DEPARTMENT OF MATHEMATICS
University of California, Berkeley

J. Marsden — Wednesday 3:15-4:00

Math 290
DYNAMICS SEMINAR — 225 Bechtel

March 21, 1984 - John Guckenheimer, U.C. Santa Cruz:

"Hopf Bifurcation with Symmetry"

March 28, 1984 - Philip J. Morrison, Institute for Fusion Studies, Austin, and Dept. of Mathematics, U.C.Berkeley:

"Canonical Lie-Poisson Brackets
(They Arise in Tokamak Models)"

CANONICAL LIE-POISSON BRACKETS

(They arise in Tokamak Models)

Philip Morrison

Math. Dept. Berkeley CA (until June.)
IFS & Physics Dept., University of Texas at Austin

March 28, 1984

Overview

I. Introduction

II. Brackets (review)

III. CLP Brackets

IV. Tokamak Models (Examples of CLP brackets)

MHD BRACKET

$$H = \int_V \left[\frac{1}{2} \rho v^2 + \rho U(\rho, s) + \frac{B^2}{2} \right] d\tau$$

$$
\begin{aligned}
[F,G] = \\
& -\int_V \Bigg\{ \left[\frac{\delta F}{\delta \rho} \nabla \cdot \frac{\delta G}{\delta \vec{v}} + \frac{\delta F}{\delta \vec{v}} \cdot \nabla \frac{\delta G}{\delta \rho} \right] \\
& + \left[\rho^{-1} (\nabla \times \vec{v}) \cdot \left(\frac{\delta G}{\delta \vec{v}} \times \frac{\delta F}{\delta \vec{v}} \right) \right] \\
& + \left[\rho^{-1} \nabla s \cdot \left(\frac{\delta F}{\delta s} \frac{\delta G}{\delta \vec{v}} - \frac{\delta G}{\delta s} \frac{\delta F}{\delta \vec{v}} \right) \right] \\
& + \vec{B} \cdot \left[\frac{1}{\rho} \frac{\delta F}{\delta \vec{v}} \cdot \nabla \frac{\delta G}{\delta \vec{B}} - \frac{1}{\rho} \frac{\delta G}{\delta \vec{v}} \cdot \nabla \frac{\delta F}{\delta \vec{B}} \right] \\
& + \vec{B} \cdot \left[\left(\nabla \frac{1}{\rho} \frac{\delta F}{\delta \vec{v}} \right) \cdot \frac{\delta G}{\delta \vec{B}} - \left(\nabla \frac{1}{\rho} \frac{\delta G}{\delta \vec{v}} \right) \cdot \frac{\delta F}{\delta \vec{B}} \right] \Bigg\} d\tau
\end{aligned}
$$

$$
\begin{aligned}
[F,G] = & -\int_V \Bigg\{ \rho \left[\frac{\delta F}{\delta \vec{M}} \cdot \nabla \frac{\delta G}{\delta \rho} - \frac{\delta G}{\delta \vec{M}} \cdot \nabla \frac{\delta F}{\delta \rho} \right] \\
& + \vec{M} \cdot \left[\frac{\delta F}{\delta \vec{M}} \cdot \nabla \frac{\delta G}{\delta \vec{M}} - \frac{\delta G}{\delta \vec{M}} \cdot \nabla \frac{\delta F}{\delta \vec{M}} \right] \\
& + \sigma \left[\frac{\delta F}{\delta \vec{M}} \cdot \nabla \frac{\delta G}{\delta \sigma} - \frac{\delta G}{\delta \vec{M}} \cdot \nabla \frac{\delta F}{\delta \sigma} \right] + \vec{B} \cdot \left[\frac{\delta F}{\delta \vec{M}} \cdot \nabla \frac{\delta G}{\delta \vec{B}} - \frac{\delta G}{\delta \vec{M}} \cdot \nabla \frac{\delta F}{\delta \vec{B}} \right] \\
& + \vec{B} \cdot \left[\left(\nabla \frac{\delta F}{\delta \vec{M}} \right) \cdot \frac{\delta G}{\delta \vec{B}} - \left(\nabla \frac{\delta G}{\delta \vec{M}} \right) \cdot \frac{\delta F}{\delta \vec{B}} \right] \Bigg\} d\tau
\end{aligned}
$$

$$\rho_t = [\rho, H] \qquad \vec{B}_t = [\vec{B}, H] \qquad \sigma_t = [\sigma, H]$$

$$\vec{M}_t = [\vec{M}, H]$$

II. BRACKETS (Review)

$\vec{\Psi}_t = \vec{L}(\vec{\Psi})$ — nonlinear evolution eq.; e.g. p.d.e.

