Overview of Mathematical Models for Plasma Dynamics

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Overall Overview of 3 Lectures

I. Magnetofluid models

II. Dissipation: metriplectic dynamics

III. Linear Vlasov as a Hamiltonian theory

LECTURE I

Hamiltonian structure and reductions of plasma models: N-Body $\rightarrow$ Kinetic $\rightarrow$ Magnetofluid

What is a Plasma?

Adding Heat:

Solid $\rightsquigarrow$ Liquid $\rightsquigarrow$ Gas $\rightsquigarrow$ Plasma

97% of visible universe is plasma

Dynamics of Charged Particles $\Leftrightarrow$ Electromagnetic Fields

- $\Rightarrow$ Complicated partial differential equations. Magnetofluid models with geometry

N-Body Action Principle

$$S[q,\phi,A] = -\sum_{i=1}^{N} \int_{t_0}^{t_f} dt\, m_i c^2 \sqrt{1 - |\dot{q}_i(t)|^2/c^2} \quad (1)$$

$$- \sum_{i=1}^{N} \int_{t_0}^{t_f} dt \int_{\Omega} d^3r\, e_i \left[\phi(r,t) - \dot{q}_i(t) \cdot A(r,t)\right] \delta(r - q_i(t)) \quad (2)$$

$$+ \int_{t_0}^{t_f} dt \int_{\Omega} d^3r \left[\frac{\epsilon_0}{2} |E(r,t)|^2 - \frac{1}{2\mu_0} |B(r,t)|^2\right] \quad (3)$$

where

$$E = -\nabla\phi - \frac{\partial A}{\partial t} \quad \text{and} \quad B = \nabla \times A$$

$i \in \mathbb{N}\,,\quad t \in [t_0, t_f] =: T \subset \mathbb{R}\,,\quad r \in \Omega \subset \mathbb{R}^3\,,\quad q_i : T \to \mathbb{R}^3\,,\quad \phi, A : \Omega \times \mathbb{R} \to \mathbb{R}, \mathbb{R}^3\,,\quad \delta$ distribution

Fréchet/Gateaux Derivative:

$$DS[q,\phi,A] \cdot (\delta q, \delta\phi, \delta A) = 0 \qquad \forall \text{ directions } \Rightarrow$$

Relativistic Maxwell's equations with sources of $N \sim 10^{23}$ interacting particles $\Rightarrow$ **all done!**

Lessons

- N body problem near useless $\rightarrow$ seek reductions
  - Kinetic theories (Hilbert, Chapman-Enskog, other) $\rightarrow$ magnetofluid theories
  - Hilbert's 6th again and Millennium Prize problems

- Plasmas at most basic level are a Hamiltonian system
  - recurring theme!

General 2-Fluid Theory for species $s \in \{\pm\}$

$$\textit{Momentum}: \quad m_s n_s \left(\frac{\partial \boldsymbol{v}_s}{\partial t} + \boldsymbol{v}_s \cdot \nabla \boldsymbol{v}_s\right) = -\nabla p_s + \nabla \cdot \Pi_s + e_s n_s \left(\boldsymbol{E} + \boldsymbol{v}_s \times \boldsymbol{B}\right) + \boldsymbol{R}_s$$

$$\textit{Continuity}: \quad \frac{\partial n_s}{\partial t} = -\nabla \cdot (n_s \boldsymbol{v}_s)$$

$$\textit{Entropy}: \quad \frac{\partial s_s}{\partial t} + \boldsymbol{v}_s \cdot \nabla s_s = -\frac{1}{n_s m_s T_s} \nabla \cdot Q_s + \frac{1}{n_s m_s T_s} \Pi_s : \nabla \boldsymbol{v}_s$$

(alternatively T_s *or* p_s)

$$\textit{Sources -- Charge and Current}: \quad \rho_e = \sum_s e_s n_s \quad \text{and} \quad \boldsymbol{J} = \sum_s e_s n_s \boldsymbol{v}_s$$

$$\textit{Local Thermo}: \quad U_s(\rho_s, s_s) \quad st \quad T_s = \frac{\partial U_s}{\partial s_s} \quad \text{and} \quad p_s = \rho_s^2 \frac{\partial U_s}{\partial \rho_s}$$

$$\textit{Maxwell}: \quad \epsilon_0 \nabla \cdot \boldsymbol{E} = \rho_e \quad \text{and} \quad \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} = \frac{1}{\mu_0} \nabla \times \boldsymbol{B} - \boldsymbol{J}$$

$$\textit{Stress and collisions}: \quad \Pi_s \sim \sum \eta_s^a \nabla \boldsymbol{v}_s \quad \text{and} \quad \sum_s \boldsymbol{R}_s = 0$$

Reduction to Extended MHD

Kepler 2-Body:

$$\boldsymbol{R}_{cm} = \frac{m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2}{m_1 + m_2} \quad \text{and} \quad \boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$$

Magnetofluid:

$$\boldsymbol{v} = \frac{m_+n_+\boldsymbol{v}_+ + m_-n_-\boldsymbol{v}_-}{m_+n_+ + m_+n_+} \quad \text{and} \quad \boldsymbol{J} = e(n_+\boldsymbol{v}_+ - n_-\boldsymbol{v}_-)$$

Coodinate Change:

$$\boxed{\textbf{Extended MHD} \ + \ \textbf{Non} - \textbf{Quasineutrality Terms}}$$

Neutrality vs. Quasineutrality:

$$\int d^3r \left(n_+(\boldsymbol{r},t) - n_-(\boldsymbol{r},t)\right) = 0 \quad \text{vs.} \quad n_+(\boldsymbol{r},t) = n_-(\boldsymbol{r},t) = n(\boldsymbol{r},t)$$

Extended MHD

Continuity equation:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \boldsymbol{V})\,, \qquad \rho = m_+ n_+ + m_- n_- = (m_+ + m_-) n$$

Momentum Equation;

$$\rho \left(\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \nabla) \boldsymbol{V} \right) = -\nabla p + \boldsymbol{J} \times \boldsymbol{B} - \frac{m_-}{e} (\boldsymbol{J} \cdot \nabla) \frac{\boldsymbol{J}}{en}\,,$$

Faraday's Law:

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times \boldsymbol{E}$$

Ohm's Law

$$\boldsymbol{E} = -\boldsymbol{V} \times \boldsymbol{B} + \frac{1}{en} (\boldsymbol{J} \times \boldsymbol{B} - \nabla p_e) + \frac{m_-}{e^2 n} \left[\frac{\partial \boldsymbol{J}}{\partial t} + \nabla \cdot (\boldsymbol{V}\boldsymbol{J} + \boldsymbol{J}\boldsymbol{V}) \right] - \frac{m_-}{e^2 n} (\boldsymbol{J} \cdot \nabla) \frac{\boldsymbol{J}}{en} + \frac{\boldsymbol{J}}{\sigma}$$

$\rightarrow$ **Model reductions** based on asymptotics, small parameter expansions, such as Hall MHD, Inertial MHD, 1960s – 1980s –, reduced MHD, gyrofluid models, Hasegawa-Mima or quasigeostrophy, various reduced fluid models ...

Ideal extended MHD is a Hamiltonian Field Theory

Momentum + Faraday + Ohm:

$$\left(\frac{\partial}{\partial t} + \pounds_{\boldsymbol{V}_\pm}\right)\mathcal{B}_\pm = 0\,, \qquad \Rightarrow \qquad \text{Frozen fluxes}$$

where

$$V_\pm = \boldsymbol{v} - \kappa_\pm \nabla\times\boldsymbol{B}/\rho\,, \quad \mathcal{B}_\pm = \boldsymbol{B}^* + \kappa_\pm\nabla\times\boldsymbol{v}\,, \quad \text{and} \quad \boldsymbol{B}^* = \boldsymbol{B} + d_e^2\nabla\times\big((\nabla\times\boldsymbol{B})/\rho\big)\,.$$

Here, $\boldsymbol{V}_\pm$ are vector fields, $\mathcal{B}_\pm$ are 2-forms, and $\pounds_{\boldsymbol{V}_\pm}$ are Lie derivative. $d_e, \kappa_\pm$ constants depending on $e, m_-, \dots$.

Maybe paying attention to Hamiltonian structure is a good idea!

Hamiltonian Review ODEs $\rightarrow$ PDEs

Poisson Bracket: $\{f, g\}$

Hamilton's Canonical Equations

Phase Space with Canonical Coordinates: (q, p)

Hamiltonian function: $H(q, p)$ $\leftarrow$ the energy

Equations of Motion:

$$\boxed{\dot{p}_i = -\frac{\partial H}{\partial q^i} \quad \text{and} \quad \dot{q}^i = \frac{\partial H}{\partial p_i}} \qquad i = 1, 2, \ldots N$$

Phase Space Coordinate Rewrite: $z = (q, p)\,, \quad \alpha, \beta = 1, 2, \ldots 2N$

$$\dot{z}^\alpha = J_c^{\alpha\beta} \frac{\partial H}{\partial z^\beta} = \{z^\alpha, H\}_c \quad \text{where} \quad (J_c^{\alpha\beta}) = \begin{pmatrix} 0_N & I_N \\ -I_N & 0_N \end{pmatrix}$$

$J_c :=$ Poisson tensor, Hamiltonian bi-vector, cosymplectic form

Noncanonical Hamiltonian Structure

Sophus Lie (1890) $\longrightarrow$ PJM (noncanonical 1980) $\longrightarrow$ Weinstein (1982) Poisson Manifolds

Noncanonical Coordinates:

$$\dot{z}^\alpha = \{z^\alpha, H\} = J^{\alpha\beta}(z)\frac{\partial H}{\partial z^\beta}$$

Noncanonical Poisson Bracket:

$$\{f,g\} = \frac{\partial f}{\partial z^\alpha} J^{\alpha\beta}(z) \frac{\partial g}{\partial z^\beta}$$

Bilinear Poisson Bracket Properties:

antisymmetry $\rightarrow$ $\{f,g\} = -\{g,f\}$ $\rightarrow$ $J^{\alpha\beta} = -J^{\beta\alpha}$

Jacobi identity $\rightarrow \{f,\{g,h\}\}+\{b,\{h,f\}\}+\{h,\{f,g\}\} = 0 \rightarrow$ Jacobiator $S^{\alpha\beta\gamma} = J^{\alpha\ell}\partial_\ell J^{\beta\gamma} + cyc \equiv 0$

Leibniz $\rightarrow$ $\{fh,g\} = f\{h,g\} + \{h,g\}f\,,$ fg pointwise

G. Darboux: $detJ \neq 0 \Longrightarrow J \rightarrow J_c$ Canonical Coordinates

Sophus Lie: $detJ = 0 \Longrightarrow$ Canonical Coordinates plus Casimirs $\leftarrow$ G. Sudarshan
(Lie's distinguished functions!)

Noncanonical Poisson Brackets – Flows on Poisson Manifolds

Definition. A Poisson manifold $\mathcal{Z}$ has bracket

$$\{\,,\,\}\colon C^\infty(\mathcal{Z}) \times C^\infty(\mathcal{Z}) \to C^\infty(\mathcal{Z})$$

st $C^\infty(\mathcal{Z})$ with $\{\,,\,\}$ is a Lie algebra realization, i.e., is

- $\mathbb{R}$-bilinear,
- antisymmetric,
- Jacobi identity
- Leibniz, i.e., acts as a derivation $\Rightarrow$ vector field.

Geometrically $C^\infty(\mathcal{Z}) \equiv \Lambda^0(\mathcal{Z})$ and $\boldsymbol{d}$ exterior derivative.

$$\{f,g\} = \langle \boldsymbol{d}f, J\boldsymbol{d}g\rangle = J(\boldsymbol{d}f, \boldsymbol{d}g) = \frac{\partial f}{\partial z^\alpha} J^{\alpha\beta} \frac{\partial g}{\partial z^\beta}\,.$$

J the Poisson tensor/operator. Flows are integral curves of noncanonical Hamiltonian vector fields, $J\boldsymbol{d}H$, i.e.,

$$\dot{z}^\alpha = J^{\alpha\beta}(z)\frac{\partial H(z)}{\partial z^\beta}\,, \qquad \mathcal{Z}'s \text{ coordinate patch } z = (z^1, \dots, z^N)$$

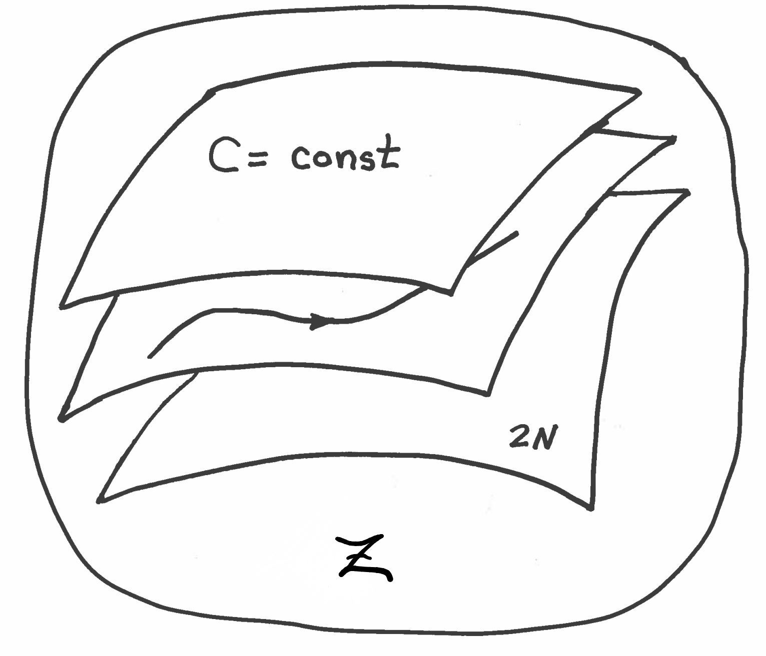
Because of degeneracy, $\exists$ functions C st $\{f, C\} = 0$ for all $f \in C^\infty(\mathcal{Z})$, are the Casimirs.

Poisson Manifold (phase space) $\mathcal{Z}$ Cartoon

Degeneracy in J $\Rightarrow$ Casimirs:

$$\{f, C\} = 0 \quad \forall\ f : \mathcal{Z} \to \mathbb{R}$$

Lie-Darboux Foliation by Casimir (symplectic) leaves:



Noncanonical Hamiltonian Field Theory I

Phase Space = functions u in Banach space $\mathcal{B}$ over $\Omega \subset \mathbb{R}^d$ st $u\colon \Omega \mapsto \mathbb{R}^N$, $d, N \in \mathbb{N}$.

