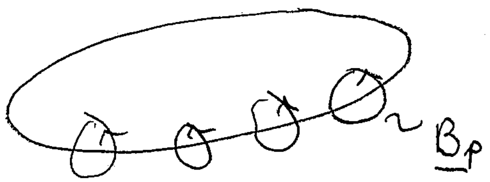
Introduction to Hamiltonian Chaos

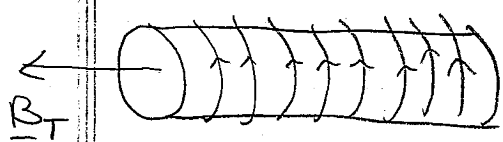
Philip Morrison
April 29, 1988
ARL

Magnetic Field as Hamiltonian System

Consider the simple configuration of a current ring. It is well-known that the $\underline{B}$ field lines near the ring are little circles

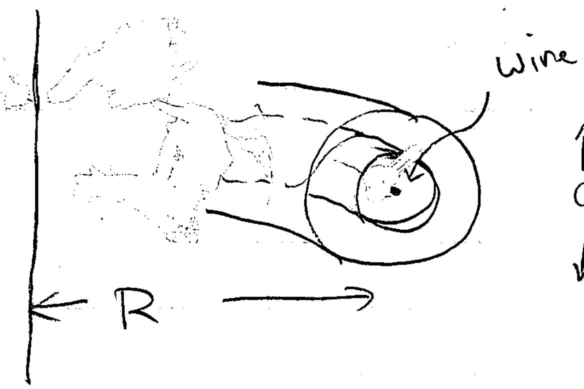


current ring



solenoid

together

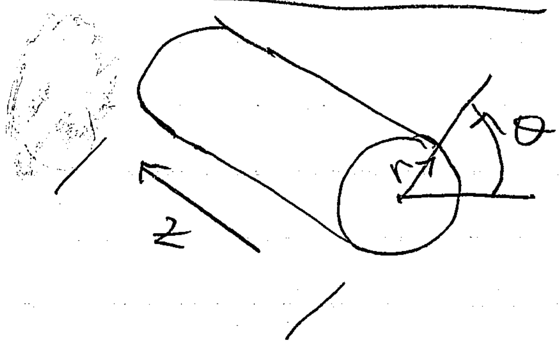


$$\underline{B} = \underline{B}_T + \underline{B}_P$$

Field lines are helical and lie on nested toroidal surfaces

Suppose $a/R \ll 1 \Rightarrow$ <u>straight torus</u>

or <u>periodic cylinder</u>



$$z = 0 \Leftrightarrow z = 2\pi$$

$$\underline{B} = B_0 \hat{z} + \underline{B}_p = B_0 \hat{z} + \nabla \times \underline{A}_p$$

$$A_p = -\psi \hat{z}$$

$$\underline{B} = B_0 \hat{z} + \hat{z} \times \nabla \psi$$

<u>Field lines</u>

$$\frac{B_z}{dz} = \frac{B_\theta}{r\,d\theta} = \frac{B_r}{dr}$$

cartesian coords.

$$\frac{B_z}{dz} = \frac{B_x}{dx} = \frac{B_y}{dy}$$

$$\Rightarrow \quad \frac{d\theta}{dz} = \frac{B_\theta}{B_0} = \frac{1}{r}\frac{\partial \psi}{\partial r}$$

$$\frac{dr}{dz} = \frac{B_r}{B_0} = -\frac{1}{r}\frac{\partial \psi}{\partial \theta}$$

Define $\boxed{P_\theta = r^2/2}$

$$\frac{d\theta}{dz} = \frac{\partial \psi}{\partial P_\theta} \qquad \frac{dP_\theta}{dz} = -\frac{\partial \psi}{\partial \theta}$$

Hamilton's Equations

z plays the role of time
ψ " " " " Hamiltonian

Before using this simple system to ~~introduce~~ a couple of ideas - let me remind you that for the configuration described $B_r = 0 \Rightarrow \psi(r) = H_0(P_\theta)$ only