$\Psi_i : D \longrightarrow \mathbb{R}$ — $D \subset \mathbb{R}^{2n}$

suppose: $\vec{\Psi} \in V \equiv$ Real vector space with $\mathbb{L}^2$ inner pdt. $\langle , \rangle$

consider: $\mathcal{V} \equiv$ Space of Frechet differentiable functionals; e.g. $F : V \longrightarrow \mathbb{R}$

write as $F[\vec{\Psi}]$; e.g. $F[\vec{\Psi}] = \int_D \mathcal{F}(\Psi)\, dz$

Frechet Derivative:

$$\vec{DF} \cdot \vec{\Phi} = \frac{d}{d\varepsilon} F[\vec{\Psi} + \varepsilon \vec{\Phi}]\Big|_{\varepsilon=0} = \left\langle \frac{\delta F}{\delta \vec{\Psi}}, \vec{\Phi} \right\rangle$$

exists & linear in $\vec{\Phi}$

$$= \left\langle \frac{\delta F}{\delta \Psi_i}, \Phi_i \right\rangle$$

$\frac{\delta F}{\delta \vec{\Psi}}$ is a nonlinear operator on V "gradient"

Generalized Poisson Bracket :

Bilinear operator, $\{\ ,\ \} : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$

of the form

$$\{F, G\} = \left\langle \frac{\delta F}{\delta \psi_i}, \mathcal{O}^{ij} \frac{\delta G}{\delta \psi_j} \right\rangle$$

where $\mathcal{O}^{ij}$ is a nonlinear operator on V

Also must satisfy

(1) $\{F, F\} = 0 \quad \forall F \in \mathcal{V}$ $\qquad \overleftrightarrow{\mathcal{O}}^{\dagger} = -\overleftrightarrow{\mathcal{O}}$

(2) Jacobi identity

$$\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0$$

$\forall F, G, H \in \mathcal{V}$

$\mathcal{G} = \{\mathcal{V}, \{\ ,\ \}\}$ Lie algebra of Functionals ("outer") "big"

Hamiltonian Evolution Eqs. :

$\Psi_t = \{\Psi, H\}$ $\qquad \Psi[\vec{\psi}] \in \mathcal{G}$ any functional of $\vec{\psi}$

$H[\vec{\psi}] \equiv$ Hamiltonian funct

Jacobi Identity

must take $\frac{\delta}{\delta\psi_i}\{F, G\}$?

Lemma. Second variation has following symmetry:

$$\left\langle \frac{\delta^2 F}{\delta\psi_i \delta\psi_j}\phi, \Theta \right\rangle = \left\langle \phi, \frac{\delta^2 F}{\delta\psi_j \delta\psi_i}\Theta \right\rangle$$

Proof.

$$\frac{d}{d\varepsilon} F[\psi_i + \varepsilon\phi]\Big|_{\varepsilon=0} = \left\langle \frac{\delta F}{\delta\psi_i}, \phi \right\rangle \equiv G_\phi[\vec{\Psi}]$$

$$\frac{d}{d\varepsilon} G_\phi[\psi_j + \varepsilon\Theta]\Big|_{\varepsilon=0} = \left\langle \frac{\delta^2 F}{\delta\psi_j \delta\psi_i}\Theta, \phi \right\rangle$$

Interchange order; Θ first ϕ second

∎

<u>Theorem.</u> When taking $\frac{\delta}{\delta\psi_i}\{F,G\}$, only the $\mathcal{O}^{ij}$ contribution matters in Jacobi. The res[t] automatically vanishes.

