On Metriplectic relaxation to equilibrium

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Based on the paper:

C. Bressan[a], M. Kraus[a], O. Maj[a], and pjm[b], *Metriplectic relaxation to equilibria*, Commun Nonlinear Sci Numer Simulat **161**, 110076 (2026)(59pp).

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Overview

Part 1: Review of noncanonical Hamiltonian and metriplectic dynamics

Part 2: Metriplectic relaxation as a computational tool: Examples

Part 3: Final comments

Motivation

Metriplectic* dynamics dynamically solves the equilibrium variational principle by relaxation

$$min\{S(u)\colon u \in V, H(u) = H(u_0)\}.$$

S is the entropy/Lyapunov function, H is the Hamiltonian/energy, u dynamical variables.

Here we construct fake dynamics to solve for equilibria.

(* part degenerate grad flow METRI and part Hamiltonian PLECTIC)

Part 1: Hamiltonian and Metriplectic Review

Hamilton's Canonical Equations

Phase Space with Canonical Coordinates: (q, p)

Hamiltonian function: $H(q, p)$ ← the energy

Equations of Motion:

$$\boxed{\dot{p}_i = -\frac{\partial H}{\partial q^i} \quad \text{and} \quad \dot{q}^i = \frac{\partial H}{\partial p_i}} \qquad i = 1, 2, \ldots N$$

Phase Space Coordinate Rewrite: $z = (q, p)\,, \quad \alpha, \beta = 1, 2, \ldots 2N$

$$\dot{z}^\alpha = J_c^{\alpha\beta} \frac{\partial H}{\partial z^\beta} = \{z^\alpha, H\}_c \quad \text{where} \quad (J_c^{\alpha\beta}) = \begin{pmatrix} 0_N & I_N \\ -I_N & 0_N \end{pmatrix}$$

$J_c :=$ <u>Poisson tensor</u>, Hamiltonian bi-vector, cosymplectic form

Noncanonical Hamiltonian Structure

Sophus Lie (1890) $\longrightarrow$ PJM (noncanonical 1980) $\longrightarrow$ Weinstein (1982) Poisson Manifolds

Noncanonical Coordinates:

$$\dot{z}^\alpha = \{z^\alpha, H\} = J^{\alpha\beta}(z)\frac{\partial H}{\partial z^\beta}$$

Noncanonical Poisson Bracket:

$$\{f,g\} = \frac{\partial f}{\partial z^\alpha} J^{\alpha\beta}(z) \frac{\partial g}{\partial z^\beta}$$

Bilinear Poisson Bracket Properties:

antisymmetry $\rightarrow$ $\{f,g\} = -\{g,f\}$ $\rightarrow$ $J^{\alpha\beta} = -J^{\beta\alpha}$

Jacobi identity $\rightarrow$ $\{f,\{g,h\}\}+\{b,\{h,f\}\}+\{h,\{f,g\}\} = 0$ $\rightarrow$ Jacobiator $S^{\alpha\beta\gamma} = J^{\alpha\ell}\partial_\ell J^{\beta\gamma} + cyc \equiv 0$

Leibniz $\rightarrow$ $\{fh,g\} = f\{h,g\} + \{h,g\}f\,,$ fg pointwise

G. Darboux: $detJ \neq 0 \Longrightarrow J \rightarrow J_c$ Canonical Coordinates

Sophus Lie: $detJ = 0 \Longrightarrow$ Canonical Coordinates plus Casimirs $\leftarrow$ G. Sudarshan
(Lie's distinguished functions!)

Noncanonical Poisson Brackets – Flows on Poisson Manifolds

Definition. A Poisson manifold $\mathcal{Z}$ has bracket

$$\{\,,\,\}\colon C^\infty(\mathcal{Z}) \times C^\infty(\mathcal{Z}) \to C^\infty(\mathcal{Z})$$

st $C^\infty(\mathcal{Z})$ with $\{\,,\,\}$ is a Lie algebra realization, i.e., is

- $\mathbb{R}$-bilinear,
- antisymmetric,
- Jacobi identity
- Leibniz, i.e., acts as a derivation ⇒ vector field.