For $\mathcal{F} \in C^1(\mathcal{B})$, $D\mathcal{F}(u)$ its Fréchet derivative at $u \in \mathcal{B}$.

Let $\mathcal{B}'$ be another Banach space with a nondegenerate pairing $\langle \cdot, \cdot \rangle_{\mathcal{B}\times\mathcal{B}'}\colon \mathcal{B} \times \mathcal{B}' \to \mathbb{R}$,

For $\mathcal{F} \in \mathcal{B}$, the functional derivative $\delta\mathcal{F}/\delta u \in \mathcal{B}'$ unique element of $\mathcal{B}$ (if it exists) such that

$$D\mathcal{F}(u)v = \left\langle v, \frac{\delta \mathcal{F}(u)}{\delta u} \right\rangle_{\mathcal{B}\times\mathcal{B}'}, \quad \forall v \in \mathcal{B}.$$

Assume $\mathcal{B} \subseteq L^2(\Omega, \mu; \mathbb{R}^N)$, $\mathcal{B}' = L^2(\Omega, \mu; \mathbb{R}^N)$, and measure $d\mu$ on Ω,

$$(u, v)_{L^2} = \int_\Omega u(x) \cdot v(x) d\mu(x), \quad u, v \in L^2(\Omega, \mu; \mathbb{R}^N).$$

Noncanonical Hamiltonian Field Theory II

A *Poisson bracket* on $\mathcal{B}$ is a bilinear *antisymmetric* map

$$\{\cdot,\cdot\}\colon C^\infty(\mathcal{B}) \times C^\infty(\mathcal{B}) \to C^\infty(\mathcal{B}),$$

st for any $\mathcal{F}, \mathcal{G}, \mathcal{H} \in C^\infty(\mathcal{B})$,

$$\{\mathcal{F}, \mathcal{G}\mathcal{H}\} = \{\mathcal{F}, \mathcal{G}\}\mathcal{H} + \mathcal{G}\{\mathcal{F}, \mathcal{H}\}\,,$$
$$\Big\{\mathcal{F}, \{\mathcal{G}, \mathcal{H}\}\Big\} + \Big\{\mathcal{G}, \{\mathcal{H}, \mathcal{F}\}\Big\} + \Big\{\mathcal{H}, \{\mathcal{F}, \mathcal{G}\}\Big\} = 0.$$

i.e. Leibniz identity and Jacobi identities. Thus we have a Lie algebra realization on $C^\infty(\mathcal{B})$ with derivation.

Poisson bracket:

$$\{\mathcal{F}, \mathcal{G}\} = \sum_{i,j=1}^{N} \int_\Omega \int_\Omega \frac{\delta \mathcal{F}(u)}{\delta u_i}(x)\, \mathcal{J}_{ij}(u; x, x')\, \frac{\delta \mathcal{G}(u)}{\delta u_j}(x')\, d\mu(x')\, d\mu(x),$$

$\mathcal{J}_{ij}(u; x, x')$ Poisson operator.

Example: 2D Euler

(also Vlasov-Poisson, Hasegawa-Mima, ...)

Scalar Vorticity:

$$\omega : \mathbb{T}^2 \times \mathbb{R} \to \mathbb{R}\,, \qquad\qquad \omega = \hat{z} \cdot \nabla \times \boldsymbol{v}$$

Equation of motion:

$$\frac{\partial \omega}{\partial t} = [\omega, \phi] = \partial_x \omega\, \partial_y \phi - \partial_y \omega\, \partial_x \phi\,, \qquad\qquad \omega = \Delta \phi$$

Hamiltonian

$$H[\omega] = \frac{1}{2} \int d^2x |\nabla \phi|^2$$

Poisson Bracket:

$$\{\mathcal{F}, \mathcal{G}\} = \int d^2x\, \omega \left[\frac{\delta \mathcal{F}}{\delta \omega}, \frac{\delta \mathcal{G}}{\delta \omega}\right]$$

Hamiltonian Form:

$$\frac{\partial \omega}{\partial t} = \{\omega, H\}$$

Lie-Poisson Form – Inner and Outer Lie algebras

Outer, for functionals $\mathcal{F}, \mathcal{G} \in \mathfrak{G}$:

$$\{\, , \,\} : \mathfrak{G} \times \mathfrak{G} \to \mathfrak{G}$$

Inner, for functions $f, g \in \mathfrak{g}$

$$[\, , \,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

Pairing:

$$\langle\, , \,\rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$$

Lie-Poisson Bracket:

$$\{\mathcal{F}, \mathcal{G}\} = \left\langle \omega, \left[\frac{\delta \mathcal{F}}{\delta \omega}, \frac{\delta \mathcal{G}}{\delta \omega}\right]\right\rangle$$

Example: 2D MHD – a Reduction of Extended MHD

Magnetic Field:

$$\boldsymbol{B} = \nabla\psi \times \hat{z}\,, \qquad\qquad \psi : \mathbb{T}^2 \times \mathbb{R} \to \mathbb{R}\,,$$

Equations of motion:

$$\frac{\partial \omega}{\partial t} = [\omega, \phi] + [\psi, J] \quad \text{and} \quad \frac{\partial \psi}{\partial t} = [\psi, \phi]\,, \qquad J = \Delta\psi \ \& \ \omega = \Delta\phi$$

Hamiltonian

$$H[\omega] = \frac{1}{2}\int d^2x \left(|\nabla\phi|^2 + |\nabla\psi|^2\right)$$

Poisson Bracket:

$$\{\mathcal{F}, \mathcal{G}\} = \int d^2x \left\{\omega \left[\frac{\delta \mathcal{F}}{\delta \omega}, \frac{\delta \mathcal{G}}{\delta \omega}\right] + \psi\left(\left[\frac{\delta \mathcal{F}}{\delta \psi}, \frac{\delta \mathcal{G}}{\delta \omega}\right] + \left[\frac{\delta \mathcal{F}}{\delta \omega}, \frac{\delta \mathcal{G}}{\delta \psi}\right]\right)\right\}$$

Hamiltonian Form:

$$\frac{\partial \omega}{\partial t} = \{\omega, H\} \qquad \text{and} \qquad \frac{\partial \psi}{\partial t} = \{\psi, H\}$$

3-Field and 4-Field Models

Daughters of 2-fluid Theory

3-Field (H-M ∪ RMHD):

$$\frac{\partial \omega}{\partial t} = [\omega, \phi] + [\psi, J]\,, \qquad \frac{\partial \psi}{\partial t} = [\psi, \phi] + \alpha[\chi, \psi]\,, \qquad \frac{\partial \chi}{\partial t} = [\chi, \phi] + [\psi, J]$$

Hamiltonian

$$H[\omega] = \frac{1}{2}\int d^2x \left(|\nabla\phi|^2 + |\nabla\psi|^2 + \alpha\chi^2\right)$$

4-Field:

$$\frac{\partial(\psi - d_e^2\nabla^2\psi)}{\partial t} + [\phi, \psi - d_e^2\nabla^2\psi] - d_\beta[\psi, p] = 0\,, \qquad \frac{\partial v}{\partial t} + [\phi, v] - c_\beta[p, \psi] = 0$$

$$\frac{\partial p}{\partial t} + [\varphi, p] - c_\beta[v, \psi] - d_\beta[\nabla^2\psi, \psi] = 0\,, \qquad \frac{\partial \nabla^2\phi}{\partial t} + [\phi, \nabla^2\phi] + [\nabla^2\psi, \psi] = 0$$

Hamiltonian:

$$H = \frac{1}{2}\int d^2x \,(d_e^2 J^2 + |\nabla\psi|^2 + |\nabla\phi|^2 + v^2 + p^2)$$

Lie Algebra Extensions

Hamiltonian Form:

$$\frac{\partial \xi^\lambda}{\partial t} = \{\xi^\lambda, H\}, \qquad \lambda = 1, \cdots, n,$$

Lie-Poisson Brackets on $\mathfrak{g}^n$. Let n-tuple $\bar{g} = (g_1, g_2, \ldots g_n) \in \mathfrak{g}^n$

$$[\bar{g}, \bar{h}] = ([\bar{g}, \bar{h}]_1, [\bar{g}, \bar{h}]_2, \ldots, [\bar{g}, \bar{h}]_n)$$

For constants $W_\lambda^{\mu\nu} = W_\lambda^{\nu\mu}$

$$[\bar{g}, \bar{h}]_\lambda = \sum_{\mu\nu=1}^{n} W_\lambda^{\mu\nu} [g_\mu, h_\nu]$$

Jacobi iff

$$W_\lambda^{\sigma\tau} W_\sigma^{\mu\nu} = W_\lambda^{\sigma\nu} W_\sigma^{\tau\mu}$$

$$\left[W^{(\nu)}\right]_\lambda^{\mu} = W_\lambda^{\nu\mu} \quad \rightarrow \quad W^{(\nu)} W^{(\sigma)} = W^{(\sigma)} W^{(\nu)}$$ ← pairwise commutation

Poisson Bracket:

$$\{\mathcal{F}, \mathcal{G}\}_\pm = \pm \sum_{\lambda,\mu,\nu=1}^{n} W_\lambda^{\mu\nu} \left\langle \xi^\lambda, \left[\frac{\delta \mathcal{F}}{\delta \xi^\mu}, \frac{\delta \mathcal{G}}{\delta \xi^\nu}\right]\right\rangle$$

Why 2D? The Guide Field and Weakly 3D

For $\boldsymbol{B} = B_0\hat{z} + \nabla\psi \times \hat{z}$ operator

$$\boldsymbol{B}\cdot\nabla f = B_0\frac{\partial f}{\partial z} + \nabla\psi\times\hat{z}\cdot\nabla f = B_0\frac{\partial f}{\partial z} - \hat{z}\cdot\nabla\psi\times\nabla f = B_0\frac{\partial f}{\partial z} - [\psi, f]$$

Ordering $B_0 = \mathcal{O}(1)$, $\partial/\partial z = \mathcal{O}(\epsilon)$, $\boldsymbol{B}_\perp = \mathcal{O}(\epsilon)$, $\nabla_\perp = \mathcal{O}(\epsilon) \Rightarrow$ weakly 3D theories.

Weakly 3D RMHD:

$$\frac{\partial\omega}{\partial t} + \frac{\partial J}{\partial z} = [\omega, \phi] + [\psi, J] \quad \text{and} \quad \frac{\partial\psi}{\partial t} + \frac{\partial\phi}{\partial z} = [\psi, \phi]\,, \qquad J = \Delta_\perp\psi \;\&\; \omega = \Delta_\perp\phi$$

Hamiltonian

$$H[\omega] = \frac{1}{2}\int d^3x\,\left(|\nabla_\perp\phi|^2 + |\nabla_\perp\psi|^2\right)$$

Poisson Bracket:

$$\{\mathcal{F}, \mathcal{G}\} = \int d^3x\,\left\{\omega\left[\frac{\delta\mathcal{F}}{\delta\omega}, \frac{\delta\mathcal{G}}{\delta\omega}\right] + \psi\left(\left[\frac{\delta\mathcal{F}}{\delta\psi}, \frac{\delta\mathcal{G}}{\delta\omega}\right] + \left[\frac{\delta\mathcal{F}}{\delta\omega}, \frac{\delta\mathcal{G}}{\delta\psi}\right]\right) + \frac{\delta\mathcal{F}}{\delta\psi}\frac{\partial}{\partial z}\frac{\delta\mathcal{G}}{\delta\omega} - \frac{\delta\mathcal{G}}{\delta\omega}\frac{\partial}{\partial z}\frac{\delta\mathcal{F}}{\delta\omega}\right\}$$

Theory of Weakly 3D Poisson Brackets

Poisson Brackets:

$$\{\mathcal{F},\mathcal{G}\} = \{\mathcal{F},\mathcal{G}\}_{\perp} + \{\mathcal{F},\mathcal{G}\}_{||} = \int d^3x\, W^{\mu\nu}_{\lambda}\ \xi^{\lambda}\, [\mathcal{F}_{\mu},\ \mathcal{G}_{\nu}] + \int d^3x\, A^{\mu\nu} \mathcal{F}_{\mu} \frac{\partial}{\partial z} \mathcal{G}_{\nu}$$

Jacobi iff:

$$A^{\lambda\delta} W^{\mu\nu}_{\lambda} = A^{\lambda\nu} W^{\delta\mu}_{\lambda} = A^{\lambda\mu} W^{\nu\delta}_{\lambda}$$

There exist many weakly 3D daughters of extended MHD, gyroviscosity, etc.

Uses

- Catagorization 2D,Weakly 3D, 3D+
- Conservation of Energy automatic
- Structure preserving integration
- Stability:

Poisson brackets $\Rightarrow$ Casimirs $\Rightarrow$

Variational Principles for equilibria:

$$\frac{\partial \xi^\lambda}{\partial t} = 0 = \{\xi^\lambda, H\} = \{\xi^\lambda, H + C\} = \mathcal{J}^{\lambda\alpha} \frac{\delta(H + C)}{\delta \xi^\alpha}$$

Energy-Casimir is Dirichlet's Theorem of Mechanics:

$$\delta^2(H + C) = \text{positive definite quadratic form} \rightarrow \text{“stability”}$$

Gardner, Oberman & Kruskal 1950s, Arnold 1960s, generalizations 1980s $\rightarrow$. Get for free!

Thank You!

References I

My articles are available at http://www.ph.utexas.edu/∼morrison

There are many general sources on plasma physics, an early, very original, and comprehensive one that discusses the kinetic to fluid theory procedures of Hilbert, Chapman-Enskog, etc. is

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Overall Overview of 3 SNS Lectures

I. Magnetofluid models $\leftarrow$ Hamiltonian

II. Dissipation: metriplectic dynamics

III. Linear Vlasov as a Hamiltonian theory

LECTURE II

Metriplectic dynamical systems and the unified thermodynamic (UT) algorithm

Metriplectic 4-Bracket:
Michael Updike (geometry), undergrad, now Princeton grad student,
Azeddine Zaidni (UT Algorithm), grad student intern now PhD, UM6, Marrakesh-Safi, Morocco;
Naoki Sato (collision operators), NIFS, junior faculty, Nagoya, Japan,
William Barham (numerics), UT Austin → LANL Director's posdoc.