The functional form of this depends upon the current dist. inside

$$\underline{J} = J(r)\,\hat{z}$$

We will be interested in one where

$$H_0 \propto r^4$$

although the form is not that important ⇒

$$\boxed{H_0 = \frac{P_\theta^2}{2m}} \quad m \text{ some constant.}$$

Two Ideas

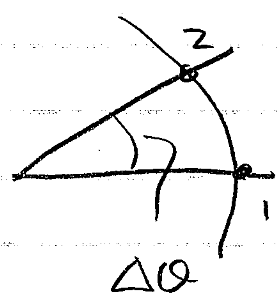
1. Surface of Section

If we consider the $\rho_0 - \theta$ plane at $z=0$ and plot the pts where the field line intersects we get a s of s. (puncture plot)

$\nabla \cdot \underline{B} = 0 \Rightarrow$ Area preserving Poincaré section.

Picture 1

There are two types of surfaces depending on the pitch of the helix. One important quantity is the rotational transform ι. Roughly speaking it is the change in θ after one transit



$$\Delta\theta \sim \iota$$

More precisely

$$\lim_{n\to\infty} \left(\frac{\Delta\theta}{\Delta z}\right)\frac{1}{n} = \frac{\iota}{2\pi} \equiv \bar{\iota} \qquad \left(\text{recall } \hbar = \frac{h}{2\pi}\right)$$

The quantity t characterizes the surface. From the picture we observe that there are two types of surfaces.

(i) t rational $\Rightarrow$ the field line closes upon itself $\Rightarrow$ fixed points; i.e. an entire surface of them.

(ii) t irrational $\Rightarrow$ ergodic coverage of surface.

2. Integrability

The picture we have painted so far describes what is referred to as an integrable Hamiltonian system. This is the exceptional case. Usually all the field lines don't close upon themselves in a nice way. Consider the following def.

Integrable

An n degree of freedom Ham. System is integrable if $\exists$ n indep. consts. in involution (n "q's")

Define P.B.

$$[F_i, H] = 0 \qquad [F_i, F_j] = 0 \qquad i,j = 1, \cdots n$$

If the surface in $2n$ dim. phase space defined by $F_i = c_i$ is connected & compact $\Rightarrow$ phase space foliated by invariant n-Tori

$\exists$ Action-Angle coordinates

$$(q, p) \longrightarrow (J, \theta)$$

$$H(q,p) = H(J)$$

$$\dot{\theta} = \frac{\partial H}{\partial J} \equiv \omega_0(J) \qquad \dot{J} = \frac{\partial H}{\partial \theta} = 0$$

$$J = \text{const} \;; \quad \theta = \omega_0 t + \theta_0$$

Ripple

Let us now return to the mag. field problem and add a new twist to the configuration. Suppose e.g. that the coils that create B_T are not perfect. The wires themselves may have a helical pitch, with irregularity and gaps too. We will represent this "ripple" by adding a piece to the vector potential, which recall is the Hamiltonian

$$\psi_R = \sum_{m,n}{}' \psi_{m,n}(r) \cos(m\theta - nz)$$

$$\Rightarrow \qquad r^2/2 = P_\theta$$

Non perfection of toroid.

$$H = H_0 + H_1$$

$$H = \frac{P_\theta^2}{2m} + \sum_{m,n}{}' \psi_{m,n}(P_\theta) \cos(m\theta - nz)$$

This is a Ham. for a typical, i.e. non-integrable system

Print sequence 2 → Slides

KAM theorem & Perturbation Theory

The goal of pert. theory is to transfer to a new system of canonical coords. where the Hamiltonian becomes ignorable, i.e doesn't depend upon the coords.