Geometrically $C^\infty(\mathcal{Z}) \equiv \Lambda^0(\mathcal{Z})$ and $\boldsymbol{d}$ exterior derivative.

$$\{f,g\} = \langle \boldsymbol{d}f, J\boldsymbol{d}g\rangle = J(\boldsymbol{d}f,\boldsymbol{d}g) = \frac{\partial f}{\partial z^\alpha} J^{\alpha\beta} \frac{\partial g}{\partial z^\beta}\,.$$

J the Poisson tensor/operator. Flows are integral curves of noncanonical Hamiltonian vector fields, $J\boldsymbol{d}H$, i.e.,

$$\dot z^\alpha = J^{\alpha\beta}(z)\frac{\partial H(z)}{\partial z^\beta}\,, \qquad \mathcal{Z}'s \text{ coordinate patch } z = (z^1,\dots,z^N)$$

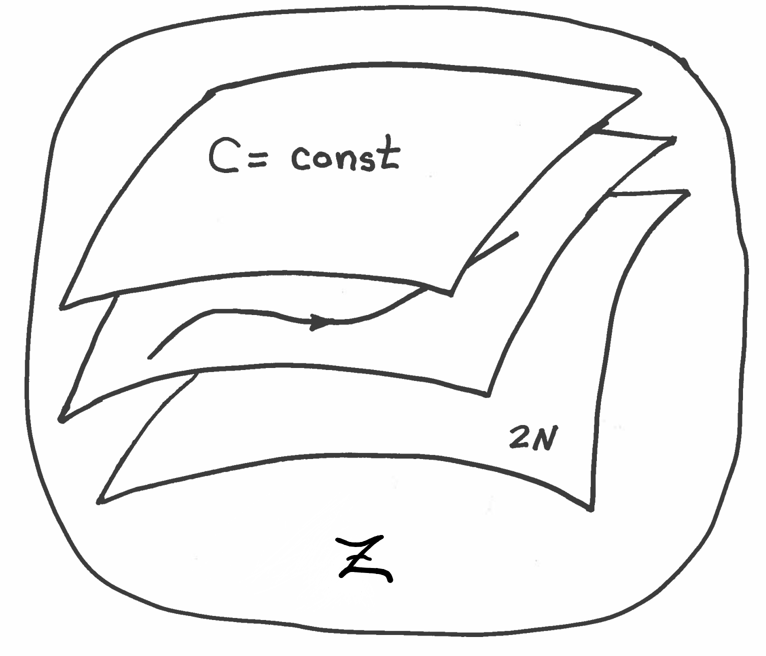
Because of degeneracy, ∃ functions C st $\{f,C\} = 0$ for all $f \in C^\infty(\mathcal{Z})$, called Casimir invariants. **Casimirs are candidate entropies!**

Poisson Manifold (phase space) $\mathcal{Z}$ Cartoon

Degeneracy in J $\Rightarrow$ Casimirs:

$$\{f, C\} = 0 \quad \forall\ f : \mathcal{Z} \to \mathbb{R}$$

Lie-Darboux Foliation by Casimir (symplectic) leaves:



Introduction: Hamiltonian systems - Poisson manifolds cont.

- Invariants of Hamiltonian systems:

$$\{F,G\} = -\{G,F\} \implies \dot{H} = \{H,H\} = 0, \quad \text{(energy)},$$
$$\{F,C\} = 0 \text{ for any } F \in C^\infty(\mathcal{Z}) \implies \dot{C} = \{C,H\} = 0, \quad \text{(Casimir invariants)}.$$

- An example: The Vlasov equation,

$$\mathcal{Z} = \text{ space of distributions } f(x,v), \qquad H(f) = \int f\Big(\frac{1}{2}v^2 + \frac{q}{m}\phi(x)\Big)dxdv,$$
$$\{F,G\} = \int f\Big[\frac{\delta F(f)}{\delta f}, \frac{\delta G(f)}{\delta f}\Big]dxdv, \qquad DF(f)g = \int \frac{\delta F(f)}{\delta f}\, g\, dxdv,$$
$$\text{equations of motion:} \quad \dot{f} = -v\cdot\nabla_x f - \frac{q}{m}\Big[-\nabla\phi(x) + \frac{v\times B_0(x)}{c}\Big]\cdot\nabla_v f,$$
$$\text{Casimir invariants:} \quad C(f) = \int \varphi(f)dxdv, \quad \varphi \in C^\infty(\mathbb{R},\mathbb{R}).$$

- Other notable examples: Maxwell eqs., Vlasov-Maxwell system, compressible Euler eqs., reduced Euler eqs., ideal magnetohydrodynamics (MHD), reduced MHD, etc...