Other Collaborators:
C. Bressan, O. Maj, M. Kraus, E. Sonnendrücker; T. Ratiu, A. Bloch, B. Coquinot & M. Materassi.

Theory of Thermodynamically Consistent Theories

theory $=$ dynamical system $= \mathfrak{X}(\mathcal{Z})$

Finite dimensions $\exists$ rigor. Infinite dimensions $\exists$ wishful thinking.

Dynamics vs. Thermodynamics

Dynamical thermodynamics (nonequilibrium thermodynamics) $\neg$ thermodynamics

$$\frac{\partial}{\partial t} \quad \leftarrow \quad \text{yes!} \qquad \text{vs.} \qquad \frac{\partial}{\partial T} \quad \leftarrow \quad \text{no!}$$

Finite dimensions $\exists$ rigor. Infinite dimensions $\exists$ wishful thinking.

Goal minimal geometric structure: Not too much, not too little!

Overview

I. Motivation and Review

II. Metriplectic 4-Bracket

III. Unified Thermodynamic (UT) Algorithm and Fluid Theories

IV. Collision Operator Examples

V. Final Comments

I. Motivation

Thermodynamic Consistency – Examples

Navier-Stokes is **inconsistent:**

$$\partial_t \boldsymbol{v} = -\boldsymbol{v}\cdot\nabla\boldsymbol{v} - \frac{1}{\rho_0}\nabla p + \nu\nabla^2\boldsymbol{v}\,, \qquad \nabla\cdot\boldsymbol{v} = 0 \quad \Rightarrow \quad p[\boldsymbol{v}]$$

$H = \int_\Omega \rho_0\,|\boldsymbol{v}|^2/2$ and $\dot{H} \le 0\,,$ $\nexists$ any thermodynamics!

Navier-Stokes-Fourier (NSF) is **consistent:** (Eckart 1940):

$$\partial_t \boldsymbol{v} = -\boldsymbol{v}\cdot\nabla\boldsymbol{v} - \frac{1}{\rho}\nabla p + \frac{1}{\rho}\nabla\cdot\mathcal{T}$$

viscous stress tensor is $\mathcal{T}$

$$\partial_t \rho = -\nabla\cdot(\rho\boldsymbol{v})$$

$$\partial_t s = -\boldsymbol{v}\cdot\nabla s - \frac{1}{\rho T}\nabla\cdot\boldsymbol{q} + \frac{1}{\rho T}\mathcal{T}:\nabla\boldsymbol{v}$$

heat flux & viscous heating

$H = \int_\Omega \rho\,|\boldsymbol{v}|^2/2 + \rho u(\rho, s)\,,$ $\dot{H} = 0$ and $S = \int_\Omega \rho s \;\rightarrow\; \dot{S} \ge 0$

Example of **Thermodynamic Completion**, i.e. NS $\rightarrow$ NSF.

Thermodynamic Consistency

The realization in a **dynamical system** of the first and second laws of thermodynamics:

First Law is energy conservation:

$$\dot{H} = 0$$

Second Law is entropy production:

$$\dot{S} \geq 0$$

Good models lift thermodynamics to **dynamical systems.** They have two functions H, S.

Theories & Models as Dynamical Systems

Main Scientific Goal:
Predict the future or explain the past $\Rightarrow$

$$\dot{z} = V(z)\,, \qquad \text{dynamical variable } z \in \mathcal{Z} \text{ the Phase Space}$$

Ultimately a dynamical system. Vector fields on manifolds and Cauchy problem (IVP).

Examples: Maps, ODEs, PDEs, etc. finite-dimensional, infinite-dimensional (field theories)

Whence vector field V?

• Fundamental parent theory (microscopic, N interacting gravitating or charged particles, BBGKY hierarchy, Vlasov-Maxwell system, ...). Identify small parameters, limits, rigorous asymptotics, Hilbert's 6th $\rightarrow$ Reduced Computable Model for V.

• Phenomena based modeling using known properties, constraints, symmetries, etc. used to intuit $\rightarrow$ Reduced Computable Model V. $\leftarrow$ Here metriplectic structure can be useful.

Vector Field Splitting

$$V(z) = V_{nondissipative} + V_{dissipative}$$

How?

Vector Field Splitting

$$V(z) = V_{nondissipative} + V_{dissipative}$$

How?

$V_{nondissipative} \equiv$ Hamiltonian and $V_{dissipative} \equiv$?

What is Dissipation?

- Not all conservative systems are Hamiltonian
- Not all Hamiltonian systems are conservative
- Not all reversible systems are Hamiltonian
- All finite dynamical systems (autonomous ODEs) are reversible (1 parameter Lie group)
- Some infinite systems (PDEs) are reversible and some irreversible (group vs. semigroup)
- Not all Hamiltonian systems have time reversal symmetry
- Not all systems with time reversal symmetry are Hamiltonian
- $\exists$ systems with time reversible symmetry and asymptotic stability

Thermodynamically Consistent Dissipation:

Energy conserving systems with an increasing entropy that implies global asymptotic stability.

Such systems have a 'vector field' that naturally splits in Hamiltonian and dissipative parts. Hamiltonian is an unambiguous way to define nondissipative. The metriplectic 4-bracket is an unambiguous way to define dissipative. Together they $\Rightarrow$ thermodynamic consistency.

Toward a Thermodynamically Consistent Split

$$V(z) = V_H + V_D$$

Hamiltonian Form:

$$V_H = \{z, H\} = J\,\frac{\partial H}{\partial z} \qquad \text{where} \quad J^T = -J$$

where $J(z)$ is Poisson tensor/operator and H is the Hamiltonian. Has product decomposition.

Dissipative Form:

$$V_D = \ldots\ ? \qquad \rightarrow \qquad V_D = (z, G) = G\,\frac{\partial F}{\partial z} \qquad \text{where} \quad G^T = G$$

General <u>degenerate</u> 'metric tensor' G of some kind for <u>gradient system</u>?

Metriplectic Dynamics

Metric $\cup$ Symplectic Flows (pjm 1986) $\leftrightarrow V_D + V_H$

- Formalism for natural split of vector fields

- Enforces thermodynamic consistency: $\dot{H} = 0$ the 1st Law and $\dot{S} \geq 0$ the 2nd Law.

- Other invariants? E.g., collision operators preserve, mass, momentum, ... There exists some theory for building in, but won't discuss today.

- **Encompassing 4-bracket**: Entropy is a Casimir is & "curvature" is dissipation rate

Ideas of Casimirs are candidates for entropy, multibracket, curvature, etc. in pjm (1984). Metriplectic in pjm (1986).

Metriplectic 4-Bracket Dynamics

Dynamical System (finite or infinite):

$$\dot{z} = \{z, H\} + (z, H; S, H) = \{z, H\} + (z, S)H$$

Dynamics for any observable (functional of dynamical variables), z, is generated by multi-linear brackets, Poisson bracket + 4-bracket (2024), with Hamiltonian H and entropy = Casimir S.

Hamiltonian Review

Poisson Bracket: $\{f, g\}$

Hamilton's Canonical Equations

Phase Space with Canonical Coordinates: (q, p)

Hamiltonian function: $H(q, p)$ $\leftarrow$ the energy

Equations of Motion:

$$\boxed{\dot{p}_\alpha = -\frac{\partial H}{\partial q^\alpha}\,, \qquad \dot{q}^\alpha = \frac{\partial H}{\partial p_i}\,,} \qquad \alpha = 1, 2, \ldots N$$

Phase Space Coordinate Rewrite: $z = (q, p)\,, \quad i, j = 1, 2, \ldots 2N$

$$\dot{z}^i = J_c^{ij}\frac{\partial H}{\partial z^j} = \{z^i, H\}_c\,, \qquad (J_c^{ij}) = \begin{pmatrix} 0_N & I_N \\ -I_N & 0_N \end{pmatrix}\,,$$

$J_c :=$ Poisson tensor, Hamiltonian bi-vector, cosymplectic form

Noncanonical Hamiltonian Structure

S. Lie (1890) $\longrightarrow$ pjm (noncanonical 1980) $\longrightarrow$ Poisson Manifolds etc.

Noncanonical Coordinates:

$$\dot{z}^i = \{z^i, H\} = J^{ij}(z)\frac{\partial H}{\partial z^j}$$

Noncanonical Poisson Bracket:

$$\{f, g\} = \frac{\partial f}{\partial z^i} J^{ij}(z) \frac{\partial g}{\partial z^j}$$

Bilinear Poisson Bracket Properties:

antisymmetry $\rightarrow$ $\{f, g\} = -\{g, f\}$ $\rightarrow$ $J^{ij} = -J^{ji}$

Jacobi identity $\rightarrow$ $\{f,\{g,h\}\}+\{b,\{h,f\}\}+\{h,\{f,g\}\} = 0$ $\rightarrow$ Jacobiator $S^{ijk} = J^{i\ell}\partial_\ell J^{jk} + cyc \equiv 0$

Leibniz $\rightarrow$ $\{fh, g\} = f\{h, g\} + \{h, g\}f\,,$ fg pointwise

G. Darboux: $detJ \neq 0 \Longrightarrow J \rightarrow J_c$ Canonical Coordinates

Sophus Lie: $detJ = 0 \Longrightarrow$ Canonical Coordinates plus Casimirs $\leftarrow$ G. Sudarshan

(Lie's distinguished functions!)

Noncanonical Poisson Brackets – Flows on Poisson Manifolds

Definition. A Poisson manifold $\mathcal{Z}$ has bracket

$$\{\,,\}\colon C^\infty(\mathcal{Z})\times C^\infty(\mathcal{Z})\to C^\infty(\mathcal{Z})$$

st $C^\infty(\mathcal{Z})$ with $\{\,,\}$ is a Lie algebra realization, i.e., is

- $\mathbb{R}$-bilinear,
- antisymmetric,
- Jacobi identity
- Leibniz, i.e., acts as a derivation $\Rightarrow$ vector field.

Geometrically $C^\infty(\mathcal{Z})\equiv\Lambda^0(\mathcal{Z})$ and $\boldsymbol{d}$ exterior derivative.

$$\{f,g\}=\langle \boldsymbol{d}f, J\boldsymbol{d}g\rangle = J(\boldsymbol{d}f,\boldsymbol{d}g)=\frac{\partial f}{\partial z^i}J^{ij}\frac{\partial g}{\partial z^j}\,.$$

J the Poisson tensor/operator. Flows are integral curves of noncanonical Hamiltonian vector fields, $J\boldsymbol{d}H$, i.e.,

$$\dot z^i = J^{ij}(z)\frac{\partial H(z)}{\partial z^j}\,, \qquad \mathcal{Z}'s \text{ coordinate patch } z=(z^1,\dots,z^N)$$

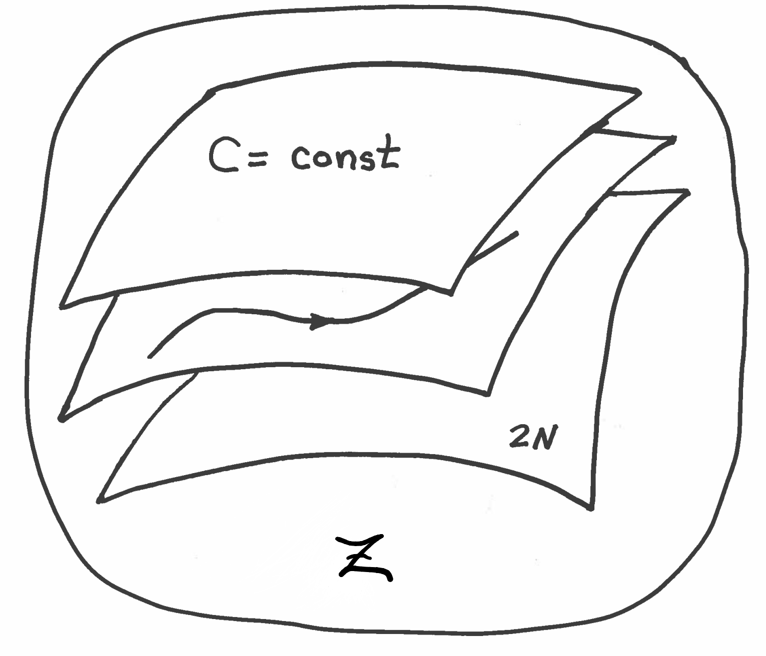
Because of degeneracy, $\exists$ functions C st $\{f,C\}=0$ for all $f\in C^\infty(\mathcal{Z})$, called Casimir invariants. **Casimir are candidate entropies!**

Poisson Manifold (phase space) $\mathcal{Z}$ Cartoon

Degeneracy in J $\Rightarrow$ Casimirs:

$$\{f, C\} = 0 \quad \forall\ f : \mathcal{Z} \to \mathbb{R}$$

Lie-Darboux Foliation by Casimir (symplectic) leaves:



Hamiltonian Structure of Vlasov-Poisson (pjm 1980)

Hamiltonian:

$$H = \frac{1}{2}\int |v|^2 f(z)\, dz + \frac{1}{2}\int |E|^2\, dx \qquad \text{where } z = (x, v) \in \mathbb{R}^6$$
$$= \frac{1}{2}\int |v|^2 f\, dz + \frac{1}{2}\int\int G(x|x')f(z)f(z')\; dz\, dz',$$

Bracket:

$$\{F, G\} = \int f\left(\nabla_x F_f \cdot \nabla_v G_f - \nabla_x G_f \cdot \nabla_v F_f\right) dz = \int f\left[F_f, G_f\right] dz$$

where $F_f = \delta F/\delta f$ means functional derivative of F with respect to f etc.

Equation of Motion:

$$\frac{\partial f}{\partial t} = \{f, H\},$$

Casimirs invariants:

$$C[f] = \int \mathcal{C}(f)\, dz, \qquad st \quad \{F, C\} = 0 \quad \forall F.$$

where $\mathcal{C}$, an arbitrary function; thus C that includes entropy.

II. Metriplectic 4-Bracket: $(f, k; g, n)$ or $(F, K; G, N)$

Finite dimensions (functions):

$$f, k, g, n \in C^{\infty}(\mathcal{Z})$$

Infinite dimensions (functionals):

$$F, K, G, N : \mathcal{B} \to \mathbb{R}$$

where $\mathcal{B}$ is some function space.

Why a 4-Bracket?

• One slot for dynamical variables (observables), z.

