate, one fixed point on $H=$ constant.

Infinite dimensions: $\mu \to M\,, \quad \sigma \to \Sigma$.

Lie Algebra Based Metriplectic 4-Brackets

- For structure constants ${c^{kl}}_s$:

$$(f, k; g, n) = {c^{ij}}_r {c^{kl}}_s \, g^{rs} \frac{\partial f}{\partial z^i} \frac{\partial k}{\partial z^j} \frac{\partial g}{\partial z^k} \frac{\partial n}{\partial z^l} \, .$$

Lacks cyclic symmetry, but ∃ procedure to remove torsion (Bianchi identity) for any symmetric 'metric' g^{rs}. Dynamics does not see torsion, but manifold does.

- For $g^{rs}_{CK} = {c^{rl}}_k \, {c^{sk}}_l$ the Cartan-Killing metric, torsion vanishes automatically. Completely determined by Lie algebra. For $\mathfrak{so}(3)$ reproduces relaxing free rigid body (pjm 1986).

- Covariant connection $\nabla\colon \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$. A contravariant connection $D\colon \Lambda^1(\mathcal{Z}) \times \Lambda^1(\mathcal{Z}) \to \Lambda^1(\mathcal{Z})$ satisfying Koszul identities, but Leibniz becomes $D_\alpha(f\gamma) = f D_\alpha \gamma + J(\alpha)[f]\gamma$ where $J(\alpha)[f] = \alpha_i J^{ij} \partial f/\partial z^j$ is a 0-form that replaces the term $\mathbf{X}(f)$ (Fernandes, 2000). Here $\alpha, \beta, \gamma \in \Lambda^1(\mathcal{Z})$, $f \in \Lambda^0(\mathcal{Z})$. Build 4-bracket like curvature from connection ⇒?

III. Unified Thermodynamic (UT) Algorithm

UT Algorithm is an algorithm for constructing metriplectic systems! Akin to building Lagrangians. Applied to many systems. **So far UT Algorithm either reproduces, corrects, or extends for every case considered!**

Examples: Cahn-Hilliard-Navier-Stokes, Brenner-Navier-Stokes, Generalized Brenner-Navier-Stokes, generalization of Landau collision operator, special phase space collision operators, ...

Four Steps of the UT Algorithm: General fluid example

1. Identify dynamical variables defined on $\Omega \subset \mathbb{R}^3$; e.g. for NSF

$$\boldsymbol{\xi} = (\boldsymbol{m} = \rho \boldsymbol{v}, \rho, \sigma = \rho s)$$

2. Propose energy and entropy functionals, $H[\boldsymbol{\xi}]$ and $S[\boldsymbol{\xi}]$; for NSF

$$H = \int_\Omega \frac{|\mathbf{m}|^2}{2\rho} + \rho u(\rho, \sigma/\rho) \quad \text{and} \quad S = \int_\Omega \sigma$$

3. Find Poisson bracket $\{F, G\}$ for which entropy S is a Casimir invariant, $\{F, S\} = 0\, \forall\, F$

4. Construct metriplectic 4-bracket $(F, K; G, N)$ via Kulkarni-Nomizu product. How? **We now have new method that separates local thermodynamics from phenomenological quantities**, giving the EoMs as Poisson bracket + 4-bracket:

$$\partial_t \boldsymbol{\xi} = \{\boldsymbol{\xi}, H\} + (\boldsymbol{\xi}, H; S, H)$$

Result automatically Thermodynamically consistent!

3. For NSF Ideal Fluid Poisson Bracket Dynamics

Hamiltonian:

$$H = \int_\Omega \frac{\rho|\mathbf{v}|^2}{2} + \rho u\,(\rho, s)\;, \qquad T = \frac{\partial u}{\partial s}\,, \qquad p = \rho^2 \frac{\partial u}{\partial \rho}\,.$$

Lie-Poisson Bracket (pjm-Greene, 1980):

$$\{F, G\} = -\int_\Omega \mathbf{m}\cdot[F_{\boldsymbol{m}}\cdot\nabla G_{\boldsymbol{m}} - G_{\boldsymbol{m}}\cdot\nabla F_{\boldsymbol{m}}] + \rho\,[F_{\boldsymbol{m}}\cdot\nabla G_\rho - G_{\boldsymbol{m}}\cdot\nabla F_\rho] \\ + \sigma\,[F_{\boldsymbol{m}}\cdot\nabla G_\sigma - G_{\boldsymbol{m}}\cdot\nabla F_\sigma]\,.$$