Introduction: Accounting for dissipation

Definition

Metriplectic[5] system $(\mathcal{Z}, \{\cdot,\cdot\}, (\cdot,\cdot), H, S)$:

- $(\mathcal{Z}, \{\cdot,\cdot\}, H)$ is a Hamiltonian system on the Poisson manifold $(\mathcal{Z}, \{\cdot,\cdot\})$.
- The **metric bracket** $(\cdot,\cdot) : C^\infty(\mathcal{Z}) \times C^\infty(\mathcal{Z}) \to C^\infty(\mathcal{Z})$ is bilinear, **symmetric**, and

$$(F, F) \geq 0, \text{ for any } F \in C^\infty(\mathcal{Z}).$$

- The Hamiltonian $H \in C^\infty(\mathcal{Z})$ and entropy $S \in C^\infty(\mathcal{Z})$ satisfy the compatibility conditions

$$\{F, S\} = 0, \quad (F, H) = 0, \quad \text{for any } F \in C^\infty(\mathcal{Z}).$$

- The equations of motion are generated by

$$\dot{F} = \{F, H\} - (F, S), \quad \text{for any } F \in C^\infty(\mathcal{Z}).$$

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[5] Morrison, Center for Pure and Applied Mathematics Report, UC Berkeley (1984); Physica D 18 (1986) 410–419.

Introduction: Accounting for dissipation cont.

- Thermodynamic consistency of metriplectic systems:

$$(H,F) = 0,\ \{S,F\} = 0,\ (F,F) \geq 0 \text{ for any } F \in C^1(\mathcal{Z}) \implies$$

$$\dot{H} = \{H,H\} - (H,S) = 0, \qquad \text{(energy conservation)},$$

$$\dot{S} = \{S,H\} - (S,S) = -(S,S) \leq 0, \qquad \text{(monotonic entropy)}.$$

- An example: Boltzmann equation with Coulomb collisions,

$$\big(\mathcal{Z}, \{\cdot,\cdot\}, H\big) \text{ as for the Vlasov eq.,} \quad S(f) = \int f \log f \, dxdv,$$

$$(F,G) = \frac{k}{2} \int \Big(\nabla_v \frac{\delta F}{\delta f} - \nabla_v \frac{\delta F}{\delta f'}\Big) \cdot U_L(v - v')\Big(\nabla_v \frac{\delta F}{\delta f} - \nabla_v \frac{\delta F}{\delta f'}\Big) ff' \, dxdvdv',$$

$$k = \text{constant}, \quad U_L(u) = \frac{1}{|u|}\Big[I - \frac{u \otimes u}{|u|^2}\Big],$$

$$\text{equation of motion:} \quad \dot{f} = -v \cdot \nabla_x f - \frac{q}{m}\Big[-\nabla\phi(x) + \frac{v \times B_0(x)}{c}\Big] \cdot \nabla_v f + \underbrace{C_L(f)}_{\text{Landau op.}} .$$

Part 2: Metriplectic Relaxation as a tool: Examples

Part 2: Metriplectic relaxation

- If $z(t)$, $t \geq 0$, is a solution of a metriplectic system with $z(0) = z_0$, then

$$\begin{aligned} H_0 &:= H(z_0) = H\big(z(t)\big), && \text{for all } t \geq 0, \\ S_0 &:= S(z_0) \geq S\big(z(t_1)\big) \geq S\big(z(t_2)\big), && \text{for all} \quad 0 \leq t_1 \leq t_2. \end{aligned}$$

- If invariants $I^1(z), \dots, I^{k-1}(z)$ exist, then $I(z) = \big(I^1(z), \dots, I^{k-1}(z), I^k(z) = H(z)\big)$ and

$$I_0 := I(z_0) = I\big(z(t)\big), \quad \text{for all } t \geq 0.$$

- It is natural to ask whether a solution
  1. exists globally in time;
  2. has a limit $z(t) \to z_*$ for $t \to +\infty$;
  3. the limit satisfies
$$S(z_*) = \inf \big\{S(z) \colon z \in \mathcal{Z},\ I(z) = I_0\big\}.$$

- Relevance:
  - Long-time limit of physical systems.
  - Use metriplectic systems as relaxation methods to solve constrained minimization problems.