<u>Consider:</u> $$\frac{d}{d\varepsilon}\{F,G\}[\psi_i+\varepsilon\phi]\Big|_{\varepsilon=0} =$$

$$\frac{d}{d\varepsilon}\left\langle \frac{\delta F}{\delta\psi_i}[\psi_i+\varepsilon\phi],\ \mathcal{O}^{ij}[\psi_i+\varepsilon\phi]\frac{\delta G}{\delta\psi_j}[\psi_i+\varepsilon\phi]\right\rangle_{\varepsilon}$$

<u>Obtain:</u> $$\frac{\delta\{F,G\}}{\delta\psi_k} = \frac{\delta^2 F}{\delta\psi_i\,\delta\psi_k}\mathcal{O}^{ij}\frac{\delta G}{\delta\psi_j}$$

$$- \frac{\delta^2 G}{\delta\psi_j\,\delta\psi_k}\mathcal{O}^{ij}\frac{\delta F}{\delta\psi_i} + \underbrace{\frac{1}{\phi}\left\langle\frac{\delta F}{\delta\psi_i},\frac{d}{d\varepsilon}\mathcal{O}^{ij}[\psi_k+\varepsilon\phi]\frac{\delta G}{\delta\psi_j}\right\rangle}$$

$\overleftrightarrow{\mathcal{O}}^{+} = -\overleftrightarrow{\mathcal{O}}$ & Lemma $\Longrightarrow$ only surviving term

III CANONICAL LIE POISSON BRACKETS

LP Brackets : Have underlying Lie group
O^{ij} linear in ψ^i – w/ derivatives

Definitions :

$G \equiv$ Lie group

$\mathfrak{g} \equiv$ Lie algebra of G ; $\mathfrak{g} \xrightarrow{\exp} G$.
$\mathfrak{g}$ has bracket $[\,,\,]$

$\mathfrak{g}^* \equiv$ dual of $\mathfrak{g}$

$\langle\,,\,\rangle \equiv$ pairing between $\mathfrak{g}$ & $\mathfrak{g}^*$

$$DF(\vec{\Psi})\cdot\vec{\Phi} = \left\langle \vec{\Phi}, \frac{\delta F}{\delta \vec{\Psi}} \right\rangle$$

$\vec{\psi} \in \mathfrak{g}^*$

$\frac{\delta F}{\delta \vec{\psi}} \in \mathfrak{g}$

Bracket :
$F, G : \mathfrak{g}^* \to \mathbb{R}$

$$\{F, G\} = \left\langle \vec{\psi}, \left[\frac{\delta F}{\delta \vec{\psi}}, \frac{\delta G}{\delta \vec{\psi}} \right] \right\rangle$$

"Big" "outer" algebra

"inner"
"little" algebra

CLP Brackets : Underlying canonical group

$G \equiv$ group of canonical transformations of $\mathbb{R}^{2n}$

$\mathfrak{g} \equiv$ algebra associated w/ G

functions on $\mathbb{R}^{2n}$ with bracket defined by

$$[f, g] = \sum_{i=1}^{n} \left[\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial x_i} \right]$$

assume $\mathfrak{g}$ has $\mathbb{L}^2$ inner product

(identify $\mathfrak{g}$ & $\mathfrak{g}^*$; restrict)

Bracket on Functionals :

$$\{ F, G \} = \langle \psi , [\frac{\delta F}{\delta \psi} , \frac{\delta G}{\delta \psi}] \rangle$$

$F, G : \mathfrak{g} \to \mathbb{R}$

Brackets by extension

$\mathfrak{g}^m \equiv$ m-fold Cartesian product of $\mathfrak{g}$

$$\vec{\Psi} = (\psi_1, \psi_2, \ldots, \psi_m)$$

s.t. $\psi_i \in \mathfrak{g} \quad \forall i$

bracket: (on $\mathfrak{g}^m$)

$$[\vec{\Psi}, \vec{\Phi}] = \vec{\chi} \quad ; \quad \vec{\Psi}, \vec{\Phi}, \vec{\chi} \in \mathfrak{g}^m$$

$$\chi_i = c^i_{jk} [\psi_j, \phi_k] \quad ; \quad c^i_{jk} \in \mathbb{R}$$

CLP bracket:

$$\{F, G\} = \left\langle \vec{\Psi}, \left[\frac{\delta F}{\delta \vec{\Psi}}, \frac{\delta G}{\delta \vec{\Psi}} \right] \right\rangle$$

coordinate form:

$$\{F, G\} = \int_D \psi_i \, C^i_{jk} \left[\frac{\delta F}{\delta \psi_j}, \frac{\delta G}{\delta \psi_k} \right] dz$$