• Two slots for two fundamental functions: Hamiltonian, H, and Entropy (Casimir), S.

• There remains one slot for $\mathcal{F}$, free energy like generator $\mathcal{F} = H - TS$. Better argument: Needed to have multilinearity.

Comments:

• Provides natural reductions to other bilinear & binary brackets.

• The three slot brackets of pjm 1984 were not trilinear. Four needed to be <u>multilinear</u>.

The Metriplectic 4-Bracket

4-bracket on 0-forms (functions):

$$(\,\cdot\,,\,\cdot\,;\,\cdot\,,\,\cdot\,)\colon \Lambda^0(\mathcal{Z})\times\Lambda^0(\mathcal{Z})\times\Lambda^0(\mathcal{Z})\times\Lambda^0(\mathcal{Z})\to\Lambda^0(\mathcal{Z})$$

For functions $f,k,g,n\in\Lambda^0(\mathcal{Z})$ in a coordinate patch the 4-bracket has the form:

$$(f,k;g,n) = R^{ijkl}(z)\frac{\partial f}{\partial z^i}\frac{\partial k}{\partial z^j}\frac{\partial g}{\partial z^k}\frac{\partial n}{\partial z^l}\,. \qquad \leftarrow \text{quadravector?}$$

• Metriplectic manifolds have both Poisson tensor, J^{ij}, and compatible quadravector R^{ijkl}, where S (selected from set of Casimirs) and H comes from Hamiltonian part.

A blend of my previous early ideas 1980s: Two important functions H and S, symmetries, curvature idea, multi-brackets.

Metriplectic 4-Bracket Properties

(i) $\mathbb{R}$-linearity in all arguments, e.g, for $\lambda \in \mathbb{R}$

$$(f + \lambda h, k; g, n) = (f, k; g, n) + \lambda(h, k; g, n)$$

(ii) algebraic identities/symmetries

$$(f, k; g, n) = -(k, f; g, n)\,, \qquad (f, k; g, n) = -(f, k; n, g)\,, \qquad (f, k; g, n) = (g, n; f, k)$$

(iii) derivation in all arguments, e.g.,

$$(fh, k; g, n) = f(h, k; g, n) + (f, k; g, n)h$$

where as usual, fh denotes pointwise multiplication.

Symmetries of algebraic curvature without torsion identity. **Minimal Metriplectic.**

Observation: Often see $R^{l}{}_{ijk}$ or R_{lijk} but not R^{lijk}! Never 4-bracket, i.e. action on 1-forms?

Properties – Existence – General Construction Methods

• Thermodynamic Consistency Built-in:

$$\dot{H} = \{H, H\} + (H, H; S, H) = 0 \qquad \text{and} \qquad \dot{S} = (S, H; S, H) \geq 0$$

Reduces to metriplectic 2-bracket (1984): $(F, G)_H = (F, H; G, H)$.

• For any Riemannian manifold ∃ metriplectic 4-bracket. This means there is a wide class of them, but the bracket tensor does not need to come from Riemann tensor only needs to satisfy the bracket properties.

• If Riemannian, entropy production rate is positive contravariant sectional curvature. For closed $\sigma, \eta \in \Lambda^1(\mathcal{Z})$, entropy production by

$$\dot{S} = K(\sigma, \eta) := (S, H; S, H) \geq 0\,,$$

where the second equality follows from $\sigma = \boldsymbol{d}S$ and $\eta = \boldsymbol{d}H$.

• Two methods of construction? **Kulkarni-Nomizu** (K-N) product and **Lie algebra** based. $K(\sigma, \eta) \geq 0$ automatic for K-N and easily made minimally degenerate!

Methods of Construction

Construction via Kulkarni-Nomizu Product

Given σ and μ, two symmetric rank-2 tensor fields operating on 1-forms (assumed exact) $\boldsymbol{d}f, \boldsymbol{d}k$ and $\boldsymbol{d}g, \boldsymbol{d}n$, the K-N product is

$$\begin{aligned} \sigma \owedge \mu\,(\boldsymbol{d}f, \boldsymbol{d}k, \boldsymbol{d}g, \boldsymbol{d}n) &= \sigma(\boldsymbol{d}f, \boldsymbol{d}g)\,\mu(\boldsymbol{d}k, \boldsymbol{d}n) - \sigma(\boldsymbol{d}f, \boldsymbol{d}n)\,\mu(\boldsymbol{d}k, \boldsymbol{d}g) \\ &+ \mu(\boldsymbol{d}f, \boldsymbol{d}g)\,\sigma(\boldsymbol{d}k, \boldsymbol{d}n) - \mu(\boldsymbol{d}f, \boldsymbol{d}n)\,\sigma(\boldsymbol{d}k, \boldsymbol{d}g)\,. \end{aligned}$$

Metriplectic 4-bracket:

$$(f, k; g, n) = \sigma \owedge \mu(\boldsymbol{d}f, \boldsymbol{d}k, \boldsymbol{d}g, \boldsymbol{d}n)\,.$$

In coordinates:

$$R^{ijkl} = \sigma^{ik}\mu^{jl} - \sigma^{il}\mu^{jk} + \mu^{ik}\sigma^{jl} - \mu^{il}\sigma^{jk}\,.$$

If σ or μ defines inner product, then minimally degenerate, one fixed point on H= constant.

Infinite dimensions: $\mu \to M\,, \quad \sigma \to \Sigma$ ‘operators’.

Lie Algebra Based Metriplectic 4-Brackets

- For structure constants $c^{kl}{}_s$:

$$(f,k;g,n) = c^{ij}{}_r c^{kl}{}_s \, g^{rs} \frac{\partial f}{\partial z^i}\frac{\partial k}{\partial z^j}\frac{\partial g}{\partial z^k}\frac{\partial n}{\partial z^l}\,.$$

Lacks cyclic symmetry, but $\exists$ procedure to remove torsion (Bianchi identity) for any symmetric 'metric' g^{rs}. Dynamics does not see torsion, but manifold does.

- For $g^{rs}_{CK} = c^{rl}{}_k \, c^{sk}{}_l$ the Cartan-Killing metric, torsion vanishes automatically. Completely determined by Lie algebra. For $\mathfrak{so}(3)$ reproduces relaxing free rigid body (pjm 1986).

- Covariant connection $\nabla\colon \mathfrak{X}\times\mathfrak{X}\to\mathfrak{X}$. A contravariant connection $D\colon \Lambda^1(\mathcal{Z})\times\Lambda^1(\mathcal{Z})\to\Lambda^1(\mathcal{Z})$ satisfying Koszul identities, but Leibniz becomes $D_\alpha(f\gamma) = fD_\alpha\gamma + J(\alpha)[f]\gamma$ where $J(\alpha)[f] = \alpha_i J^{ij}\partial f/\partial z^j$ is a 0-form that replaces the term $\mathbf{X}(f)$ (Fernandes, 2000). Here $\alpha,\beta,\gamma\in\Lambda^1(\mathcal{Z})$, $f\in\Lambda^0(\mathcal{Z})$. Build 4-bracket like curvature from connection $\Rightarrow$?

III. Unified Thermodynamic (UT) Algorithm

UT Algorithm is an algorithm or recipe for constructing metriplectic (thermodynamically consistent) systems! Akin to building Lagrangians. Applied to many systems. **So far UT Algorithm either reproduces, corrects, or extends for every case considered!**

- **Fluid Theories**:

- Cahn-Hilliard-Navier-Stokes: agrees with Anderson et al.; corrects Guo and Lin

- Brenner-Navier-Stokes: UT Algorithm produces Brenner's equations, plus corrects statements, e.g., that the results are most general.

- Generalization of Brenner-Navier-Stokes: UT Algorithm produces equations of Reddy et al. (2019). All are generalizations of Navier-Stokes-Fourier with modified dissipation.

- **Collision Operators**: Landau, Fermi-Dirac, generalization to any monotonic equilibrium, collisions in noncanonical phase space . . .

Four Steps of the UT Algorithm

1. Identify dynamical variables

$$\text{Fluid} \rightarrow \boldsymbol{\xi}(x,t) = (\boldsymbol{m} = \rho \boldsymbol{v}, \rho, \sigma = \rho s) \qquad \text{or} \qquad \text{Kinetic} \rightarrow f(z,t)$$

2. Propose energy and entropy functionals, $H[\boldsymbol{\xi}]$ and $S[\boldsymbol{\xi}]$

$$\text{Fluid} \rightarrow H = \int dx \left(\frac{|\mathbf{m}|^2}{2\rho} + \rho U(\rho, \sigma/\rho) \right) \quad \text{and} \quad S = \int dx\, \sigma$$

$$\text{Kinetic} \rightarrow H = m \int dz\, f |v|^2/2 + \int dx\, |E|^2/2 \quad \text{and} \quad S = \int dz\, f \ln f$$

3. Find Poisson bracket $\{F, G\}$ for which entropy S is a Casimir invariant, $\{F, S\} = 0\, \forall\, F$

4. Construct metriplectic 4-bracket $(F, K; G, N)$ via Kulkarni-Nomizu product via physical reasoning that **separates local thermodynamics from phenomenological quantities**, giving the EoMs as Poisson bracket + 4-bracket:

$$\partial_t \boldsymbol{\xi} = \{\boldsymbol{\xi}, H\} + (\boldsymbol{\xi}, H; S, H)$$

Result automatically **thermodynamically consistent** for **any** choices of H and S!

3. For NSF Ideal Fluid Poisson Bracket Dynamics

Hamiltonian:

$$H = \int_\Omega \frac{\rho|\mathbf{v}|^2}{2} + \rho u\,(\rho, s)\;, \qquad T = \frac{\partial u}{\partial s}\,, \qquad p = \rho^2 \frac{\partial u}{\partial \rho}\,.$$

Lie-Poisson Bracket (pjm-Greene, 1980):

$$\{F, G\} = -\int_\Omega \mathbf{m} \cdot [F_{\boldsymbol{m}} \cdot \nabla G_{\boldsymbol{m}} - G_{\boldsymbol{m}} \cdot \nabla F_{\boldsymbol{m}}] + \rho\,[F_{\boldsymbol{m}} \cdot \nabla G_\rho - G_{\boldsymbol{m}} \cdot \nabla F_\rho]$$
$$+\, \sigma\,[F_{\boldsymbol{m}} \cdot \nabla G_\sigma - G_{\boldsymbol{m}} \cdot \nabla F_\sigma]\,.$$

Equations of Motion:

$$\partial_t \mathbf{v} = \{\boldsymbol{v}, H\} = -\boldsymbol{v} \cdot \nabla \boldsymbol{v} - \nabla p/\rho\,, \quad \partial_t \rho = \{\rho, H\} = -\nabla \cdot (\rho\, \boldsymbol{v})\,, \quad \partial_t \sigma = \{\sigma, H\} = -\nabla \cdot (\sigma\, \boldsymbol{v})\,.$$

Casimir:

$$S = \int_\Omega \rho s = \int_\Omega \sigma\,.$$

Note: $F_{\boldsymbol{m}} = \delta F / \delta \boldsymbol{m}$, etc., functional derivatives.

Infinite Dimensions – Field Theories

Multi-component fields:

$$\chi(z,t) = \left(\chi^1(z,t), \chi^2(z,t), \ldots, \chi^M(z,t)\right), \qquad z \in \mathcal{D}$$

Metriplectic 4-bracket:

$$(F,G;K,N) \;=\; \int d^N z \int d^N z' \int d^N z'' \int d^N z''' \, \widehat{R}^{ijkl}(z,z',z'',z''') \times \frac{\delta F}{\delta \chi^i(z)} \frac{\delta G}{\delta \chi^j(z')} \frac{\delta K}{\delta \chi^k(z'')} \frac{\delta N}{\delta \chi^l(z''')}$$

Fréchet derivative:

$$\delta F[\chi;\eta] = \frac{d}{d\epsilon} F[\chi + \epsilon\eta]\Big|_{\epsilon=0} = \int_{\mathcal{D}} d^N z \, \frac{\delta F[\chi]}{\delta \chi^i} \eta^i$$

$\delta F/\delta \chi$ the functional (variational) derivative (a gradient)

$\widehat{R}^{ijkl}(z,z',z'',z''')$ defined as distribution, an operator (e.g. a pseudo-differential ...) acting on the functional derivatives.

4. Metriplectic 4-Bracket

Old method (early 2024): guess the K-N quantities M and Σ.

New Method

Theorem: Order dynamical variables st

$$\partial_t \xi^\alpha = \{\xi^\alpha, H\} + \nabla \cdot \boldsymbol{J}^\alpha\,, \qquad \alpha = 1, \dots, N-1\,,$$
$$\partial_t \xi^N = \{\xi^N, H\} + \nabla \cdot \boldsymbol{J}^N + \boldsymbol{Z}_\alpha \cdot \tilde{L}^{\alpha\beta} \cdot \boldsymbol{Z}_\beta\,.$$

where $\xi^N = \sigma$, the entropy density. Above splits Hamiltonian and conservative.

Then

$$\dot{S} = \int_\Omega \boldsymbol{Z}_\alpha \cdot \tilde{L}^{\alpha\beta} \cdot \boldsymbol{Z}_\beta =: \int_\Omega \dot{\sigma}^{prod} \geq 0\,.$$

and $\dot{H} \Rightarrow$

$$\boldsymbol{Z}_\alpha = \nabla H_{\xi^\alpha}\,, \qquad \boldsymbol{J}^\alpha = -H_{\xi^N} \tilde{L}^{\alpha\beta} \nabla H_{\xi^\beta} = -L^{\alpha\beta} \nabla H_{\xi^\beta}\,.$$

which leads naturally to

$$M(dF, dG) = F_{\xi^N} G_{\xi^N}\,, \qquad \Sigma(dF, dG) = \nabla(F_{\xi^\alpha}) \frac{L^{\alpha\beta}}{H_{\xi^N}} \nabla(G_{\xi^\beta})\,.$$

Important Byproduct of UT Algorithm

- Special **ordering of dynamical variables** and concomitant 'Force-Flux' relations of nonequilibrium thermodynamics:

$$\boldsymbol{J}^\alpha = L^{\alpha\beta} X_\beta \quad \rightarrow \quad \boldsymbol{J}^\alpha = -L^{\alpha\beta}\,\nabla(\delta H/\delta\xi^\beta)$$

'Forces': $\boldsymbol{X} \sim \nabla T, \nabla p, \nabla \boldsymbol{v}$ etc., UT Algorithm removes ambiguous selection of forces and provides definition of phenomenological coefficients, $L^{\alpha\beta}$, for dynamical variables ξ^β.