Part 2: An example

- Reduced Euler equations on $\mathbb{T}^2$: $x = (x_1, x_2) \in \mathbb{T}^2$ $(\cong [0, 2\pi]^2$ with periodic bnd. cond.),

$$\begin{cases} \partial_t \omega + [\omega, \phi] = 0, \\ -\Delta\phi - \omega = 0, \end{cases} \qquad \text{where} \quad [\alpha, \beta] = \partial_{x_1}\alpha\, \partial_{x_2}\beta - \partial_{x_1}\beta\, \partial_{x_2}\alpha,$$

$\phi =$ steaming potential for the fluid velocity $U = (\partial_{x_2}\phi, -\partial_{x_1}\phi)$,
$\omega =$ scalar vorticity.

- The reduced Euler equations are Hamiltonian with

$$\text{phase space } \mathcal{Z} = \{u : \mathbb{T}^2 \to \mathbb{R}\}, \qquad \omega = u - \overline{u}, \qquad \overline{u} := \frac{1}{4\pi^2}\int_{\mathbb{T}^2} u(x)\, dx,$$

$$\{F, G\} = \int_{\mathbb{T}^2} u\Big[\frac{\delta F}{\delta u}, \frac{\delta G}{\delta u}\Big]\, dx, \qquad H(u) = \frac{1}{2}\int_{\mathbb{T}^2} |\nabla\phi|^2\, dx, \qquad -\Delta\phi = u - \overline{u}.$$

- Equilibrium points:

$$-\Delta\phi = \omega = f(\phi) \quad \Longrightarrow \quad [\omega, \phi] = 0, \quad -\Delta\phi = \omega.$$

Part 2: An example – Test brackets

- For reduced Euler equations, we consider two examples of metric brackets.
- **Metric bracket 1** – Metric double bracket (selective decay[12] [13]):

$$(F, G) := \Big(\Big[\frac{\delta F}{\delta u}, \frac{\delta H}{\delta u}\Big], \Big[\frac{\delta H}{\delta u}, \frac{\delta G}{\delta u}\Big]\Big)_{L^2(\mathbb{T}^2)}.$$

- **Metric bracket 2** – Based on an L^2-projector:

$$(F, G) := \Big(\frac{\delta F}{\delta u}, \Pi_H \frac{\delta G}{\delta u}\Big)_{L^2(\mathbb{T}^2)},$$

- Notation:

$$(v, w)_{L^2(\mathbb{T}^2)} = \int_{\mathbb{T}^2} vw \, dx, \qquad \Pi_H v := v - \Big\|\frac{\delta H}{\delta u}\Big\|_{L^2(\mathbb{T}^2)}^{-2} \frac{\delta H}{\delta u}\Big(\frac{\delta H}{\delta u}, v\Big)_{L^2(\mathbb{T}^2)}.$$

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[12] Bloch, Morrison and Ratiu, in Proceedings in Mathematics and Statistics 35 (2013) 371–415.

[13] Gay-Balmaz and Holm, Nonlinearity 26 (2013) 495; Nonlinearity 27 (2014) 1747.