For example

$$H(\underline{p},\underline{q}) = H_0(\underline{p}) + \varepsilon H_1(\underline{p},\underline{q})$$

In action angle variables

$$(\underline{J},\underline{\theta}) \longleftrightarrow (\underline{p},\underline{q})$$

we want

$$H(\underline{p},\underline{q}) = \bar{H}(\underline{J})$$

$$\Rightarrow \quad \frac{\partial \bar{H}}{\partial \underline{J}} = \dot{\underline{\theta}} \equiv \underline{\omega_0(J)}$$

$$-\frac{\partial H}{\partial \underline{\theta}} = \dot{\underline{J}} = 0$$

$$\Rightarrow \begin{cases} \underline{\theta} = \omega_0(J)\, t + \theta_0 \\ \underline{J} = \text{const} \end{cases}$$

One way of generating canonical trans is via a mixed variable generating function

$$S(\underline{q}, \underline{J})$$

$\underline{q}$: old coord. — $\underline{J}$: new mom.

The trans eqs. are

$$\underline{p} = \frac{\partial S}{\partial \underline{q}} \qquad \underline{\theta} = \frac{\partial S}{\partial \underline{J}}$$

Thus

$$H\left(\frac{\partial S(\underline{q}, \underline{J})}{\partial \underline{q}}, \underline{q}\right) = \bar{H}(\underline{J})$$

↑ Eq. for S ; Hamilton–Jacobi Eq.

__H_1 small__ ⟹ near identity

$$S = \underline{q} \cdot \underline{J} + \varepsilon S_1(\underline{q}, \underline{J}) + \cdots \quad \Longrightarrow$$

$$H_0\left(\underline{J} + \varepsilon \frac{\partial S_1}{\partial \underline{q}} + \cdots\right) + \varepsilon H_1(\underline{J} + \cdots, \underline{q}) = \bar{H}(\underline{J})$$

to order ε

$$H_0(\underline{J}) + \varepsilon \frac{\partial H_0}{\partial \underline{J}}(\underline{J}) \cdot \frac{\partial S_1}{\partial q} + \varepsilon H_1(J, q)$$

$$= \bar{H}(J) = \bar{H}_0(J) + \bar{H}_1(J)\varepsilon + \ldots$$

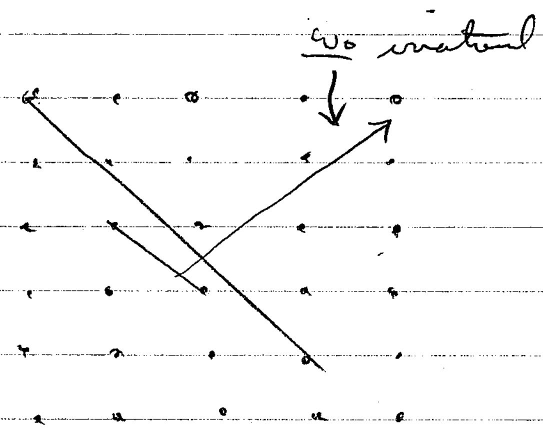
If.

$$H_1 = \sum_{\underline{m}} H_m^{(1)}(p) e^{i \underline{m} \cdot \underline{q}}$$

$$S_1 = \sum S_m^{(1)}(J) e^{i \underline{m} \cdot \underline{q}}$$

$$\Rightarrow \boxed{S(\underline{q}, \underline{J}) = \underline{q} \cdot \underline{J} + i\varepsilon \sum_{m \neq 0} \frac{H^{(1)}(\underline{J}) e^{i \underline{m} \cdot \underline{q}}}{\underline{m} \cdot \underline{\omega_0}}}$$