- Separates dependences on thermodynamical variables that come from internal energy U (local thermodynamic equilibrium) from those that appear in the phenomenological coefficients $L^{\alpha\beta}$. For example in in the Fourier heat law entropy production expression

$$\dot{\sigma}^{prod} = \nabla T \cdot \frac{\bar{\kappa}}{T^2} \cdot \nabla T$$

one T comes from Fourier's law $q = -\bar{\kappa}\nabla T/T$ while the other comes from the phenomenological coefficient.

- Physically identify the sectional curvature

$$\dot{S} = (S, H; S, H) = K(H, S) = \int_\Omega \Sigma(dH, dH) = \int_\Omega \nabla H_{\xi^\alpha} \cdot \tilde{L}^{\alpha\beta} \cdot \nabla H_{\xi^\beta} \geq 0\,.$$

4. Metriplectic 4-Bracket $\leftarrow$ Old method trial and error!

K-N Product:

$$M(F_{\boldsymbol{\xi}}, G_{\boldsymbol{\xi}}) = F_\sigma G_\sigma$$

$$\Sigma(F_{\boldsymbol{\xi}}, G_{\boldsymbol{\xi}}) = \hat{\Lambda}_{ijkl}\, \partial_j F_{M_i} \partial_k G_{M_l} + a\, \nabla F_\sigma \cdot \nabla G_\sigma$$

$\partial_i := \partial/\partial x^i$ with general isotropic Cartesian tensor of order 4

$$\hat{\Lambda}_{ikst} = \alpha \delta_{ik}\delta_{st} + \beta(\delta_{is}\delta_{kt} + \delta_{it}\delta_{ks}) + \ \gamma(\delta_{is}\delta_{kt} - \delta_{it}\delta_{ks})$$

Construct

$$(F,G)_H = (F,H;G,H) \quad \rightarrow \quad \boldsymbol{\xi}_t = \{\boldsymbol{\xi}, H\} + (\boldsymbol{\xi}, S)_H \Rightarrow$$

using $S = \int d^3x\, \rho s$ and $H = \int d^3x \Big(\rho|\boldsymbol{v}|^2/2 + \rho U(\rho, s)\Big)$

$$\partial_t \boldsymbol{v} = -\boldsymbol{v} \cdot \nabla \boldsymbol{v} - \frac{1}{\rho}\nabla p + \frac{1}{\rho}\nabla \cdot \mathcal{T} \qquad \leftarrow \ \mathcal{T} \text{ viscous stress}$$

$$\partial_t \rho = -\nabla \cdot (\rho \boldsymbol{v})$$

$$\partial_t s = -\boldsymbol{v} \cdot \nabla s - \frac{1}{\rho T}\nabla \cdot \boldsymbol{q} + \frac{1}{\rho T}\mathcal{T} : \nabla \boldsymbol{v}\,, \qquad \boldsymbol{q} = -\kappa \nabla T$$

Reproduces pjm 1984!

4. Metriplectic 4-Bracket: General and NSF

General flux expressions:

$$
\begin{aligned}
\boldsymbol{J}_\rho &= -L^{\rho\rho}\cdot\nabla H_\rho - L^{\rho\boldsymbol{m}}:\nabla H_{\boldsymbol{m}} - L^{\rho\sigma}\cdot\nabla H_\sigma\,,\\
\bar{J}_{\mathbf{m}} &= -L^{\boldsymbol{m}\rho}\otimes\nabla H_\rho - L^{\boldsymbol{m}\boldsymbol{m}}:\nabla H_{\boldsymbol{m}} - L^{\boldsymbol{m}\sigma}\otimes\nabla H_\sigma\,,\\
\boldsymbol{J}_\sigma &= -L^{\sigma\rho}\cdot\nabla H_\rho - L^{\sigma\boldsymbol{m}}:\nabla H_{\boldsymbol{m}} - L^{\sigma\sigma}\cdot\nabla H_\sigma\,,
\end{aligned}
$$

where $\boldsymbol{J}_\rho$ is mass flux, $\bar{J}_{\mathbf{m}}$ is momentum flux 2-tensor, and $\boldsymbol{J}_\sigma$ is entropy flux.

For **NSF** all zero except:

$$
L^{\boldsymbol{m}\boldsymbol{m}} = \bar{\bar{\Lambda}} \quad \text{and} \quad L^{\sigma\sigma} = \frac{\bar{\kappa}}{T}\,.
$$

$\bar{\bar{\Lambda}}$ isotropic 4-tensor, $\bar{\kappa}$ conduction 2-tensor

$$
\dot{S} = (S,H;S,H) = \int_\Omega \Sigma(dH,dH) = \int_\Omega \nabla\mathbf{v} : \frac{\bar{\bar{\Lambda}}}{T} : \nabla\mathbf{v} + \nabla T\cdot\frac{\bar{\kappa}}{T^2}\cdot\nabla T \geq 0\,.
$$

Note in $\bar{\kappa}/T^2$ one T from H one from $L^{\alpha\beta}$. Σ sectional curvature density?

4. Metriplectic 4-Bracket for NSF Generalizations

For **Brenner NSF** all zero except:

$$L^{m\rho} = \tilde{D}\rho\, \boldsymbol{m}\,, \quad L^{m\sigma} = \tilde{D}\hat{\sigma}\, \boldsymbol{m}\,, \quad L^{mm} = \bar{\bar{\Lambda}} + \tilde{D}\, \boldsymbol{m} \otimes \bar{I} \otimes \boldsymbol{m}\,.$$

$$L^{\sigma\rho} = \tilde{D}\rho\hat{\sigma}\, \bar{I}\,, \quad L^{\sigma\sigma} = \frac{\bar{\kappa}}{T} + \tilde{D}\hat{\sigma}^2\, \bar{I} \quad L^{\sigma m} = \tilde{D}\hat{\sigma}\, \bar{I} \otimes \boldsymbol{m}$$

$$\dot{S} = \int_\Omega \frac{1}{T}\left[\frac{\tilde{D}}{\kappa_T^2\rho^2}|\nabla\rho|^2 + \nabla T \cdot \frac{\bar{\kappa}}{T} \cdot \nabla T \; + \nabla \boldsymbol{v} : \bar{\bar{\Lambda}} : \nabla \boldsymbol{v}\right] \geq 0\,.$$

Generalization of Brenner by **Reddy et al.** (2019) falls out. We further generalized.

IV. Collision Operator Examples

General 4-Bracket Collision Operator

Phase space $z=(x,v)\in\mathbb{R}^6$, density $f(z,t)$

Define operator on $w\colon \mathbb{R}^6\to\mathbb{R}$ (at fixed time)

$$P[w]_i=\frac{\partial w(z)}{\partial v_i}-\frac{\partial w(z')}{\partial v_i'}$$

$$\begin{aligned}(F,K;G,N) &= \int dz\int dz'\,\mathcal{G}(z,z')\\ &\quad\times\,(\delta \owedge \delta)_{ijkl}\,P\left[F_f\right]_i P\left[K_f\right]_j P\left[G_f\right]_k P\left[N_f\right]_l\,,\end{aligned}$$

where simplest K-N

$$(\delta \owedge \delta)_{ijkl}=2(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk})\,.$$

Choose H and S

$$\frac{\partial f}{\partial t}=(f,H;SH)=(f,S)_H=G\,\frac{\delta S}{\delta f}\qquad\leftarrow\qquad \text{degenerate gradient system}$$

Landau Collision Operator

(Dropping Hamiltonian part for now.)

Choose: $H = \int dz f\, |v|^2/2$

$$(F,H;G,H) = (F,G)_H = \int dz \int dz' \left[\frac{\partial}{\partial v_i}\frac{\delta F}{\delta f(z)} - \frac{\partial}{\partial v_i'}\frac{\delta F}{\delta f(z')}\right] T_{ij}(z,z') \left[\frac{\partial}{\partial v_j}\frac{\delta G}{\delta f(z)} - \frac{\partial}{\partial v_j'}\frac{\delta G}{\delta f(z')}\right]$$

Reproduces metriplectic 2-bracket (gradient system) $(F,G)_H$ in pjm 1984. Let

$$T_{ij}(z,z') = w_{ij}(z,z') f(z) f(z')/2 \qquad \& \qquad w_{ij} = (\delta_{ij} - g_i g_j/g^2)\delta(\mathbf{x}-\mathbf{x}')/g$$

$$w_{ij}(z,z') = w_{ji}(z,z') \qquad w_{ij}(z,z') = w_{ij}(z',z) \qquad g_i w_{ij} = 0 \text{ with } g_i = v_i - v_i'$$

Choose Entropy:

$$S[f] = \int dz\, f \ln\, f$$

Landau Collision Operator:

$$\frac{\partial f}{\partial t} = (f,H;S,H) = (f,S)_H = G\,\frac{\delta S}{\delta f} \qquad \leftarrow \text{gradient system, pjm 1984}$$

Other Collision Operators

F-D and L-B Collision Operators

Fermi-Dirac well-known.

Kadomstev and Pogutse (1970) obtained same for L-B collision operator.

Entropy:

$$S[f] = \int f \ln f + (1 - f) \ln(1 - f)$$

4-Bracket reproduces according to

$$(f, H; S, H)$$

Usual H.

General Collision Operators

Formal H-Theorem to **any** monotonic distribution. pjm 1986, used in C. Bressan's PhD thesis and CB et al., 2025

Choose

$$T_{ij}(z,z') = w_{ij}(z,z')M(f(z))M(f(z'))/2$$

Entropy & Compatibility:

$$S[f] = \int dz\, s(f) \qquad \text{where} \quad M(f)\frac{d^2 s}{df^2} = 1$$

4-bracket gives

$$\begin{aligned}\frac{\partial f}{\partial t} &= (f,H;S,H) = (f,S)_H \\ &= \frac{\partial}{\partial v_i}\int w_{ij}\left[M(f(v))\frac{\partial f(v')}{\partial v'_j} - M(f(v'))\frac{\partial f(v)}{\partial v_j}\right] dv'\end{aligned}$$

Desiderata: Maxwell-Vlasov + Collisions

Desire to solve M-V with large inhomogeneous magnetic fields, B. Curse of dimensions and disparate time scales $\Rightarrow$ no way.

Motivates reductions: drift kinetic theory, gyrokinetic theory, ...

For example, make a kinetic theory where characteristics are drift orbits that remove the fast gyromotion in B. This is done by using near constancy of the magnetic moment μ, which on the orbit level is an adiabatic invariant. Removes fast time and lowers the dimension.

What collisions should be used? Still Landau?

In practice codes use a variety, some better than others.

Drift Kinetics on Noncanonical Phase Space

Drift orbits are governed by noncanonical Hamiltonians system, with Poisson tensor $J \to$ noncanonical Poisson bracket $[f,g]_{NC}$.

Noncanonical kinetic theory:

$$\frac{\partial f}{\partial t} + [f, \mathcal{E}]_{NC} = 0$$

Usual Vlasov $\mathcal{E} = v^2/2 + \phi$ and $[f,g]$ canonical. For noncanonical $[f,g]_{NC}$ PDE bracket has new Casimirs, inner Casimirs

$$\{F,G\} = \int dz\, f[F_f, G_f]_{NC}$$

$[c(z), f] = 0 \ \forall f$

$$C = \int dz\, c(z) f$$

Drift kinetic theories have broader choice 'entropies' for metriplectic formalism.

Collision Operator on Noncanonical Phase Space

Kulkarni-Nomizu with Poisson tensor:

$$\sigma^{ij} = \mu^{ij} = J^{ij} \Rightarrow$$
$$\mathcal{R}^{ijkl} = J^{ij} J^{kl} + J^{il} J^{kj} - J^{ki} J^{jl} - J^{ji} J^{kl}$$

Metriplectic 4-bracket:

$$(F, K; G, N) = \int dz \int dz' \, f f' \, \Gamma R^{ijkl} P\left[F_f\right]_i P\left[K_f\right]_j P\left[G_f\right]_k P\left[N_f\right]_l ,$$

where Γ determined by interaction potential gives Π and

$$P\left[F_f\right]_i = \frac{\partial}{\partial z^i} \frac{\delta F}{\delta f} - \frac{\partial}{\partial z'^i} \frac{\delta F}{\delta f'} ,$$

where z here is a mixture of x and v.

• Conservation laws, entropy production, equilibria generalization of Maxwellian involving Casimirs. Reduces to Landau. ...

Collision Operator on Noncanonical Phase Space – GC Kinetics

Why? Clusters appear and interact on shorter time scales. Drift kinetic theories and gyrokinetic theories are noncanonical.

Collision operator:

$$\left(\frac{\partial f}{\partial t}\right)_{\text{coll}} = \mathcal{C}(f,f) = \frac{\partial}{\partial z} \cdot \left[f J \cdot \int f' \, \Pi \left(J' \cdot \frac{\partial \log f'}{\partial z'} - J \cdot \frac{\partial \log f}{\partial z} \right) dz' \right]$$

where Π is a symmetric covariant (interaction) tensor, determined by type of binary interactions, and J is the Poisson tensor/operator of noncanonical Poisson bracket. Here $f' = f(z', t)$ and $J' = J(z')$.

H-Theorem $\Rightarrow$ relaxation to

$$f_\infty = \frac{1}{Z} \exp(-\beta(\mu B_0 + \frac{1}{2}\mu^2 + q\Phi) + g(\mu)) = f^{\text{MB}} e^{g(\mu)}$$

Actually fits experiments, e.g. RT-1 levitated dipole equilibria in Tokyo. On intermediate time scales $\mu \approx$ constant.

V. Final Comments

• Metriplectic 4-bracket describes thermodynamically consistent theories. Fluids, magnetofluids, multiphase fluids, kinetic theories, ... It is rich in geometry and produces interesting dynamical systems. Tons of interesting geometry already ... more to explore.

• The UT Algorithm based on the metriplectic 4-bracket, is a proven framework, provides a direct method for constructing thermodynamically consistent systems. Useful for constructing such models, even though complicated. See refs. for lots of them. Here kinetic guiding center theory. Operator ready to be put into gyrokinetic code. Simpler version linearized.