Part 2: An example – Dynamics and constrained entropy minima

- Equation of motion (dissipative terms only) and “entropy”:

$$\dot{F} = -(F, S), \qquad S(u) = \frac{1}{2}\int_{\mathbb{T}^2} \omega^2 \, dx = \frac{1}{2}\|u - \overline{u}\|^2_{L^2(\mathbb{T}^2)} = \text{fluid enstrophy}.$$

- Invariants of the system:

$$I^1(u) = \int_{\mathbb{T}^2} u \, dx = 4\pi^2 \overline{u}, \qquad I^2(u) = H(u).$$

- Analytical solution for constrained entropy minima: $\min\{S(u) \colon I(u) = I_0\}$ achieved for

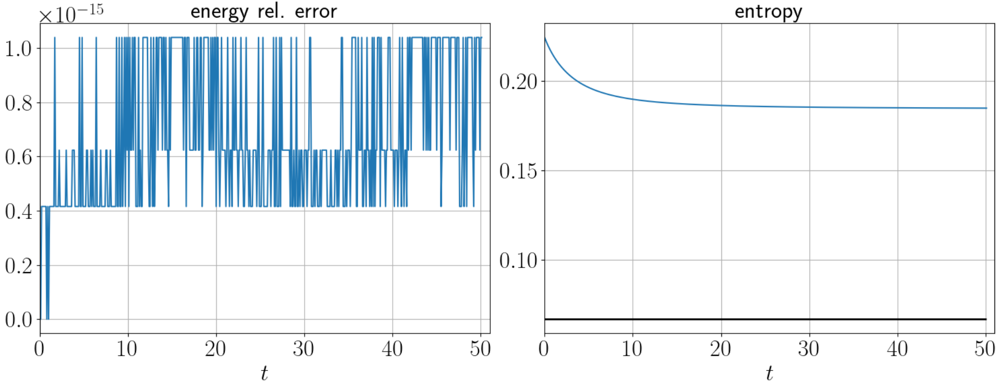
$$\omega(x) = \phi(x) = \frac{\sqrt{H_0}}{\pi}\big[\cos(\theta_0)\cos(x_1 + \theta_1) + \sin(\theta_0)\cos(x_2 + \theta_2)\big],$$

where $\theta_0, \theta_1, \theta_2 \in [0, 2\pi)$ are arbitrary phases. There is a three-parameter family of entropy minima, and the constrained entropy minimum is

$$\min\{S(u) \colon I(u) = I_0\} = H_0.$$

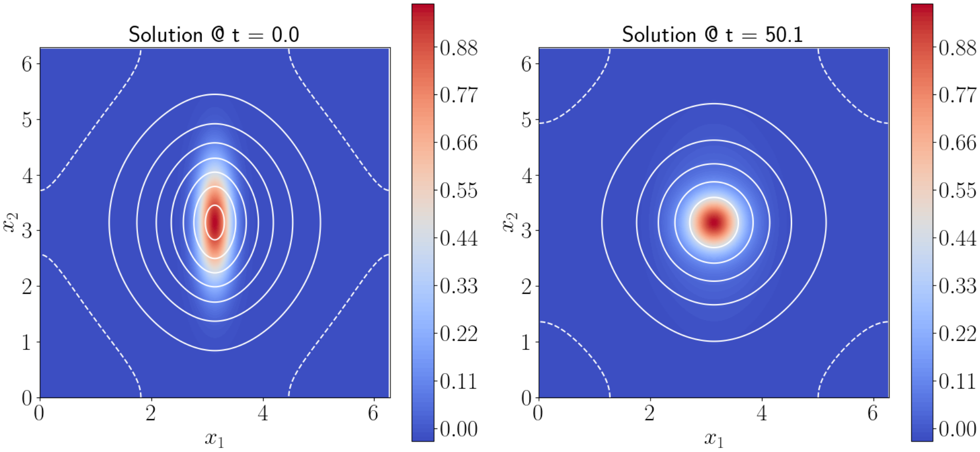
Part 2: An example – Results for Bracket 1

Conservation of the Hamiltonian and dissipation of entropy.
The value $\min\{S(u) \colon I(u) = I_0\}$ is indicated by the back horizontal line.



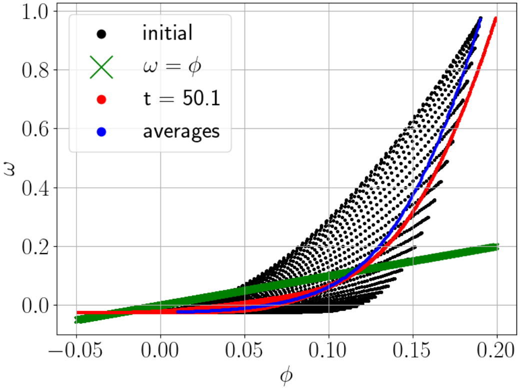
Part 2: An example – Results for Bracket 1 cont.