Equations of Motion:

$$\partial_t \mathbf{v} = \{\boldsymbol{v}, H\} = -\boldsymbol{v}\cdot\nabla\boldsymbol{v} - \nabla p/\rho\,, \quad \partial_t\rho = \{\rho, H\} = -\nabla\cdot(\rho\,\boldsymbol{v})\,, \quad \partial_t\sigma = \{\sigma, H\} = -\nabla\cdot(\sigma\,\boldsymbol{v})\,.$$

Casimir:

$$S = \int_\Omega \rho s = \int_\Omega \sigma\,.$$

Note: $F_{\boldsymbol{m}} = \delta F/\delta \boldsymbol{m}$, etc., functional derivatives. Have set thermodynamical variables.

4. Metriplectic 4-Bracket

Old method (early 2024): guess the K-N quantities M and Σ.

New Method

Theorem: Order dynamical variables st

$$\partial_t \xi^\alpha = \{\xi^\alpha, H\} + \nabla \cdot \boldsymbol{J}^\alpha\,, \qquad \alpha = 1, \ldots, N-1\,,$$
$$\partial_t \xi^N = \{\xi^N, H\} + \nabla \cdot \boldsymbol{J}^N + \boldsymbol{Z}_\alpha \cdot \tilde{L}^{\alpha\beta} \cdot \boldsymbol{Z}_\beta\,.$$

where $\xi^N = \sigma$, the entropy density. (Above splits Hamiltonian and conservative).

Then

$$\dot{S} = \int_\Omega \boldsymbol{Z}_\alpha \cdot \tilde{L}^{\alpha\beta} \cdot \boldsymbol{Z}_\beta =: \int_\Omega \dot{\sigma}^{prod} \geq 0\,.$$

and $\dot{H} = 0$ if

$$\boldsymbol{Z}_\alpha = \nabla H_{\xi^\alpha}\,, \qquad \boldsymbol{J}^\alpha = -H_{\xi^N} \tilde{L}^{\alpha\beta} \nabla H_{\xi^\beta} = -L^{\alpha\beta} \nabla H_{\xi^\beta}\,. \qquad \Box$$

This leads naturally to

$$M(dF, dG) = F_{\xi^N} G_{\xi^N}\,, \quad \Sigma(dF, dG) = \nabla(F_{\xi^\alpha}) \frac{L^{\alpha\beta}}{H_{\xi^N}} \nabla(G_{\xi^\beta})\,.$$

An Aside: Whence Phenomenological Relations?

Local thermodynamic equilibrium $\Rightarrow$

$$\text{internal energy} = U(\rho, s) = c\, e^{\lambda s} \rho^{\gamma} \quad st \quad \partial U / \partial s = T \quad \text{and} \quad p = \rho^2 \partial U / \partial \rho$$

Phenomenological (Onsager) coefficients $L^{\alpha\beta}$?

Calculation (difficult), experiment, ... Green-Kubo statistics ...

In practice they come from different places.

Important By-Product of UT Algorithm

- Special **ordering of dynamical variables** and concomitant 'Force-Flux' relations of nonequilibrium thermodynamics:

$$\boxed{\boldsymbol{J}^\alpha = L^{\alpha\beta} X_\beta \quad \rightarrow \quad \boldsymbol{J}^\alpha = -L^{\alpha\beta}\, \nabla(\delta H/\delta \xi^\beta)}$$

'Forces': $\boldsymbol{X} \sim \nabla T, \nabla p, \nabla \boldsymbol{v}$ etc., UT Algorithm removes ambiguous selection of forces and provides definition of phenomenological coefficients, $L^{\alpha\beta}$, for dynamical variables ξ^β.

- Separates dependences on thermodynamical variables that come from internal energy U (local thermodynamic equilibrium) from those that appear in the phenomenological coefficients $L^{\alpha\beta}$. For example in in the Fourier heat law entropy production expression

$$\dot{\sigma}^{prod} = \nabla T \cdot \frac{\bar{\kappa}}{T^2} \cdot \nabla T$$

one T comes from Fourier's law $q = -\bar{\kappa} \nabla T / T$ while the other comes from the phenomenological coefficient.