• Metriplectic 4-brackets are easy to discretize while maintaining symmetries. First numerical implementation via 4-bracket discretization (Barham et al. 2025) for 1-D Navier-Stokes-Fourier. Finite element projection of PDE to thermodynamically consistent finite-dimensional 4-bracket, i.e., ODEs. For example, for the density $\rho(x,t)$

$$\rho_h(x,t) = \sum_{i=1}^{N} \rho_i(t)\phi_i(x) \quad \to \quad \dot{\rho}_i(t) = \{\rho_i, H\} + (\rho_i, H; S, H) \ldots$$

Results use Firedrake library, implicit midpoint, Irksome module ...

References II

My articles are available at http://www.ph.utexas.edu/∼morrison

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• P. J. Morrison and Yong-Geun Oh, "Metriplectic dynamical systems on contact manifolds," (2026). (to be arXived very soon).

Overall Overview of 3 Lectures

I. Magnetofluid models

II. Dissipation: metriplectic dynamics

III. <u>Linear Vlasov as a Hamiltonian theory</u>

LECTURE III

Integral Transform for a Class of Mean-Field Theories: Action-Angle Variables for the Continuous Spectrum III

Goal: Solve linear Vlasov like one would a finite-dimensional Hamiltonian system.

G-Transform (coordinate change) Solution of LVP

Linear Vlasov:

$$\frac{\partial f_k}{\partial t} + ikvf_k - ik\phi_k \frac{e}{m}\frac{\partial f_e}{\partial v} = 0 \qquad \text{and} \qquad k^2\phi_k = 4\pi e \int_{\mathbb{R}} f_k(v,t)\,dv \qquad \text{(LVP)}$$

G transform:

$$g(u) = \widehat{G}(f(v))(u) \qquad \text{and} \qquad G \circ \widehat{G} = Id$$

$$\widehat{G}(\text{Vlasov}) \quad \to \quad \frac{\partial g_k}{\partial t} + ikug_k = 0 \quad \to \quad f_k(v,t) = G(\mathring{g}e^{-ikut}) = G\Big(\widehat{G}(\mathring{f}_k)e^{-ikut}\Big)$$

Landau Damping:

$$\rho_k(t) = \int_{\mathbb{R}} dv\, f_k(v,t) = \int_{\mathbb{R}} du\, h_k(u)e^{-ikut} \qquad \to \qquad \lim_{t\to\infty} \rho_k \sim e^{-\gamma t}$$

Riemann-Lebesgue $\Rightarrow \gamma$ closest pole of h_k in LHP.

G-Transform and Fourier Transform

- Fourier Transform for *Theory of Heat*. It turned out to be <u>more useful.</u>

- G-Transform solves LVP. It turns out to be more useful.

  ⋆ G-Transform Fokker Planck cross-over to dissipative damping. (Heninger and pjm)

  ⋆ G-Transform lab and space data. (Skiff)

Finite Hamiltonian Systems vs. Infinite Hamiltonian (Field) Theories

SHO (Finite):

Hamiltonian : $H = \frac{1}{2}\left(|\boldsymbol{v}|^2 + |\boldsymbol{x}|^2\right)$; Poisson Bracket : $[f, g] = \nabla_{\boldsymbol{x}} f \cdot \nabla_{\boldsymbol{v}} g - \nabla_{\boldsymbol{x}} g \cdot \nabla_{\boldsymbol{v}} f$

EoMs : $\dot{\boldsymbol{x}} = [\boldsymbol{x}, H] = \boldsymbol{v}$ and $\dot{\boldsymbol{v}} = [\boldsymbol{x}, H] = -\boldsymbol{x}$ $\Rightarrow$ $\ddot{\boldsymbol{x}} = -\boldsymbol{x}$

1D Wave Eq. (Infinite):

Hamiltonian : $H = \frac{1}{2}\int dx\left(\pi^2 + \psi_x^2\right)$; Poisson Bracket : $\{F, G\} = \int dx \left(\frac{\delta F}{\delta \psi}\frac{\delta G}{\delta \pi} - \frac{\delta G}{\delta \psi}\frac{\delta F}{\delta \pi}\right)$

EoMs : $\psi_t = \{\psi, H\} = \frac{\delta H}{\delta \pi} = \pi$ and $\pi_t = \{\pi, H\} = -\frac{\delta H}{\delta \psi}$ $\Rightarrow$ $\psi_{tt} = \psi_{xx}$

No functional derivative $\Rightarrow$ not Hamiltonian structure!

Finite Hamiltonian Systems → Hamiltonian Field Theories

∃ large amount of finite degree-of-freedom (DOF) Hamiltonian systems lore:

1 DOF	Integrable
2 DOF	Nonintegrable, broken tori, chaos
3 DOF	Tori are not barriers, "diffusion" around tori
N DOF	Linear & nonlinear bifurcation theory, e.g. Hamiltonian – Hopf & Krein – Moser Theorem, invariant measure, symplectic invariants, ...
.	
.	
.	
∞ DOF	All of above plus?

What doesn't carry over?

→ **Continuous Spectrum**

Hamiltonian Field Theories $\rightarrow$ Finite Hamiltonian Systems
• Vlasov Equation (1980) $\rightarrow$ Poisson Geometry

Definition. A Poisson manifold $\mathcal{Z}$ is differentiable manifold with bracket

$$\{\,,\,\}\colon C^\infty(\mathcal{Z}) \times C^\infty(\mathcal{Z}) \to C^\infty(\mathcal{Z})$$

st $C^\infty(\mathcal{Z})$ with $\{\,,\,\}$ is a Lie algebra realization, i.e., is

i) bilinear, ii) antisymmetric, iii) Jacobi, and iv) Leibniz, i.e., acts as a derivation.

Flows are integral curves of noncanonical Hamiltonian vector fields, JdH.

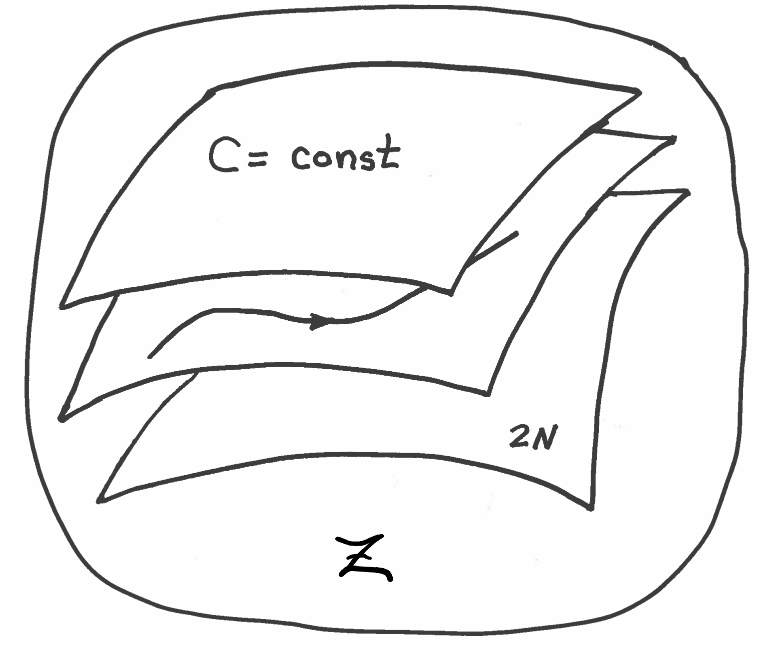
Because of degeneracy, $\exists$ functions C st $\{F, C\} = 0$ for all $F \in C^\infty(\mathcal{Z})$. Called Casimir invariants (Lie's distinguished functions!).

Poisson Manifold (phase space) $\mathcal{Z}$ Cartoon

Degeneracy in J $\Rightarrow$ Casimirs:

$$\{F, C\} = 0 \quad \forall\ F : \mathcal{Z} \to \mathbb{R}$$

Lie-Darboux Foliation by Casimir (symplectic) leaves:



Lie-Poisson Brackets

Lie-Poisson brackets are special kind of noncanonical Poisson bracket that are associated with any Lie algebra, say $\mathfrak{g}$.

Natural phase space $\mathfrak{g}^*$. For $f, g \in C^\infty(\mathfrak{g}^*)$ and $z \in \mathfrak{g}^*$.

Lie-Poisson bracket has the form

$$\begin{aligned} \{f, g\} &= \langle z, [\nabla f, \nabla g] \rangle \\ &= \frac{\partial f}{\partial z^i} c^{ij}{}_k \, z_k \frac{\partial g}{\partial z^j}, \qquad i, j, k = 1, 2, \ldots, \dim \mathfrak{g} \end{aligned}$$

Pairing $< \, , \, >$: $\mathfrak{g}^* \times \mathfrak{g} \rightarrow \mathbb{R}$, z^i coordinates for $\mathfrak{g}^*$, and $c^{ij}{}_k$ structure constants of $\mathfrak{g}$.

Vlasov Lie-Poisson Bracket:

$$\{F, G\} = \left\langle f, \left[\frac{\delta F}{\delta f}, \frac{\delta F}{\delta f}\right]\right\rangle = \int_{\mathcal{Z}} d^6 z \, f \left[\frac{\delta F}{\delta f}, \frac{\delta F}{\delta f}\right]$$

General Class of Mean-Field Hamiltonian Theories

Density:

$$\zeta(q, p, t) \qquad \text{s.t.} \qquad \zeta\colon \mathcal{Z} \times \mathbb{R} \to \mathbb{R}$$

Phase Space:

$$z := (q, p) \in \mathcal{Z} = \mathbb{T} \times \mathbb{R}$$

Equation of Motion:

$$\frac{\partial \zeta}{\partial t} + [\zeta, \mathcal{E}] = 0$$

Particle Poisson Bracket:

$$[f, g] = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial g}{\partial q}\frac{\partial f}{\partial p}$$

Particle Energy:

$$\mathcal{E}[\zeta] = \frac{\delta H}{\delta \zeta} \qquad \mathcal{E} \;\Rightarrow\; \text{nonlinear and not the Hamiltonian} H \text{ H is!}$$

Lie-Poisson Hamiltonian Structure

Hamiltonian (energy):

$$H[\zeta] = H_1 + H_2 + \cdots = \int_{\mathcal{Z}} d^2z\, h_1(z)\, \zeta(z) + \frac{1}{2}\int_{\mathcal{Z}} d^2z \int_{\mathcal{Z}} d^2z'\, \zeta(z) h_2(z,z')\, \zeta(z') + \ldots$$

Lie-Poisson Bracket:

$$\{F,G\} = \int_{\mathcal{Z}} d^2z\; \zeta\, \left[\frac{\delta F}{\delta \zeta}, \frac{\delta G}{\delta \zeta}\right] \qquad \leftarrow \text{arbitrary inner algebra}$$

Equation of Motion:

$$\frac{\partial \zeta}{\partial t} = \{\zeta, H\} = -\left[\zeta, \frac{\delta H}{\delta \zeta}\right] = -[\zeta, \mathcal{E}]$$

Casimir Invariants:

$$C[\zeta] = \int_{\mathcal{Z}} d^2z\; \mathcal{C}(\zeta)$$

Lie-Poisson Mean-Field Examples

Vlasov Poisson: $z=(x,p=mv)\,,\qquad \zeta\to f(x,p,t)\;=$ phase space density

$$
\begin{aligned}
H[f] &= \int_{\mathbb{T}\times\mathbb{R}} dx dp\, \frac{p^2}{2m}\, f(x,p) + \frac{1}{8\pi}\int_{\mathbb{R}} dx\, E^2 \\
&= \int_{\mathbb{T}\times\mathbb{R}} dx dp\, \frac{p^2}{2m}\, f(x,p) + c\int_{\mathbb{T}\times\mathbb{R}} dx dp \int_{\mathbb{T}\times\mathbb{R}} dx' dp'\, f(x,p)|x-x'|f(x',p') \\
\mathcal{E} &= \frac{\delta H}{\delta f} = \frac{p^2}{2m} + e\phi[f](x)
\end{aligned}
\tag{5}
$$

2D Euler: $z=(x,y)\,,\qquad \zeta\to\omega(x,p,t)\;=$ scalar vorticity

$$
\begin{aligned}
H[\omega] &= \int_{\mathbb{T}^2} dx dy\, \frac{v^2}{2} = \int_{\mathbb{T}^2} dx dy\, \frac{|\nabla\psi|^2}{2} \\
&= c\int_{\mathbb{T}^2} dx dy \int_{\mathbb{T}^2} dx' dy'\, \omega(x,y)\, \ln[(x-x')^2+(y-y')^2]\, \omega(x',y') \\
\mathcal{E} &= \frac{\delta H}{\delta \omega} = \psi[\omega](x,y)
\end{aligned}
$$

Other: Jeans equation, quasigeostrophy, Hasegawa-Mima, ...

Linear Normal Form with Continuous Spectrum

Stable Normal Form:

$$H = \sum_i^N \frac{\sigma_i|\omega_i|}{2}\left(p_i^2 + q_i^2\right) = i\sum_i^N \omega_i\, Q_i\, P_i = \sum_i^N \sigma_i|\omega_i| J_i \;\rightarrow\; \int du\, \sigma(u)|\omega(u)|\, J(u)$$

Stable when $\exists$ a canonical transformation to Action-Angle variables. Note important signature: $\sigma_i \in \{-1, 1\}$. Negative energy modes and Krein-Moser.