Initial and final state: the color map shows the vorticity ω, white contours the potential ϕ.



Part 2: An example – Results for Bracket 1 cont.

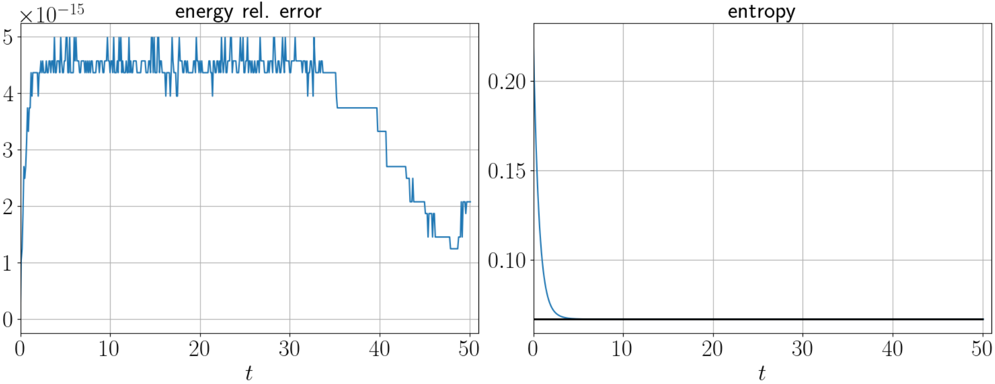
Functional relation $\omega = \phi$.



Conclusion: the system relaxes to an equilibrium (ω is a function of ϕ), but not to a constrained minimum entropy.

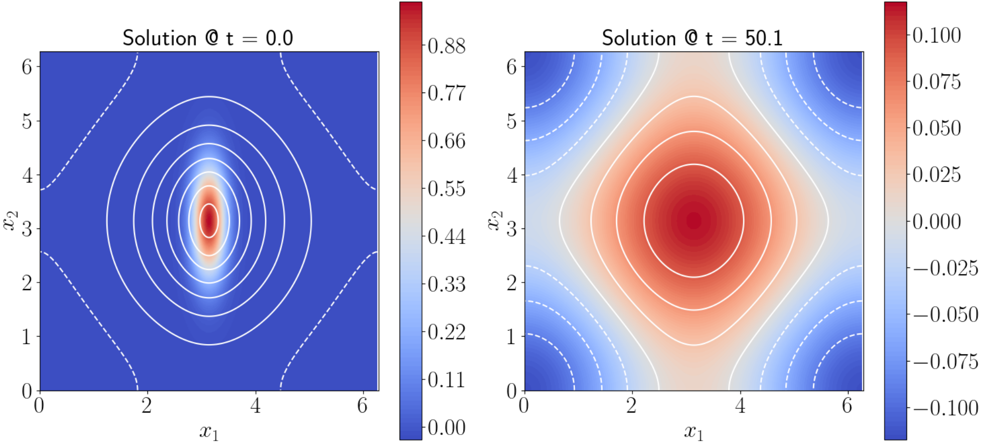
Part 2: An example – Results for Bracket 2

Conservation of the Hamiltonian and dissipation of entropy.
The value $\min\{S(u) \colon I(u) = I_0\}$ is indicated by the back horizontal line.



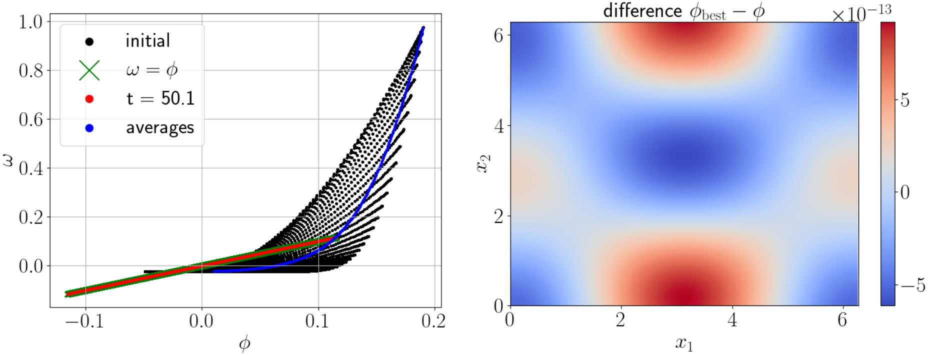
Part 2: An example – Results for Bracket 2 cont.