Jacobi is satisfied for both "inner" and "outer" algebras provided

$$(1) \quad C^i_{jk} C^k_{\ell m} + C^i_{\ell k} C^k_{mj} + C^i_{mk} C^k_{j\ell} = 0$$

$\forall i$ & $(j \ell m)$ not all equal

Symmetry requires

$$(2) \quad C^i_{jk} = + C^i_{kj}$$

Example $(\mathfrak{g} \times V_{\mathfrak{g}})$

$$\{F, G\} = \int_D \left(\psi_1 \left[\frac{\delta F}{\delta \psi_1}, \frac{\delta G}{\delta \psi_1} \right] + \psi_2 \left(\left[\frac{\delta F}{\delta \psi_1}, \frac{\delta G}{\delta \psi_2} \right] + \left[\frac{\delta F}{\delta \psi_2}, \frac{\delta G}{\delta \psi_1} \right] \right) \right) dz$$

LIE-POISSON BRACKETS

Lie-Poisson brackets are those for which the σ^{ij} is linear in ψ^i. Derivatives on ψ^i can occur.

Definitions:

$G \equiv$ Lie group

$\mathfrak{g} \equiv$ Lie algebra of G. $\mathfrak{g} \xrightarrow{\text{Exp}} G$.
$\mathfrak{g}$ has bracket $[\,,\,]$

$\mathfrak{g}^* \equiv$ dual to $\mathfrak{g}$

$\langle\,,\,\rangle \equiv$ pairing (contraction) between $\mathfrak{g}^*$ & $\mathfrak{g}$.

$\frac{\delta F}{\delta \psi}$ is defined by $DF(\psi)\cdot\psi^* = \langle \psi^*, \frac{\delta F}{\delta\psi} \rangle$

Note $\frac{\delta F}{\delta \psi} \in \mathfrak{g}$ & $\psi \in \mathfrak{g}^*$

L.P. Bracket:

$$\underbrace{\{F, G\}}_{\text{Big Algebra}} = \langle \psi, \underbrace{[\frac{\delta F}{\delta\psi}, \frac{\delta G}{\delta\psi}]}_{\text{little Algebra}} \rangle$$

$F, G : \mathfrak{g}^* \to \mathbb{R}$

CASIMIR INVARIANTS

Recall. Casimirs commute with all functions;
i.e. $\{\mathcal{C}, F\} = 0 \quad \forall F$
casimir

Recall. The collection of all casimirs forms an ideal called the center of the Lie algebra – in this case "outer"

Remark. By linear imbedding we can find Casimirs of "outer" algebra from those of "inner" algebra.

Remark. g^m has trivial center $\Rightarrow$ No induced casimirs.

Question. Given "inner" algebra what can we say about "outer" algebra?

Answer. $\mathfrak{g}^m \subset \mathcal{Y}$

Consider the linear imbedding

$h : \mathfrak{g}^m \longrightarrow \mathcal{L}$ where elements of $\mathcal{L}$ have the form

$$h(\vec{f}) = \int_D \vec{\Psi} \cdot \vec{f} \, dz \in \mathcal{L}$$

$$\vec{f} \in \mathfrak{g}^m$$

h is a homomorphism

$$h([\vec{f}, \vec{g}]) = \int_D \vec{\Psi} \cdot [\vec{f}, \vec{g}] \, dz$$

$$\shortparallel$$

$$\{h(\vec{f}), h(\vec{g})\} = \int_D \vec{\Psi} \cdot [\vec{f}, \vec{g}] \, dz$$

EXAMPLE : L-P BRACKET WITH INNER CASIMIRS

"$(L\text{-}P)^2$ BRACKET"

inner bracket: $[f, g] = \vec{s} \cdot \vec{\nabla}_s f \times \vec{\nabla}_s g$

$\vec{s} = (s_1, s_2, s_3) \in \mathbb{R}^3$; $f, g : \mathbb{R}^3 \to \mathbb{R}$

$\vec{\nabla}_s = \left(\frac{\partial}{\partial s_1}, \frac{\partial}{\partial s_2}, \frac{\partial}{\partial s_3} \right)$; casimirs: $c(s^2)$

free rigid body - spin system (rotation group

outer bracket: $\{F, G\} = \int \delta(\vec{s}) \left[\frac{\delta F}{\delta \delta}, \frac{\delta G}{\delta \delta} \right] d^3 s$

spin density

$$= \int \delta(\vec{s})\, \vec{s} \cdot \vec{\nabla}_s \frac{\delta F}{\delta \delta} \times \vec{\nabla}_s \frac{\delta G}{\delta \delta}\, d^3 s$$