Two Complications: Noncanonical & ∞-dimensional $\rightarrow$ Continuous Spectrum

Noncanonical: $\dot{z} = \mathcal{J}(z)\partial H/\partial z = \mathcal{J}(z)\partial(H+C)/\partial z$; $\delta(H+C) = 0 \Rightarrow z_e$. Set $z = z_e + \hat{z}$

$$\dot{\hat{z}} = \mathcal{J}(z_e)\partial H_L/\partial\hat{z} \qquad \text{where} \qquad H_L = \hat{z}^T \cdot D^2F(z_e)\cdot\hat{z}/2$$

$\rightarrow$ easy matrix calculation to reduce to Casimir leaf

Infinite Dimensions: Integral transform/Coordinate Change $\rightarrow$

$$g = \widehat{G}[f]$$

$\rightarrow$ general class of transforms for Lie – Poisson brackets with CS

Example: Linear Vlasov-Poisson

Equilibrium & Linearization: $\delta(H + C) = 0 \Rightarrow f_e(v)$

$$f = f_e(v) + \hat{f}(x, v, t)$$

Linearized EOM:

$$\frac{\partial \hat{f}}{\partial t} + v\frac{\partial \hat{f}}{\partial x} - \frac{e}{m}\frac{\partial \hat{\phi}[x, t; \hat{f}]}{\partial x}\frac{\partial f_e}{\partial v} = 0$$

$$\hat{\phi}_{xx} = -4\pi e \int_{\mathbb{R}} dv\, \hat{f}(x, v, t)$$

Linearized Energy (Kruskal and Oberman):

$$H_L = -\frac{m}{2} \int_{\mathbb{T}\times\mathbb{R}} dv dx\, \frac{v\, \hat{f}^2}{f_e'} + \frac{1}{8\pi} \int_{\mathbb{T}} dx\, \hat{\phi}_x^2$$

Vlasov Lie-Poisson Bracket:

$$\{F, G\}_L = \int_{\mathbb{T}\times\mathbb{R}} dv dx\, f_e(v) \left[\frac{\delta F}{\delta \hat{f}}, \frac{\delta G}{\delta \hat{f}}\right]$$

Example: Linear Vlasov-Poisson Canonization

Fourier Series: $\hat{f} = \sum_{k\in\mathbb{Z}} f_k(v,t)e^{ikx}$ and $\hat{\phi} = \sum_{k\in\mathbb{Z}} \phi_k(t)e^{ikx}$

Linearized EOM:

$$\begin{aligned} \frac{\partial f_k}{\partial t} &+ ikvf_k - ik\phi_k \frac{e}{m}\frac{\partial f_e}{\partial v} = 0 \\ k^2\phi_k &= 4\pi e \int_{\mathbb{R}} f_k(v,t)\,dv \end{aligned} \qquad \text{(LVP)}$$

Canonical Poisson Bracket:

$$\{F,G\}_L = \sum_{k=1}^{\infty} \frac{ik}{m}\int_{\mathbb{R}} dv\ f_e' \left(\frac{\delta F}{\delta f_k}\frac{\delta G}{\delta f_{-k}} - \frac{\delta G}{\delta f_k}\frac{\delta F}{\delta f_{-k}}\right) = \sum_{k=1}^{\infty} \int_{\mathbb{R}} dv \left(\frac{\delta F}{\delta q_k}\frac{\delta G}{\delta p_k} - \frac{\delta G}{\delta q_k}\frac{\delta F}{\delta p_k}\right)$$

where $q_k(v,t) = \frac{m}{ikf_e'} f_k(v,t)$ and $p_k(v,t) = f_{-k}(v,t)$

Linear (K-O) Hamiltonian:

$$H_L = -\frac{m}{2}\sum_k \int_{\mathbb{R}} dv\, \frac{v}{f_e'}\,|f_k|^2 + \frac{1}{8\pi}\sum_k k^2|\phi_k|^2 = \sum_{k,k'}\int_{\mathbb{R}} dv \int_{\mathbb{R}} dv'\, f_k(v)\,\mathcal{H}_{k,k'}(v|v')\,f_{k'}(v')$$

IVP for LVP: Good Equilibria and Initial Conditions

Definition (VP1). A function $f_e(v)$ is a good equilibrium if $f_e'(v)$ satisfies

(i) $f_e' \in L^q(\mathbb{R}) \cap C^{0,\alpha}(\mathbb{R})$, q st $1 < q < \infty$ and α st $0 < \alpha < 1$,

(ii) $\exists\, v* > 0$ st $|f_e'(v)| < A|v|^{-\mu}\ \forall\, |v| > v*$, where $A > 0$ and $\mu > 0$, and

(iii) $f_e'/v < 0 \quad \forall\, v \in \mathbb{R}$ or f_e is Penrose stable. Assume $f_e'(0) = 0$.

Definition (VP2). A function, $\mathring{f}_k(v)$, is a good initial condition if it satisfies

(i) $\mathring{f}_k(v), v\mathring{f}_k(v) \in L^p(\mathbb{R})$,

(ii) $\int_{\mathbb{R}} \mathring{f}_k(v)\, dv < \infty$.

Good equilibria imply only continuous spectrum, while good initial conditions are physically reasonable and make theorems work. Not optimal.

Hilbert Transform Review

Hilbert transform:

$$H[g](x) := \frac{1}{\pi} \fint_{\mathbb{R}} dt \, \frac{g(t)}{t - x}$$

$\exists$ theorems about Hilbert transforms in $L^p(\mathbb{R})$ and $C^{0,\alpha}(\mathbb{R})$. Plemelj, M. Riesz, Zygmund, and Titchmarsh $\cdots$ Can be extracted from Calderón-Zygmund theory.

Theorem (H1).

(ii) $H\colon L^p(\mathbb{R}) \to L^p(\mathbb{R})$, for $1 < p < \infty$, is a bounded linear operator:

$$\|H[g]\|_p \leq A_p \, \|g\|_p \, ,$$

A_p depends only on p,

(ii) H has an inverse on $L^p(\mathbb{R})$, given by

$$H[H[g]] = -g \, ,$$

(iii) $H\colon L^p(\mathbb{R}) \cap C^{0,\alpha}(\mathbb{R}) \to L^p(\mathbb{R}) \cap C^{0,\alpha}(\mathbb{R})$.

Hilbert Transform Review Continued

Theorem (H2). If $g_1 \in L^p(\mathbb{R})$ and $g_2 \in L^q(\mathbb{R})$ with $\frac{1}{p} + \frac{1}{q} < 1$, then

$$H\Big[g_1 H[g_2] + g_2 H[g_1]\Big] = H[g_1]H[g_2] - g_1 g_2 \,.$$

The proof, based on the Hardy-Poincaré-Bertrand theorem, is due to Tricomi.

Lemma (H3). If $vg \in L^p(\mathbb{R})$, then

$$H[vg](u) = u\, H[g](u) + \frac{1}{\pi} \int_{\mathbb{R}} g \, dv \,.$$

prf. $\frac{v}{v-u} = \frac{u+v-u}{v-u} = \frac{u}{v-u} + 1$

G-Transform

Definition (G1). The G-transform is defined by

$$\begin{aligned} f(v) &= G[g](v) \\ &:= \epsilon_R(v)\, g(v) + \epsilon_I(v)\, H[g](v)\,, \end{aligned}$$

where

$$\epsilon_I(v) = -\pi \frac{\omega_p^2}{k^2} \frac{\partial f_e(v)}{\partial v}\,, \qquad \epsilon_R(v) = 1 + H[\epsilon_I](v)\,, \qquad \text{Both } \mathbb{R} \mapsto \mathbb{R}\,.$$

Remarks.

- We suppress the dependence of $\epsilon_{I,R}$ on k throughout. Note, $\omega_p^2 := 4\pi n_0 e^2/m$ is the plasma frequency corresponding to an equilibrium of number density n_0.

- $\epsilon = \epsilon_R + i\epsilon_I$ (complex extended) is the plasma dispersion relation s.t. vanishing $\Rightarrow$ discrete normal eigenmodes. When $\epsilon \neq 0$ $\exists$ only continuous spectrum; i.e. no dispersion relation.

- $\epsilon_I \propto f_e' \in L^q(\mathbb{R}) \cap C^{0,\alpha}(\mathbb{R}) \Rightarrow \epsilon_R - 1 \in L^q(\mathbb{R}) \cap C^{0,\alpha}(\mathbb{R})$, and since $\lim_{|v|\to\infty} \epsilon_I = 0$, $\lim_{|v|\to\infty} \epsilon_R = 1$, both $\epsilon_R\,, \epsilon_I \in L_\infty(\mathbb{R})$.

G-Transform Properties

Theorem (G2). $G\colon L^p(\mathbb{R}) \to L^p(\mathbb{R})$, $1 < p < \infty$, is a bounded linear operator:

$$\|G[g]\|_p \le B_p \, \|g\|_p \,,$$

where B_p depends only on p.

Theorem (G3). If f_e is a good equilibrium, then $G[g]$ has an inverse,

$$\widehat{G}\colon L^p(\mathbb{R}) \to L^p(\mathbb{R})\,,$$

for $1/p + 1/q < 1$, given by

$$\begin{aligned} g(u) &= \widehat{G}[f](u) \\ &:= \frac{\epsilon_R(u)}{|\epsilon(u)|^2}\, f(u) - \frac{\epsilon_I(u)}{|\epsilon(u)|^2}\, H[f](u)\,. \end{aligned}$$

where $|\epsilon|^2 := \epsilon_R^2 + \epsilon_I^2$.

(G3) Proof

That $\widehat{G}$ is the inverse follows directly upon inserting $G[g]$ of (G1) into $g = \widehat{G}[G[g]]$, and using (H2) and $\epsilon_R(v) = 1 + H[\epsilon_I]$.

$$
\begin{aligned}
g(u) &= \widehat{G}[f](u) = \frac{\epsilon_R(u)}{|\epsilon(u)|^2}\, f(u) - \frac{\epsilon_I(u)}{|\epsilon(u)|^2}\, H[f](u) \\
&= \frac{\epsilon_R(u)}{|\epsilon(u)|^2}\, [\epsilon_R(u)\, g(u) + \epsilon_I(u)\, H[g](u)] - \frac{\epsilon_I(u)}{|\epsilon(u)|^2}\, H\left[\epsilon_R(u')\, g(u') + \epsilon_I(u')\, H[g](u')\right](u) \\
&= \frac{\epsilon_R^2(u)}{|\epsilon(u)|^2}\, g(u) + \frac{\epsilon_R(u)\epsilon_I(u)}{|\epsilon(u)|^2} H[g](u) - \frac{\epsilon_I(u)}{|\epsilon(u)|^2} H[g](u) - \frac{\epsilon_I(u)}{|\epsilon(u)|^2} H\left[H[\epsilon_I]\, g + \epsilon_I\, H[g]\right](u) \\
&= \frac{\epsilon_R^2(u)}{|\epsilon(u)|^2}\, g(u) + \frac{\epsilon_R(u)\epsilon_I(u)}{|\epsilon(u)|^2} H[g](u) - \frac{\epsilon_I(u)}{|\epsilon(u)|^2} H[g](u) - \frac{\epsilon_I(u)}{|\epsilon(u)|^2}\, [H[\epsilon_I](u) H[g](u) - g(u)\, \epsilon_I(u)] \\
&= g(u) + \frac{\epsilon_R(u)\epsilon_I(u)}{|\epsilon(u)|^2} H[g] - \frac{\epsilon_I(u)}{|\epsilon(u)|^2} H[g] - \frac{\epsilon_I(u)}{|\epsilon(u)|^2} H[\epsilon_I] H[g] \\
&= g(u) + \frac{\epsilon_R(u)\epsilon_I(u)}{|\epsilon(u)|^2} H[g](u) - \frac{\epsilon_I(u)}{|\epsilon(u)|^2} H[g](u)\, [1 + H[\epsilon_I](u)] \\
&= g(u) + \frac{\epsilon_R(u)\epsilon_I(u)}{|\epsilon(u)|^2} H[g](u) - \frac{\epsilon_I(u)}{|\epsilon(u)|^2} H[g](u) \epsilon_R(u) = g(u)
\end{aligned}
$$

G - Transform Properties Continued

Lemma (G4). If ϵ_I and ϵ_R are as above, then

(i) for $vf \in L^p(\mathbb{R})$,

$$\widehat{G}[vf](u) = u\,\widehat{G}[f](u) - \frac{\epsilon_I}{|\epsilon|^2}\frac{1}{\pi}\int_{\mathbb{R}} f\,dv\,,$$

(ii) $\widehat{G}[\epsilon_I](u) = \frac{\epsilon_I(u)}{|\epsilon|^2(u)}$

(iii) and if $f(u,t)$ and $g(v,t)$ are strongly differentiable in t; i.e. the mapping $t \mapsto f(t) = f(t,\cdot) \in L^p(\mathbb{R})$ is differentiable, (the usual difference quotient converges in the L^p sense), then

a) $\widehat{G}\left[\frac{\partial f}{\partial t}\right] = \frac{\partial \widehat{G}[f]}{\partial t} = \frac{\partial g}{\partial t}$,

b) $G\left[\frac{\partial g}{\partial t}\right] = \frac{\partial G[g]}{\partial t} = \frac{\partial f}{\partial t}$.

prf. (i) goes through like (H3), (ii) follows from $\epsilon_R = 1 + H[\epsilon_I]$, and (iii) follows because G is bounded and linear.

G-Morphism?

$$G[f] = \epsilon_R(v)\, f(v) + \epsilon_I(v)\, H[f](v)$$

$$\widehat{G}[f](u) = \frac{\epsilon_R(u)}{\epsilon_R^2 + \epsilon_I^2}\, f(u) - \frac{\epsilon_I(u)}{\epsilon_R^2 + \epsilon_I^2}\, H[f](u)$$

Compare with $z \in \mathbb{C}$: complex numbers $i^2 = -1$, while $H \circ H = -I$.

$$z = x + iy$$

$$z^{-1} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}$$

Algebraic complex structure?

Diagonalize via Mixed Variable Generating Function

Generating Functional:

$$\begin{aligned}\mathcal{F}[q,P] &= \sum_{k=1}^{\infty} \int_{\mathbb{R}} q_k(v)\, G[P_k](v)\, dv \\ &= \sum_{k=1}^{\infty} \left(\int_{\mathbb{R}} \epsilon_R(v)\, q_k(v)\, P_k(v)\, dv + \frac{1}{\pi} ⨍_{\mathbb{R}} \int_{\mathbb{R}} \frac{\epsilon_I(v)}{u-v}\, q_k(v)\, P_k(u)\, dv du \right)\end{aligned}$$

Canonical Coordinate change $(q,p) \longleftrightarrow (Q,P)$:

$$p_k(v) = \frac{\delta \mathcal{F}[q,P]}{\delta q_k(v)} = G[P_k](v) \qquad Q_k(u) = \frac{\delta \mathcal{F}[q,P]}{\delta P_k(u)} = G^{\dagger}[q_k](u)$$

Hamiltonian in new coords:

$$H_L = \sum_{k=1}^{\infty} \int_{\mathbb{R}} i\, \omega_k(u)\, Q_k(u)\, P_k(u)\, du$$

where $\omega_k(u) = ku$.

Signature and Action-Angle variables

Elementary Coord. Change:

$$(Q_k, P_k) \quad \longleftrightarrow \quad (\theta_k, J_k)$$

Hamiltonian in new coords.:

$$H_L = \sum_{k=1}^{\infty} \int_{\mathbb{R}} \sigma_k(u)\, \omega_k(u) J_k(u,t)\, du\,,$$

where $\omega_k(u) := |ku|$ and $\sigma_k(u) := \text{sign}(ku\epsilon_I)$.