small divisor



m-lattice

Weirstrass
Poincare
↓
Kolmogorov
{ Arnold
{ Moser

Convergence

$\underline{\omega_0}$ sufficiently irrational

$\omega_0 = (\omega_{01}, \omega_{02})$

$\sigma = \frac{\omega_{01}}{\omega_{02}}$

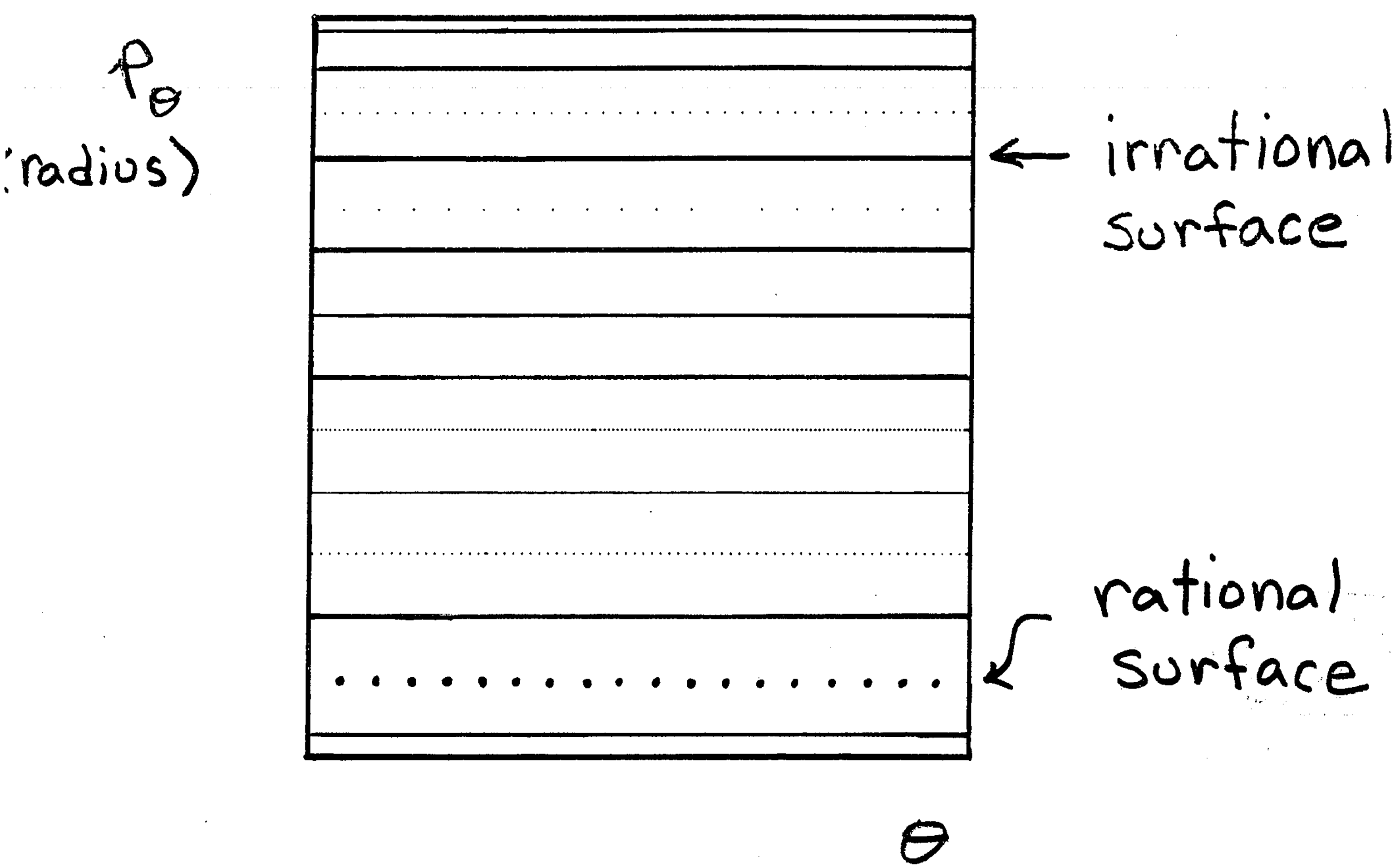
$\left| \sigma - \frac{r}{s} \right| < \frac{1}{s^{\alpha}}$ $\alpha \approx 2.5$

Continued fraction expansion

$\left| \sigma - \frac{r_n}{s_n} \right| < \frac{1}{s_n^2}$ all #'s

Hard ones to approx survive.