- Physically identify the sectional curvature

$$\dot{S} = (S, H; S, H) = K(H, S) = \int_\Omega \Sigma(dH, dH) = \int_\Omega \nabla H_{\xi^\alpha} \cdot \tilde{L}^{\alpha\beta} \cdot \nabla H_{\xi^\beta} \geq 0\,.$$

4. Metriplectic 4-Bracket: General and NSF

General flux expressions:

$$\boldsymbol{J}_\rho = -L^{\rho\rho}\cdot\nabla H_\rho - L^{\rho \boldsymbol{m}}:\nabla H_{\boldsymbol{m}} - L^{\rho\sigma}\cdot\nabla H_\sigma\,,$$
$$\bar{J}_{\mathbf{m}} = -L^{\boldsymbol{m}\rho}\otimes\nabla H_\rho - L^{\boldsymbol{mm}}:\nabla H_{\boldsymbol{m}} - L^{\boldsymbol{m}\sigma}\otimes\nabla H_\sigma\,,$$
$$\boldsymbol{J}_\sigma = -L^{\sigma\rho}\cdot\nabla H_\rho - L^{\sigma \boldsymbol{m}}:\nabla H_{\boldsymbol{m}} - L^{\sigma\sigma}\cdot\nabla H_\sigma\,,$$

where $\boldsymbol{J}_\rho$ is mass flux, $\bar{J}_{\mathbf{m}}$ is momentum flux 2-tensor, and $\boldsymbol{J}_\sigma$ is entropy flux.

For **NSF** all zero except:

$$L^{\boldsymbol{mm}} = \bar{\bar{\Lambda}} \quad \text{and} \quad L^{\sigma\sigma} = \frac{\bar{\kappa}}{T}\,.$$

$\bar{\bar{\Lambda}}$ isotropic 4-tensor, $\bar{\kappa}$ conduction 2-tensor

$$\dot{S} = (S,H;S,H) = \int_\Omega \Sigma(dH,dH) = \int_\Omega \nabla\mathbf{v} : \frac{\bar{\bar{\Lambda}}}{T} : \nabla\mathbf{v} + \nabla T\cdot\frac{\bar{\kappa}}{T^2}\cdot\nabla T \geq 0\,.$$

Note in $\bar{\kappa}/T^2$ one T from H one from $L^{\alpha\beta}$. Σ sectional curvature density?

4. Metriplectic 4-Bracket for NSF Generalizations

For **Brenner NSF** all zero except:

$$L^{m\rho} = \tilde{D}\rho\, \boldsymbol{m}\,, \quad L^{m\sigma} = \tilde{D}\hat{\sigma}\, \boldsymbol{m}\,, \quad L^{mm} = \bar{\bar{\Lambda}} + \tilde{D}\, \boldsymbol{m} \otimes \bar{I} \otimes \boldsymbol{m}\,.$$

$$L^{\sigma\rho} = \tilde{D}\rho\hat{\sigma}\, \bar{I}\,, \quad L^{\sigma\sigma} = \frac{\bar{\kappa}}{T} + \tilde{D}\hat{\sigma}^2\, \bar{I} \quad L^{\sigma m} = \tilde{D}\hat{\sigma}\, \bar{I} \otimes \boldsymbol{m}$$

$$\dot{S} = \int_\Omega \frac{1}{T}\left[\frac{\tilde{D}}{\kappa_T^2 \rho^2}|\nabla\rho|^2 + \nabla T \cdot \frac{\bar{\kappa}}{T} \cdot \nabla T \; + \nabla \boldsymbol{v} : \bar{\bar{\Lambda}} : \nabla \boldsymbol{v}\right] \geq 0\,.$$

Generalization of Brenner by **Reddy et al.** (2019) falls out. We further generalized.

Contact vs. Metriplectic Structure

A Finite-Dimensional Example

pjm and Y-G. Oh, “Metriplectic dynamical systems on contact manifolds,” (2026)

Contact Structures Hamiltonian Dynamics

Contact Manifold:
$(\mathcal{Z}, \lambda \in \Lambda^1)$ where $\mathcal{Z}$ dimension $2n+1$; vectors in $\ker\lambda$ maximally nonintegrable hyperplane field.

Darboux Coordinates:
$\lambda = dz - pdq \quad \rightarrow \quad d\lambda = -dp \wedge dq$

Reeb Vector Field:
$\lambda(R) = 1 \qquad \text{and} \qquad d\lambda(R, \cdot\,) = 0$

Reeb vector field is analog of Hamiltonian vector field.

Canonical Contact Hamiltonian System:

$$\dot{q} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial q} - p\frac{\partial H}{\partial z}, \qquad \dot{z} = -H + p \cdot \frac{\partial H}{\partial p}$$

where $H(q, p, z)$

Contact Hamiltonian Dynamics Properties

Energy Dissipation:

$$\dot{H} = -H\frac{\partial H}{\partial z}$$

does not vanish unless $H = 0$ or $\partial H/\partial z = 0$.