Poisson bracket:

$$\{F, G\}_L = \sum_{k=1}^{\infty} \int_{\mathbb{R}} \left(\frac{\delta F}{\delta \theta_k} \frac{\delta G}{\delta J_k} - \frac{\delta G}{\delta \theta_k} \frac{\delta F}{\delta J_k} \right) du\,.$$

Continuum eigenmodes have signature. Finite DOF, Krein-Moser says opposite signature needed for bifurcations: colliding $\omega_i(\lambda) \rightarrow$ instability. Vlasov: unstable modes emerge where signatures meet.

Works for Class of Lie-Poisson Hamiltonian Systems: Recall

Hamiltonian (energy):

$$H[\zeta] = H_1 + H_2 = \int_{\mathcal{Z}} d^2z\, h_1(z)\, \zeta(z) + \frac{1}{2} \int_{\mathcal{Z}} d^2z \int_{\mathcal{Z}} d^2z'\, \zeta(z) h_2(z,z')\, \zeta(z')$$

Lie-Poisson Bracket:

$$\{F,G\} = \int_{\mathcal{Z}} d^2z\ \zeta\ \left[\frac{\delta F}{\delta \zeta}, \frac{\delta G}{\delta \zeta}\right]$$

Equation of Motion:

$$\frac{\partial \zeta}{\partial t} = \{\zeta, H\} = -\left[\zeta, \frac{\delta H}{\delta \zeta}\right] = -[\zeta, \mathcal{E}]$$

Casimir Invariants:

$$C[\zeta] = \int_{\mathcal{Z}} d^2z\ \mathcal{C}(\zeta)$$

Lie-Poisson Hamiltonian Normal Form

Equilibria:

$$\frac{\partial\zeta}{\partial t} = 0 = \{\zeta, H\} = \left[-\zeta, \frac{\partial H}{\partial \zeta}\right] = -[\zeta_e, \mathcal{E}_e]$$

1 DOF Integrability:

$$z := (q,p) \longleftrightarrow (\theta, J) \quad \Rightarrow (\zeta_e(J), \mathcal{E}_e(J)) \quad \text{or} \quad \zeta_e(\mathcal{E}_e) \Rightarrow$$

Hammerstein IE:

$$\mathcal{E}_e(z) = h_1(z) + \int_{\mathcal{Z}} d^2z'\, h_2(z,z')\, \zeta_e(\mathcal{E}_e(z'))$$

Linear EOM:

$$\frac{\partial\hat{\zeta}}{\partial t} + [\hat{\zeta}, \mathcal{E}_e] + [\zeta_e, \hat{\mathcal{E}}] = 0 \quad \text{or} \quad \frac{\partial\hat{\zeta}}{\partial t} + \Omega(J)\frac{\partial\hat{\zeta}}{\partial\theta} = \frac{d\zeta_e}{dJ}\frac{\partial\hat{\mathcal{E}}}{\partial\theta}\,.$$

where $\Omega(J) := d\mathcal{E}_e/dJ$, $\hat{\mathcal{E}} = \int_{\mathcal{Z}} d^2z'\, h_2(z,z')\, \hat{\zeta}(z')$ in terms of (θ, J)

Linear Operator Problem

Fourier series:

$$\hat{\zeta} = \sum_k \zeta_k(J)\; e^{ik\theta - ik\omega t}$$

Eigenvalue problem:

$$\mathcal{L}_k \zeta_k := \Omega(J)\zeta_k - \zeta_e'\, \mathcal{E}_k[\zeta_k] = \omega\, \zeta_k\,,$$

with

$$\mathcal{E}_k(J) = \sum_{k'} \int \mathcal{H}_{k,k'}(J,J')\, \zeta_{k'}(J')\, dJ'\,,$$

where $\mathcal{L}_k\colon \mathcal{B} \to \mathcal{B}$, Banach space $\mathcal{B}$, eigenvalue ω, and $\mathcal{H}_{k,k'}(J,J')$ comes from h_2.

Partition the spectrum of $\mathcal{L}_k$: $\sigma = \sigma_p \cup \sigma_c \cup \sigma_r$. (i) $\omega \in \sigma_p$, point spectrum, if $\mathcal{L}_k - \omega\mathcal{I}$ is not one-one, where $\mathcal{I}$ is the identity operator. (ii) $\omega \in \sigma_R$, residual spectrum, if the range of $\mathcal{L}_k - \omega\mathcal{I}$ is not dense in $\mathcal{B}$. (iii) $\omega \in \sigma_c$, continuous spectrum, if the inverse of $(\mathcal{L}_k - \omega\mathcal{I})$, defined on its range, is unbounded.

This partition convenient because if σ_r is null, then the approximate or Weyl spectrum corresponds to $\sigma_p \cup \sigma_c$. Assume purely σ_c , via energy-Casimir, e.g.

Generalized G-transform

Associate Integral Equation:

$$\mathcal{E}_k(J, J_\omega) = \sum_{k'} \mathcal{H}_{k,k'}(J, J_\omega) + \sum_{k'} \int \mathcal{E}_{k'}(J', J_\omega)\, \mathcal{F}_{k,k'}(J, J', J_\omega)\, dJ' \,,$$

where

$$\mathcal{F}_{k,k'}(J, J', J_\omega) := \left[\frac{\mathcal{H}_{k,k'}(J, J') - \mathcal{H}_{k,k'}(J, J_\omega)}{\Omega(J') - \Omega(J_\omega)}\right] \zeta_e'(J')$$

well-behaved enough for Fredholm theory.

Transform:

$$G_k[g_k](J,t) := \epsilon_k^R(J)\, g_k(J,t) + \int \frac{\zeta_e'(J)\mathcal{E}_k(J, J_\omega)}{\Omega(J) - \Omega(J_\omega)}\, g_k(J_\omega, t)\, dJ_\omega \,,$$

with

$$\epsilon_k^R(J_\omega) := 1 - \int \frac{\zeta_e'(J)\mathcal{E}_k(J, J_\omega)}{\Omega(J) - \Omega(J_\omega)}\, dJ \,.$$

Generalized G-transform: Inverse & Identities

Transform Inverse:

$$\widehat{G}_k[f_k](J_\omega,t) := \frac{1}{|\epsilon_k(J_\omega)|^2}\left[\epsilon_k^R(J_\omega)f_k(J_\omega,t) + ⨍ \frac{\zeta_e'(J_\omega)\mathcal{E}_k(J,J_\omega)}{\Omega(J)-\Omega(J_\omega)}f_k(J,t)\,dJ\right],$$

where $|\epsilon_k(J)|^2 := (\epsilon_k^R)^2 + (\epsilon_k^I)^2$ and $\epsilon_k^I(J_\omega) := \pi\mathcal{E}_k(J_\omega,J_\omega)\zeta_e'(J_\omega)/\Omega'(J_\omega)$.

That $\widehat{G}\circ G = Id$ follows from Poincaré-Bertrand theorem on the interchange of the order of integration for singular integrals.

Transform Identities:

$$\widehat{G}_k[\Omega\,\zeta_k](J_\omega) = \Omega(J_\omega)\widehat{G}_k[\zeta_k](J_\omega) + \frac{\zeta_e'(J_\omega)}{|\epsilon_k|^2(J_\omega)}⨍\,\zeta_k(J,t)\,\mathcal{E}_k(J,J_\omega)\,dJ\,,$$

and

$$\widehat{G}_k[\zeta_e'\mathcal{E}_k](J_\omega) = \frac{\zeta_e'(J_\omega)}{|\epsilon_k|^2(J_\omega)}⨍\,\zeta_k(J)\,\mathcal{E}_k(J,J_\omega)\,dJ\,.$$

Shown by techniques similar to those used for verifying the inverse.

General Canonization and Diagonalization

Hamiltonian:

$$\begin{aligned} H_L &= \delta^2 H + \frac{1}{2}\int dJ d\theta\, \mathcal{C}''(\zeta_e)\,(\delta\zeta)^2 = \delta^2 H - \frac{1}{2}\int dJ d\theta\, \frac{\mathcal{E}_e'(J)}{\zeta_e'(J)}\,(\delta\zeta)^2 \\ &= \frac{1}{2}\sum_{k,k'}\int\int dJ dJ'\,\zeta_k(J)\,\mathcal{H}_{k,k'}(J,J')\,\zeta_{k'}(J') - \frac{1}{2}\sum_k \int dJ\,\frac{\mathcal{E}_e'(J)}{\zeta_e'(J)}\,\zeta_{-k}\zeta_k\,. \end{aligned}$$

Poisson bracket:

$$\{F,G\}_L = \int d\theta dJ\,\zeta_e(J)\left[\frac{\delta F}{\delta\hat{\zeta}},\frac{\delta G}{\delta\hat{\zeta}}\right] = \sum_{k=1}^{\infty} ik\int dJ\,\zeta_e'\left(\frac{\delta F}{\delta\zeta_k}\frac{\delta G}{\delta\zeta_{-k}} - \frac{\delta G}{\delta\zeta_k}\frac{\delta F}{\delta\zeta_{-k}}\right)\,.$$

Linear dynamics:

$$\frac{\partial\hat{\zeta}}{\partial t} = \{\hat{\zeta}, H_L\}_L\,.$$

General Canonization and Diagonalization Cont.

Canonization:

$$q_k(J,t) := \zeta_k(J,t)\,,p_k(J,t) = \frac{\zeta_{-k}(J,t)}{ik\zeta_e'} \quad \rightarrow \quad \{F,G\}_L = \sum_{k=1}^{\infty}\int dJ\left(\frac{\delta F}{\delta q_k}\frac{\delta G}{\delta p_k} - \frac{\delta G}{\delta q_k}\frac{\delta F}{\delta p_k}\right).$$

Diagonalization:

$$\mathcal{F}[q,P] = \sum_{k=1}^{\infty}\int dJ\,P_k(J)\,\widehat{G}[q_k](J) \qquad (q_k,p_k) \longleftrightarrow (Q_k,P_k)$$

Type-2 mixed variable generating functional again.

$$p_k(J) = \frac{\delta\mathcal{F}[q,P]}{\delta q_k(J)} = \widehat{G}^\dagger[P_k](J) \quad \text{and} \quad Q_k(J) = \frac{\delta\mathcal{F}[q,P]}{\delta P_k(J)} = \widehat{G}[q_k](J)\,.$$

Hamiltonian in New Coords:

$$\begin{aligned} H_L &= \sum_{k=1}^{\infty} ik\int dJ\,p_k\left[\zeta_e'\mathcal{E}_k - q_k\mathcal{E}_e'\right] = \sum_{k=1}^{\infty} ik\int dJ\,P_k\left(\widehat{G}[\zeta_e'\mathcal{E}_k] - \widehat{G}[\mathcal{E}_k' G[Q_k]]\right) \\ &= -\sum_{k=1}^{\infty}\int dJ\,ik\,\Omega(J)Q_k(J)\,P_k(J)\,. \end{aligned}$$

Continuation

- Investigation of the consequences of the signature of the continuous spectrum; *i.e.* proof of a kind of Krein-Moser theorem in a Banach space setting where embedded discrete modes emerge from negative σ_c. (George Hagstrom Ph.D. 2011)
- Investigate the theory of adiabatic invariants in this infinite dimensional Hamiltonian context by e.g. adding explicit time dependence to the Hamiltonian.
- Develop analog of Birkhoff's <u>nonlinear</u> normal forms for our class of infinite dimensional Hamiltonian systems with continuous spectra. (Thomas Yudichak Ph.D. 2001)
- Obtain our class of infinite dimensional Hamiltonian mean-field systems by reduction from kinetic theory BBGKY, other.
- Investigate the role played by G-transform in an infinite-dimensional setting of functional phase space tangent bundle geometry. Symplectomorphism algebra? $\mathbb{C}$-Morphism?

Thank You!

References III

My articles are available at http://www.ph.utexas.edu/∼morrison

The G-transform for diagonalizing the Vlasov-Poisson system was introduce for physicists in

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• P. J. Morrison and B. Shadwick, "Canonization and diagonalization of an infinite dimensional noncanonical Hamiltonian system: linear Vlasov theory," Acta Phys. Pol., **85**, 759–769 (1994).

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Treatment of a stable equilibrium with an eigenvalue embedded in the point spectrum was given in

• B .A. Shadwick and P. J. Morrison, On Neutral Plasma Oscillations," Phys. Lett. A **184**, 277–282 (1994).

The more rigorous version of the G-transform was published in

• P. J. Morrison. "Hamiltonian description of Vlasov dynamics: Action-angle variables for the continuous spectrum," Trans. Theory and Stat. Phys. **29**, 397–414 (2000).

The technique for assigning a signature to the continuous spectrum was introduced in the refs above, with a rigorous analog of the Hamiltonian-Hopf (Krein-Moser) bifurcation with a signed continuous spectrum was given in the following:

• G. I. Hagstrom and P. J. Morrison, "On Krein-like theorems for noncanonical Hamiltonian systems with continuous spectra: Application to Vlasov-Poisson," Trans. Theory and Stat. Phys. **39**, 466–501 (2010).

References III (cont.)

A tutorial treatment (including waterbags) of the G-transform is given in

• P. J. Morrison and G. Hagstrom, "Continuum Hamiltonian–Hopf bifurcation i," in O. Kirillov and D. Pelinovsky, editors, Nonlinear Physical Systems – Spectral Analysis, Stability and Bifurcations. (Wiley, 2014).

• G. Hagstrom and P. J. Morrison "Continuum Hamiltonian–Hopf bifurcation ii," in O. Kirillov and D. Pelinovsky, editors, Nonlinear Physical Systems – Spectral Analysis, Stability and Bifurcations. (Wiley, 2014).

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• N. J. Balmforth and P. J. Morrison, "Hamiltonian description of shear flow," in J. Norbury and I. Roulstone, editors, Large-Scale Atmosphere-Ocean Dynamics 2: Geometric Methods and Models, (Cambridge University Press, Cambridge, U.K., 2001).

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• P. J. Morrison, "Hamiltonian description of fluid and plasma systems with continuous spectra," in O. U. Velasco Fuentes, J. Sheinbaum, and J. Ochoa, editors, Nonlinear Processes in Geophysical Fluid Dynamics, pages 53–69 (Kluwer, Dordrecht, 2003).