$F, G : \mathfrak{g} \to \mathbb{R}$

bracket has "Killing" property : $\int \delta \left[\frac{\delta F}{\delta \delta}, \frac{\delta G}{\delta \delta} \right] d^3 s$

$$= - \int \frac{\delta F}{\delta \delta} \left[\delta, \frac{\delta G}{\delta \delta} \right] d^3 s$$

<u>Casimirs for $\{F, G\}$</u>

inner Casimirs: $C_I = \int \delta(\vec{s})\, c(s^2)\, d^3 s$

outer Casimirs: $C_o = \int K(\delta)\, d^3 s$

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dynamical equation: $\delta_t = \{\delta, H\} = -[\delta, h]$

$H = \int h_1 \delta + \int h_2 \delta\delta$; $\frac{\delta H}{\delta \delta} = h$

characteristic eqs. are spin eqs.

"KILLING" FORM for CLP BRACKETS

$\langle \, , \, \rangle : \mathfrak{g}^m \longrightarrow \mathbb{R}$; symmetric, bilinear

given $\vec{f}, \vec{g}, \vec{h} \in \mathfrak{g}^m$

$$\langle \vec{f}, [\vec{g}, \vec{h}] \rangle = -\langle \vec{g}, [\vec{f}, \vec{h}]^* \rangle$$

$$\stackrel{?}{=} -\langle \vec{g}, [\vec{f}, \vec{h}] \rangle$$

$$\int f_i \, C^i_{jk} [g_j, h_k] \, dz = -\int g_i \, C^j_{ik} [f_j, h_k] \, dz$$

comparing we see that: $[\vec{f}, \vec{h}]^*_i = C^j_{ik} [f_j, h_k]$

$\stackrel{?}{=}$ follows if $\underline{C^j_{ik} = C^i_{jk}}$

EXAMPLES

$[\vec{f}, \vec{g}]$	$[\vec{f}, \vec{g}]^*$
$[f_1, g_1]$	$[f_1, g_1]$
$([f_1, g_1], [f_1, g_2] + [f_2, g_1])$	$([f_1, g_1] + [f_2, g_2], [f_2, g_1])$ Not a Lie bracket
$([f_1, g_1] + [f_2, g_2], [f_1, g_2] + [f_2, g_1])$	$([f_1, g_1] + [f_2, g_2], [f_1, g_2] + [f_2, g_1])$

Quadratic Casimirs for CLP Brackets

Let (A_{ij}) be a real constant symmetric matrix.

Consider:

$$\mathcal{C}_2 \equiv \left\langle \tfrac{1}{2} A_{ij} \psi_j \, , \psi_i \right\rangle$$

Theorem 1. $\mathcal{C}_2$ is a casimir iff the following is true:

$$A C_k - (A C_k)^T = \text{diagonal matrix} \quad \forall k \, ,$$

where

$$C_k = (c_{ijk}) \, .$$

Remark. Theorem often leads to casimirs more general than quadratic.

Proof. $\left\langle \frac{\delta G}{\delta \vec{\Psi}}, [\vec{\Psi}, A \cdot \vec{\Psi}]^* \right\rangle = 0$

CASIMIRS FOR SDP – CLP BRACKET

bracket:

$$\{F,G\} = \int \Big\{ \psi_1 \Big[\frac{\delta F}{\delta \psi_1}, \frac{\delta G}{\delta \psi_1} \Big] + \sum_{i=2}^{M} \psi_i \Big(\Big[\frac{\delta F}{\delta \psi_1}, \frac{\delta G}{\delta \psi_i} \Big] + \Big[\frac{\delta F}{\delta \psi_i}, \frac{\delta G}{\delta \psi_1} \Big] \Big) \Big\} dz$$

<u>Theorem 2</u>. The following is a casimir for SDP – CLP:

$$\mathcal{C} = \int C(\psi_2, \psi_3, \ldots, \psi_M)\, dz$$

<u>Proof</u>.

$$\{\mathcal{C}, G\} = \int \frac{\delta G}{\delta \psi_1} [\psi_i, \psi_j] \frac{\partial^2 C}{\partial \psi_i \partial \psi_j} dz = 0$$