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201 Caldwell Hall ■ Ithaca, New York 14853-2602

(607) 255-8005

Nonlinear Instability and the Vlasov Equation or Free Energy Expressions for General Maxwell-Vlasov Equilibria

by

Philip J. Morrison
Physics Department
University of Texas at Austin
Austin, TX 78712

Abstract

In this talk the free energy principle is described and free energy expressions are obtained. The principle states that if a certain well-defined free energy expression associated with an arbitrary Maxwell-Vlasov equilibrium is indefinite, then the equilibrium is either (i) linearly unstable or (ii) linearly stable, but with the presence of negative energy modes. It is argued, but not proven, that in case (ii) the system is "likely" to be nonlinearly unstable. Argument is based on analogy to few degree-of-freedom Hamiltonian systems. After reviewing well-known variational principles for equilibrium and stability, and their limitations, the general free energy quadratic forms are obtained by using the Lagrangian variable formulation of the Maxwell-Vlasov equation. These forms are shown by minimization to be indefinite for all "interesting" equilibria.

The Lagrangian Representation of the Vlasov-Poisson Equation

by

Jerry Marsden
Cornell University
Department of Mathematics
316 White Hall
Ithaca, NY 14853

and

Phil Morrison
University of Texas @ Austin
Department of Physics
Austin, TX 78713

Abstract

In Morrison and Greene [1980], Gibbons [1981], and Marsden and Weinstein [1982] it is shown that the Vlasov-Poisson equation is Hamiltonian on the space of plasma densities f(x, p, t) with respect to the Vlasov-Poisson bracket:

$$\{F,K\} = \int f\left[\frac{\delta F}{\delta f}, \frac{\delta K}{\delta f}\right]_{xp} dxdp \tag{1}$$

It is also shown that the bracket (1) is the Lie-Poisson bracket on the dual of a Lie algebra and that $\dot{F} = \{F,H\}$ is equivalent to the Vlasov-Poisson equation, where H is the usual Vlasov Hamiltonian.

The goal of this note is to relate (1) to the Lagrangian (material) representation of the plasma and to explore the geometry of this relationship. We proceed in a number of steps.

Step 1 Let S denote the group of all canonical transformations of the single particle phase space, which we denote by P; coordinates on P are (x,p), as in (1).

There is a natural canonical Poisson structure on $T^* S$. The variables in $T^* S$ are (η,π), where η tracks the particle positions in phase space and π is its conjugate momentum.

Noether's theorem associated with the particle relabelling gauge symmetry S acting on S by right composition produces a Poisson map

$$J: \quad T^* S \longrightarrow g^* \tag{2}$$

which transforms the Lagrangian representation to the spatial representation. If we use the L^2 pairing to identify elements of $T^*_\eta S$ with vector fields over η, then we write $\pi \circ \eta^{-1} = X_f$, the Hamiltonian vector field of f (an additive constant on f is usually fixed by boundary conditions) then $J(\eta,\pi) = f$.

Step 2 Define

$$\tilde{H} = H \circ J,$$

so $\tilde{H}$ is a Hamiltonian on $T^* S$ and thus the Vlasov-Poisson equation is equivalent to the canonical Hamilton equations n $T^* S$ with Hamiltonian $\tilde{H}$ and canonically conjugate variables $(\eta.\pi)$. We note that the procedure of passing from $T^* S$ go g^* is an example of **reduction** - it is how Marsden and Weinstein [1982] *derived* the Maxwell-Vlasov and other brackets.

Step 3 Attempt to Legendre transform $\tilde{H}$ to get a Lagrangian L on TS.

One actually is stuck here because $\tilde{H}$ is *degenerate*; i.e. one easily sees that $\dot{\eta} = \frac{\partial \tilde{H}}{\partial \pi}$ does not provide a nonsingular transformation to $(\eta,\dot{\eta})$ space; i.e., to TS. To overcome this difficulty, we use another device.

Let M be the set of maps ξ: $P \longrightarrow Q$ where Q is single particle position space; i.e., x-space. That is, we write $P = T^*Q$ and let $\xi = \pi_Q \circ \eta$, where $\pi_Q\colon\ T^*Q \longrightarrow Q$ is the projection. We can regard $S \subset T^*M$ and we let θ be the canonical one form on ξ and denote its conjugate variable by π_ξ.

Now $TS \subset TT^*M$ and there is enough structure to do a generalized Legendre transform, the geometry of which has been studied by Tulczyjew [1974]: Define the following Lagrangian on TS:

$$L(\eta,\dot{\eta}) = \theta(\eta) \cdot \dot{\eta} - \tilde{H}(\eta,\dot{\eta})$$

In finite dimensions, this of course is just

$$L(q^i,\ p_j,\ \dot{q}^i,\ \dot{p}_i) = p_i\dot{q}^i - \tilde{H}(q^i,p_j)$$

Note that we have a natural identification

$$TS \cong T^*S$$

and one checks that the Euler-Lagrange equations are equivalent to Hamilton's equations. In particular, these equations are equivalent to the classical Lagrange-Hamiltonian variational principle $\delta\int_b^a L\,dt = 0$ on TS; i.e., in $\eta,\dot{\eta}$ space,

Step 5 Make use of the fact that the plasma density f_t at time t is advected by the flow η_t of particle paths in P:

$$f_t = f_0 \circ \eta_t$$

(this is essentially Noether's theorem) and in fact for each t, $\eta_t \in S$. Thus, if we regard f_0 as parametrically given, H depends only on η. Thus, we can express (3) in the form

$$L(\eta,\dot{\eta}) = \theta(\eta) \cdot \dot{\eta} - H(\eta,f_0). \tag{5}$$

Keeping f_0 fixed, we see that L is degenerate since it is linear in $\dot{\eta}$.

Step 6 Use Dirac constraints in the following way. For a Lagrangian of the form (5), or in finite dimensions, of the form

$$L(q^i,\dot{q}^i) = \alpha_j\dot{q}^j - h(q^j), \tag{5'}$$

where α_j are functions of q^i, we have the primary constraints in the sense of Dirac

$$p_i = \frac{\partial L}{\partial q^i} = \alpha_j \tag{6}$$

One checks that if the two-form $d\alpha$ is non-degenerate then the Euler-Lagrange equations for (5') with the constraints (6) are equivalent to Hamilton's equations on Q with Hamiltonian h and with symplectic form $\omega = -d\alpha$ (see Bergvelt and Kerf [1985]).

Aside: For example, one can do the KdV equation this way (cf. Gotay [1988]). Note that the Euler Lagrange equations for

$$L(\psi,\dot{\psi}) = \int[\psi_x\psi + \frac{1}{2}\psi_x^3 - \frac{1}{2}(\psi_{xx})^2]dx$$

are the KdV equation

$$u_t + u\,u_x + u_{xxx} = 0, \text{ where } u = \psi_x.$$

the procedure in Step 6 produces the usual Gardner Hamiltonian structure for the KdV equation.

Thus (still keeping f_0 as a parameter), (5) gives a Hamiltonian $\hat{H}(\xi,\pi_\xi,f_0)$ on T^*M. This new Hamiltonian $\hat{H}$ is in fact non-degenerate and so gives a Lagrangian on TM which in fact is the Low Lagrangian, studied in Low [1962].

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