Is Entropy z?:

$$\dot{z} = L$$

In general has no definite sign, i.e., not a Lyapunov function, but could be?

Metriplectic Construction

Trivial Poisson Manifold (one jet bundle):

$$\mathcal{Z} = T^*\mathbb{R}^n \times \mathbb{R} \qquad \leftarrow \qquad \text{stack of symplectic hyperplanes}$$

Darboux Coordinates:

$$J = \begin{bmatrix} 0_n & I_n & 0 \\ -I_n & 0_n & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Poisson bracket:

$$\{f, g\} = J(df, dg) \quad \text{for} \quad f, g \in \Lambda^0(\mathcal{Z})$$

Casimirs:

$$C = C(z) \qquad \leftarrow z \qquad \text{could be entropy}$$

Dynamics:

$$\begin{bmatrix} \dot{q} \\ \dot{p} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 0_n & I_n & 0 \\ -I_n & 0_n & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \partial H/\partial q \\ \partial H/\partial p \\ \partial H/\partial z \end{bmatrix}.$$

Metriplectic 4-Bracket

Kulkarni-Nomizu product:

$$\mu(df,dg) = \frac{\partial f}{\partial z}\frac{\partial g}{\partial z} \qquad \text{and} \qquad \sigma(df,dg) = \frac{\partial f}{\partial p}\cdot\frac{\partial g}{\partial p}$$ $\leftarrow$ fiber derivatives for projection $T^*N \to N$

Metriplectic 4-Bracket:

$$(f,k;g,n) = \frac{\partial f}{\partial p}\cdot\frac{\partial g}{\partial p}\frac{\partial k}{\partial z}\frac{\partial n}{\partial z} - \frac{\partial f}{\partial p}\cdot\frac{\partial n}{\partial p}\frac{\partial k}{\partial z}\frac{\partial g}{\partial z} + \frac{\partial k}{\partial p}\cdot\frac{\partial n}{\partial p}\frac{\partial f}{\partial z}\frac{\partial g}{\partial z} - \frac{\partial k}{\partial p}\cdot\frac{\partial g}{\partial p}\frac{\partial f}{\partial z}\frac{\partial n}{\partial z}$$

Dynamics:

$$\dot{q} = \{q,H\} + (q,H;S,H) = \frac{\partial H}{\partial p}$$

$$\dot{p} = \{q,H\} + (p,H;S,H) = -\frac{\partial H}{\partial q} - \frac{\partial H}{\partial p}\frac{\partial H}{\partial z}$$

$$\dot{z} = (z,H;S,H) = \left|\frac{\partial H}{\partial p}\right|^2 \geq 0$$

By construction $\dot{H} = 0$ and thermodynamically consistent with entropy z.

Comparison with Contact Hamiltonian

$$\dot{q}^i = \{q^i, H\} + (q^i, H; S, H) = \frac{\partial H}{\partial p_i} = \frac{\partial H}{\partial p_i}$$

$$\dot{p}_i = \{p_i, H\} + (p_i, H; S, H) = -\frac{\partial H}{\partial q^i} - \frac{\partial H}{\partial p^i}\frac{\partial H}{\partial z} \neq -\frac{\partial H}{\partial q^i} - p_i\frac{\partial H}{\partial z}$$

$$\dot{z} = (z, H; S, H) = \left|\frac{\partial H}{\partial p}\right|^2 \neq -H + p \cdot \frac{\partial H}{\partial p}\,.$$

Not the same. **Unless**, H Euler homogeneous of degree 2, e.g., geodesic

$$H = g^{ij} p_i p_j / 2 = p_i p^i / 2$$

Duffing Oscillator – Thermodynamic Completion

Usual Equation of Motion:

$$\ddot{q} + \delta\dot{q} + \alpha q + \beta q^3 = \gamma \sin(\omega t + \phi),$$

Hamiltonian:

$$H = \frac{p^2}{2} + \frac{\alpha q^2}{2} + \frac{\beta q^4}{4} - \gamma q \sin(\omega t + \phi) + \delta z$$

Entropy:

$$S = z$$

Contact Hamiltonian Duffing Oscillator

Contact Hamiltonian system:

$$\dot{q} = p$$
$$\dot{p} = -\alpha q - \beta q^3 + \gamma \sin(\omega t + \phi) - \delta p$$
$$\dot{z} = \frac{p^2}{2} - \delta z - \frac{\alpha q^2}{2} - \frac{\beta q^4}{4} + \gamma q \sin(\omega t + \phi)$$