Surface of Section - Integrable



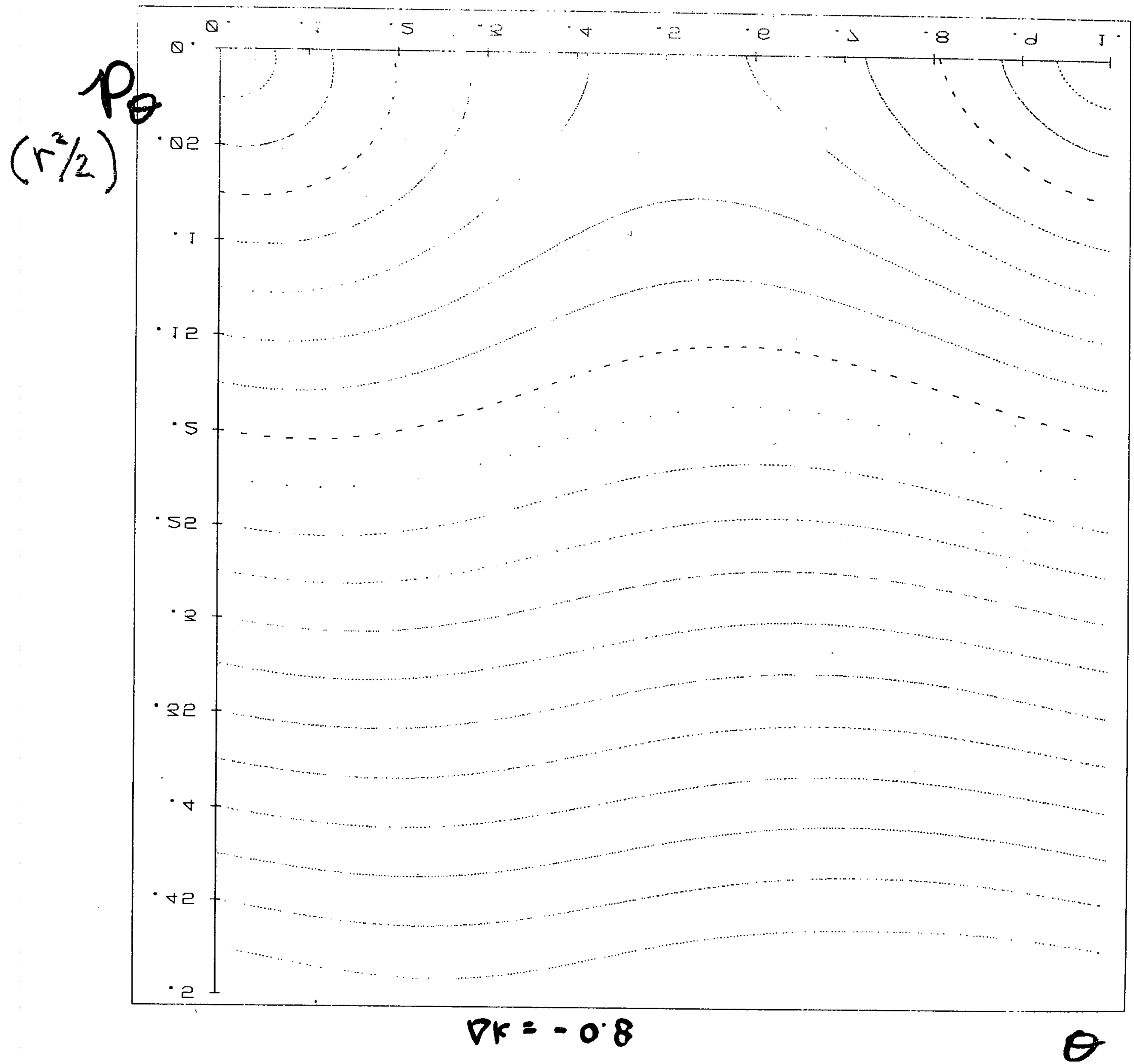
<u>Rational Surface</u>: ι rational &

every point a fixed point

<u>Irrational Surface</u>: ι irrational &

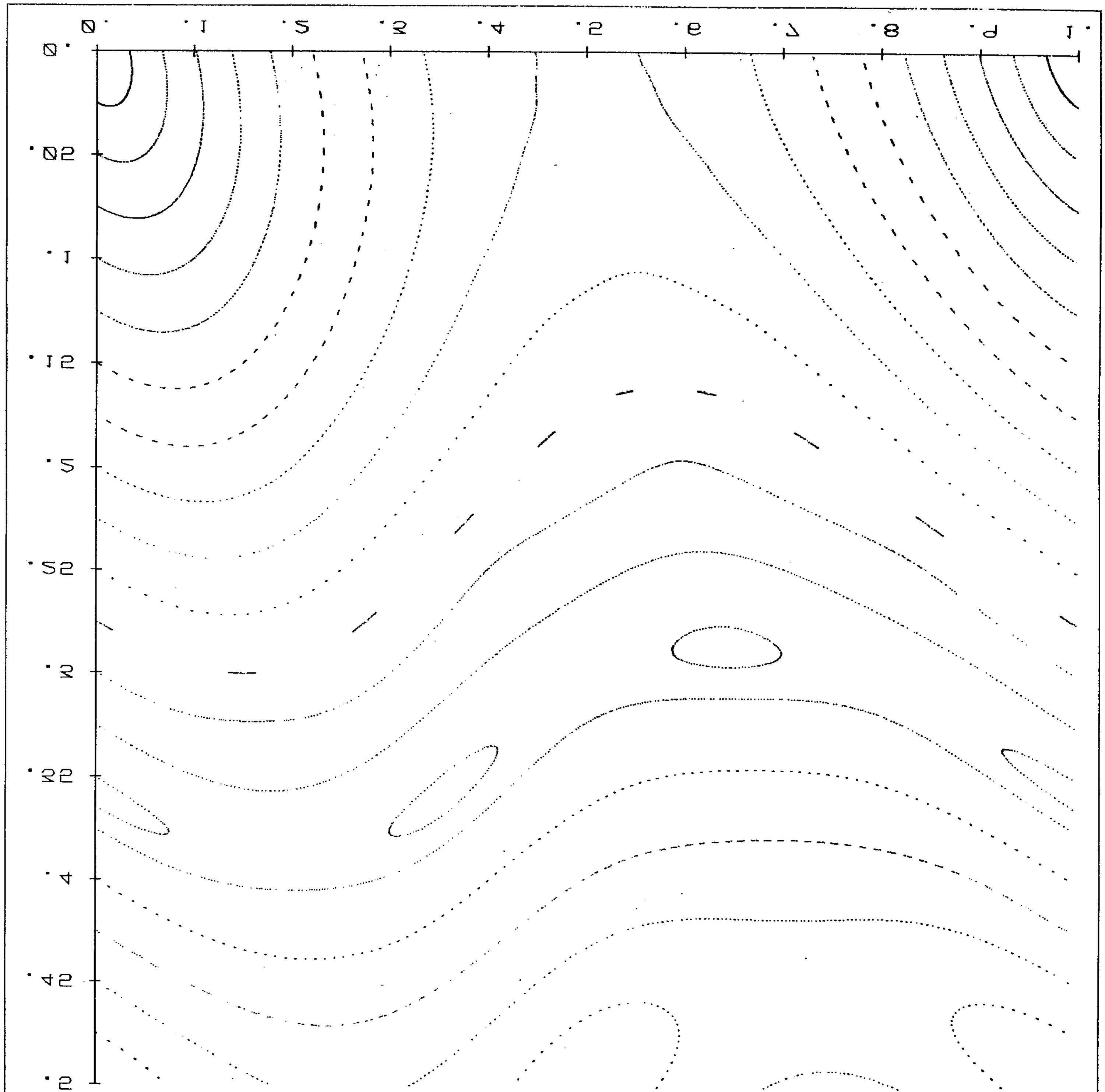