Initial and final state: the color map shows the vorticity ω, white contours the potential ϕ.



Part 2: An example – Results for Bracket 2 cont.

Functional relation $\omega = \phi$.
Error with the exact solution (best fit over the three phases $\theta_0, \theta_1, \theta_2$).



Conclusion: The projector-based bracket appears to find a constrained entropy minimum, but dissipates more vorticity.

Part 2: Relaxation and specifically degenerate brackets

Definition

A symmetric bracket $(\cdot,\cdot) : C^\infty(\mathcal{Z}) \times C^\infty(\mathcal{Z}) \to C^\infty(\mathcal{Z})$ is **specifically degenerate** with respect to $I \in C^\infty(\mathcal{Z}, \mathbb{R}^k)$, if

$$\big(F, F\big)(z) = 0 \quad \Longleftrightarrow \quad DF(z) = \sum_{\alpha=1}^{k} \lambda_\alpha DI^\alpha(z).$$

If $k = 1$ and $I = H$ the bracket is called **minimally degenerate**.

- For finite-dimensional systems ($\mathcal{Z} \subseteq \mathbb{R}^n$) and under technical conditions, we can prove[14]:

$$\text{specifically deg.} \quad \Longrightarrow \quad \lim_{t\to+\infty} z(t) = z_*, \quad S(z_*) = \min\{S(z) \colon I(z) = I(z_0)\}.$$

- Bracket 1 **is not specifically degenerate** with respect to the two invariants $I^1, I^2 = H$.
- Bracket 2 **is minimally degenerate**.

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[14] Bressan, Kraus, OM, Morrison, Commun. Nonlinear Sci. Numer. Simulat. 161 (2026) 110076, proposition 3.

Part 2: Magnetic field relaxation

- Problem: find a dynamical system that relaxes to a magnetohydrodynamic equilibrium,

$$(\operatorname{curl} B) \times B = \nabla p, \quad \operatorname{div} B = 0.$$

- Bracket formulation of the magnetic relaxation problem: force-free case (Beltrami fields),

$$B \in \mathcal{Z} \subseteq \{w \in H(\operatorname{curl}, \Omega) \times H(\operatorname{div}, \Omega) \colon \operatorname{div} w = 0, \quad n \cdot w = 0, \text{ on } \partial\Omega\}, \quad \Omega \subset \mathbb{R}^3,$$

$$(F, G) = \int_\Omega \left[\operatorname{curl} \frac{\delta F}{\delta B}, \operatorname{curl} \frac{\delta H}{\delta B}\right]_{\mathbb{R}^3} \cdot \left[\operatorname{curl} \frac{\delta H}{\delta B}, \operatorname{curl} \frac{\delta G}{\delta B}\right]_{\mathbb{R}^3} dx, \qquad [a, b]_{\mathbb{R}^3} = a \times b,$$

$$H(B) = \frac{1}{2} \int_\Omega B \cdot \operatorname{curl}^{-1} B \, dx = \text{mag. helicity}, \qquad S(B) = \frac{1}{2} \int_\Omega |B|^2 \, dx = \text{mag. energy}.$$

- Then $\dot{F} = -(F, S)$ gives a known relaxation method[15] [16] [17], and it is a gradient flow[18].

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[15] Chodura and Schlüter, J. Comput. Phys. 41 (1981) 68–88. Moffatt, J. Fluid Mech. 159 (1985) 359–378.