Energy non-Conservation:

$$\dot{H} = -H\frac{\partial H}{\partial z} + \frac{\partial H}{\partial t} = -\delta H - \gamma \omega q \cos(\omega t + \phi)$$

Autonomous case:

$$H = H_0\, e^{-\delta t}$$

Metriplectic Duffing Oscillator

Metriplectic system:

$$\dot{q} = p$$
$$\dot{p} = -\alpha q - \beta q^3 + \gamma \sin(\omega t + \phi) - \delta\, p$$
$$\dot{z} = p^2 \geq 0$$

Energy non-Conservation:

$$\dot{H} = \frac{\partial H}{\partial t} = -\gamma \omega q \cos(\omega t + \phi)$$

Autonomous case:

$$\dot{H} = 0$$

V. Final Comments

• Metriplectic dynamics is rich in geometry and produces interesting dynamical systems. Tons of interesting geometry already ... more to explore.

• The UT Algorithm based on the metriplectic 4-bracket, is a proven framework, provides a direct method for constructing thermodynamically consistent systems. Points clearly to the origin of thermodynamic dependencies.

• Metriplectic 4-brackets are easy to discretize while maintaining symmetries. First numerical implementation via 4-bracket discretization (Barham et al. 2025) for 1-D Navier-Stokes-Fourier. Finite element projection of PDE to thermodynamically consistent finite-dimensional 4-bracket, i.e., ODEs. For example, for the density $\rho(x,t)$

$$\rho_h(x,t) = \sum_{i=1}^{N} \rho_i(t)\phi_i(x) \quad \rightarrow \quad \dot{\rho}_i(t) = \{\rho_i, H\} + (\rho_i, H; S, H) \ldots$$

Results use Firedrake library, implicit midpoint, Irksome module ...

Papers I

Old:

• A. N. Kaufman and P. J. Morrison, "Algebraic Structure of the Plasma Quasilinear Equations," Physics Letters A **88**, 405–406 (1982).

• P. J. Morrison, "Bracket Formulation for Irreversible Classical Fields," Phys. Lett. A **100**, 423–427 (1984).

• P. J. Morrison, "Some Observations Regarding Brackets and Dissipation," CPAM report (1984). Available as arXiv:2403.14698v1 [mathph] 15 Mar 2024.

• P. J. Morrison, "A Paradigm for Joined Hamiltonian and Dissipative Systems," Physica D **18**, 410 (1986).

• P. J. Morrison, "Thoughts on Brackets and Dissipation: Old and New," Journal of Physics: Conference Series **169**, 012006 (12pp) (2009).

Papers II

New:

- P. J. Morrison and Y-G. Oh, "Metriplectic dynamical systems on contact manifolds," arXiv:2605.09482v1 [math.SG]
- C. Bressan, M. Kraus, O. Maj, and P. J. Morrison, "Metriplectic Relaxation to Equilibria," Commun. Nonlinear Sci. Numer. Simulat. 161 (2026) 110076 (59pp). Invited paper.
- N. Sato and P. J. Morrison, "Scattering Theory in Noncanonical Phase Space: A Drift-Kinetic Collision Operator for Weakly Collisional Plasmas," Physics of Plasmas **32**, 102306 (24pp) (2025).
- A. Zaidni and P. J. Morrison, "Metriplectic 4-Bracket Algorithm for Constructing Thermodynamically Consistent Dynamical Systems," Physical Review E **112**, 025101 (14pp) (2025).
- W. Barham, P. J. Morrison, and A. Zaidni, "A Thermodynamically Consistent Discretization of 1D Thermal-Fluid Models Using their Metriplectic 4-Bracket Structure, Communications in Nonlinear Science and Numerical Simulations **145**, 108683 (9pp)(2025).
- A. Zaidni, P. J. Morrison, and S. Benjelloun, "Thermodynamically Consistent Cahn-Hilliard-Navier-Stokes Equations Using the Metriplectic Dynamics Formalism," Physica D **468**, 134303 (11pp) (2024).
- N. Sato and P. J. Morrison, "A Collision Operator for Describing Dissipation in Noncanonical Phase Space,"Fundamental Plasma Physics **10**, 100054 (18pp) (2024).
- P. J. Morrison and M. Updike, "Inclusive Curvature-Like Framework for Describing Dissipation: Metriplectic 4-Bracket Dynamics," Physical Review E **109**, 045202 (22pp) (2024).
- B. Coquinot and P. J. Morrison, "A General Metriplectic Framework with Application to Dissipative Extended Magnetohydrodynamics,"Journal of Plasma Physics **86**, 835860302 (32pp) (2020).