field lines ergodically fill surface

Small ripple



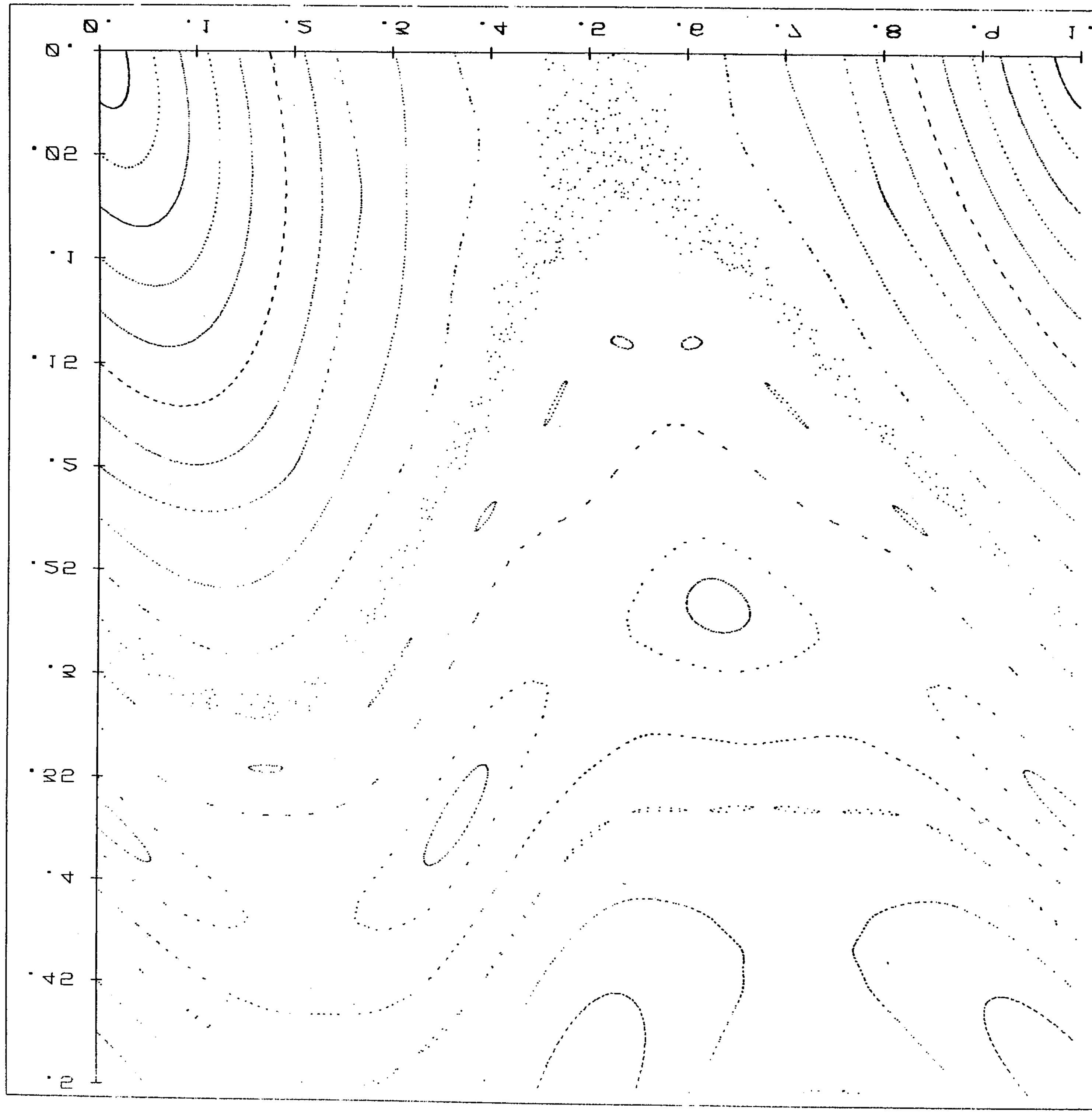
More ripple

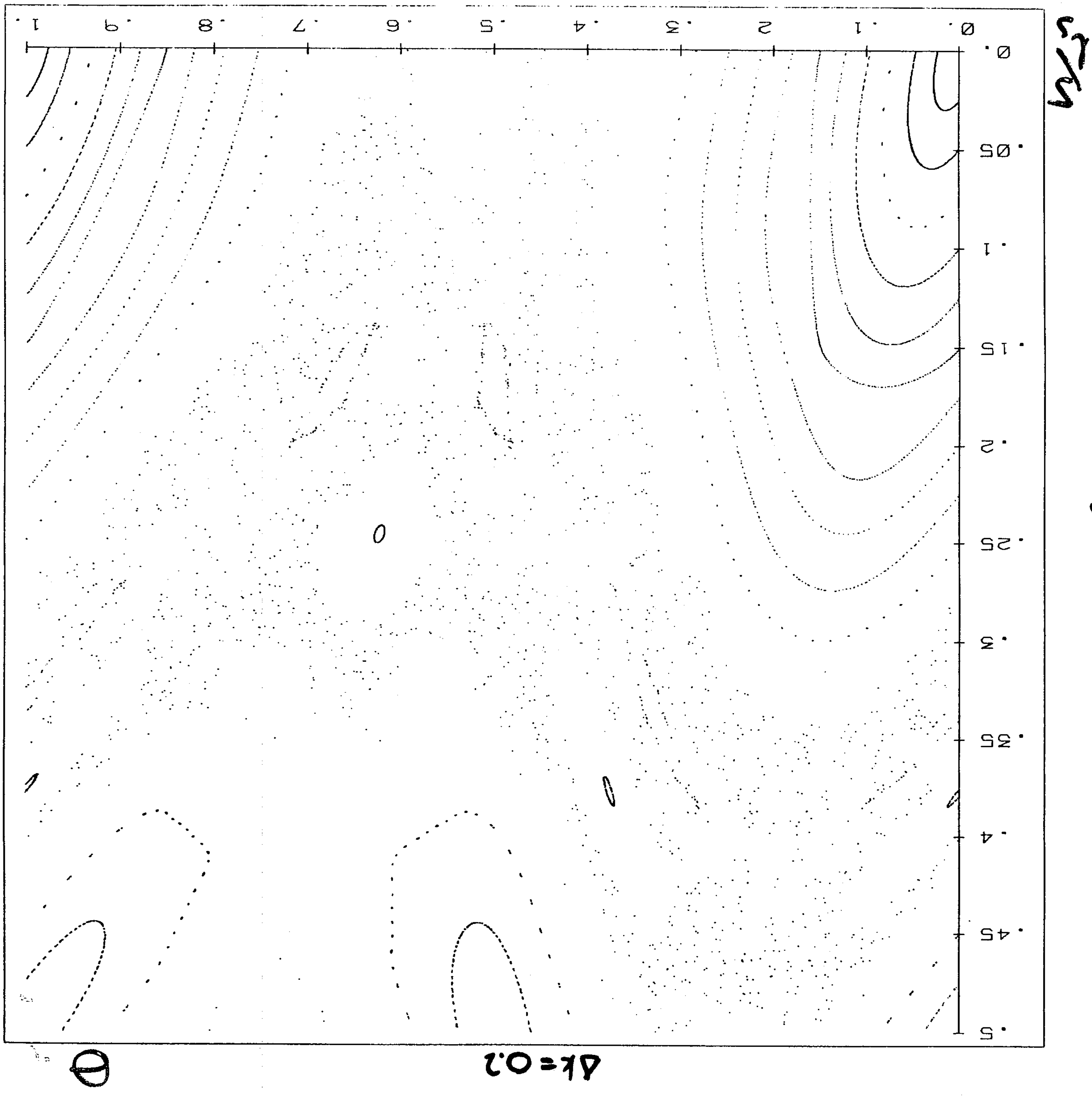
$r^2/2$



Strong ripple

$r^2/2$





Very
Strong
Ripple ⟶
Global
Stochasticity

Hamiltonian Chaos & Plasma Physics

* Charged particle orbits in <u>fixed</u> $\underline{E}$ & $\underline{B}$ fields

  confinement
  heating

* Magnetic field lines – $\nabla \cdot \underline{B} = 0$

* Nonlinear stability

  arnold diffusion

* Self-consistent problem

<u>References</u>

R. MacKay & J. Meiss, "Ham. Dyn. Systems – A reprint Selection"

M. Berry

A. Lichtenberg & M. Lieberman, "Regular & Stochastic Moti " Springer-Verlag