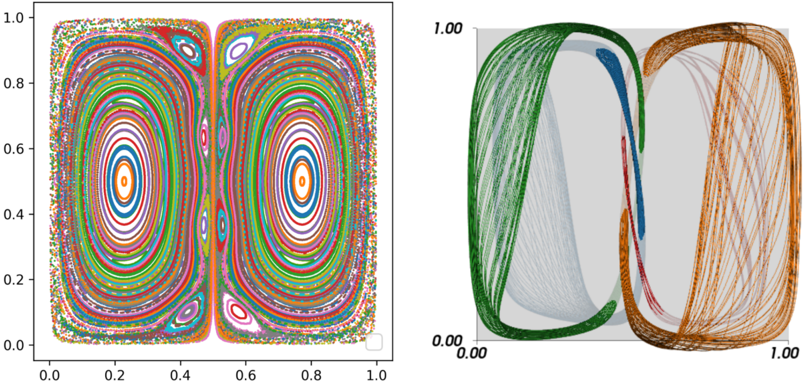
[16] Candelaresi, Pontin and Hornig, SIAM J. Sci. Comput. 36 (2014) B952–B968.

[17] Blickhan et al., arXiv:2510.26986. Beekie, Friedlander and Vicol, Comm. Math. Phys. 390 (2022) 1311.

[18] Brenier and Duan, Calculus of Variations and Partial Differential Equations 57 (2018) 125.

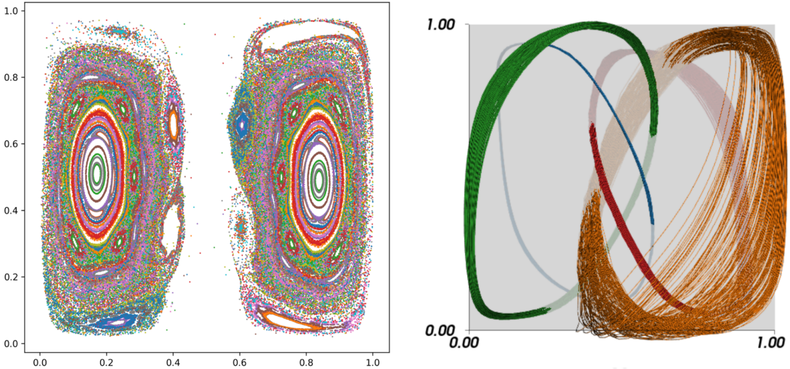
Part 2: Magnetic field relaxation – A numerical experiment

Poincaré section on $x_2 = 1/2$, and selected field lines: initial condition.



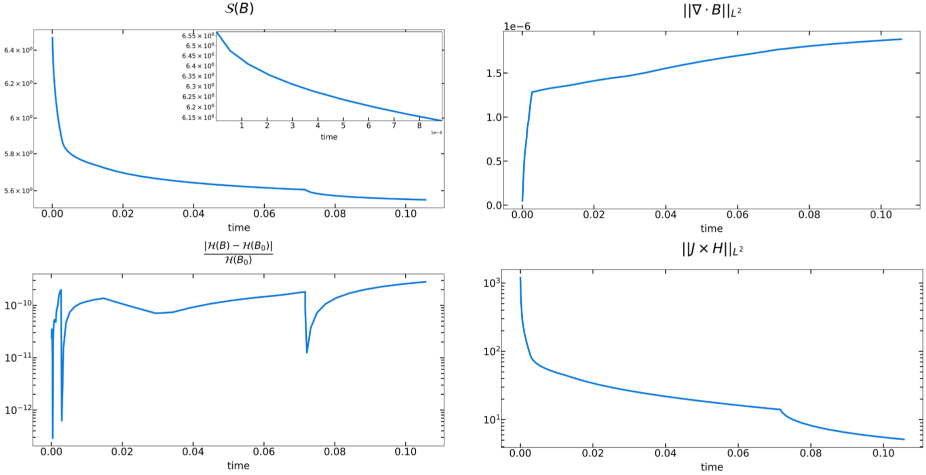
Part 2: Magnetic field relaxation – A numerical experiment

Poincaré section on $x_2 = 1/2$, and selected field lines: last point in time.



Part 2: Magnetic field relaxation – A numerical experiment

Diagnostics.



Part 3: Final comments

Metriplectic systems provide a convenient setting for both physics modeling (see my CAIMS talk on Monday) and the design of numerical schemes.

Metriplectic relaxation to equilibria:

- Importance of the null space – Specifically/Minimally degenerate brackets.
- Application to equilibrium calculations.
- How can we speed it up?