Nonlinear Instability, Negative Energy Modes & the Hamiltonian Structure of the Vlasov – Poisson Equation

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Overview

I. Conjecture of N.L. Instability of linearly stable equilibria

II. Finite degree-of-freedom Hamiltonian system w/ NEM

III. Vlasov – Poisson as Hamiltonian System (∞ degree-of-freedom – Eulerian field theory)

IV. Formal Liapunov Stability – Energy

Conjecture of Nonlinear Instability of Linearly Stable Equilibria

Vlasov-Poisson in 1 dimension : $f : \mathbb{R}^2 \times \mathbb{R}^+ \to \mathbb{R}^+$

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - E\frac{\partial f}{\partial v} = 0 \qquad \frac{\partial E}{\partial x} = n_0 - n(x)$$

$$n(x) = \int_{-\infty}^{\infty} f(x,v,t)\,dv \qquad \int_{-\infty}^{\infty}(n(x)-n_0)\,dx = 0$$

Equilibria :

$F_0(v)$ complex extension analytic in a strip about real axis ; i.e. $F_0(z)$ analytic on $A = \{z = v + iw \mid |w| < \delta, v \in$

$$\int v^n F_0\,dv < \infty \qquad n = 0,1,\cdots M \;;\; M \geq 1$$

$F_0'(v)$ has finitely many zeros, $v_1 < v_2 < \cdots v_m$ $\quad m \geq 3$

s.t. $F_0' > 0$ on $(-\infty, v_1)$

$F_0' < 0$ on (v_1, v_2)

$\vdots$

Examples :

(i) $F_0(v) = A e^{-v^2/2} + a e^{-(v-v_D)^2/2T}$ $\qquad \int F_0\,dv = n_0$

$v_D, A, a, T \in \mathbb{R}^+$

Linear Dynamics:

$$\frac{\partial \delta f}{\partial t} + v \frac{\partial \delta f}{\partial x} - \delta E \frac{\partial F_0}{\partial v} = 0$$

$$\frac{\partial \delta E}{\partial x} = -\int \delta f \, dv$$

initial data: $-\int \delta f_0 \, dx \, dv = 0$

- δf_0 L_1 in x (for Fourier transform)
- δf_0 analytic extension in strip

$\Rightarrow$ Solution in Van Kampen modes

stability: If v_D, a etc. are chosen in a certain way, (v_D small) $\nexists$ exponentially growing solutions & Van Kampen decomposition is complete.

Stable Def:

$\forall$ $L_?$ nhbds U of zero $\exists$ a V of zero, $V \subset U$, s.t. if $\delta f_0 \in V$ $\Rightarrow$ $\delta f \in U$ $\forall t$.

Conjecture:

Finite Dof Hamiltonian Systems & NEM

$$\dot{q} = \frac{\partial H}{\partial p} \qquad \dot{p} = -\frac{\partial H}{\partial q}$$

$$z = (q, p) \qquad \dot{z} = [z, H] = J_c \nabla H$$

$$[f, g] = \frac{\partial f}{\partial z^i} J_c^{ij} \frac{\partial g}{\partial z^j} \qquad J_c = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

Equilibria: $\nabla H(z_0) = 0$

Stability: $z = z_0 + \delta z$

<u>Spectral</u>

$\delta z \sim e^{i\omega t}$

ω real

$\omega \neq 0$

<u>Liapunov</u>

Use constants of motion

$\delta^2 H$ available

Standard Case: $H = T + V$

$$\delta^2 H = \frac{(\delta p)^2}{2m} + \frac{1}{2} \frac{\partial^2 V}{\partial q^2}(q_0)(\delta q)^2$$

Lagranges Theorem:

$$\frac{\partial^2 V}{\partial q^2} > 0 \implies \text{stable}$$

General Case: $H(z) \neq T + V$

$$\longrightarrow \quad \delta^2 H = \sum_{i,j} a_{ij}\, \delta z^i \delta z^j$$

$$a_{ij} = \frac{\partial^2 H}{\partial z^i \partial z^j}(z_0)$$

a_{ij} definite $\Longrightarrow$ stable

a_{ij} indefinite $\Longrightarrow$ "Nothing"

Dirichlet's Theorem

Negative Energy Mode

- $\delta^2 H$ indefinite
- linear stability (neutral)

Equilibrium is not a minimum energy state.

Nonlinearly Unstable?

Example: (Cherry 1927)

$$H_L = -\frac{\omega_1}{2}(p_1^2+q_1^2) + \frac{\omega_2}{2}(p_2^2+q_2^2)$$

$\omega_1, \omega_2 > 0$

$\omega_2 = 2\omega_1$

NEM

- linearly stable
- $\delta^2 H$ indef.

$$H_c = H_L + \frac{\alpha}{2}\left[q_2(q_1^2 - p_1^2) - 2q_1 p_1 p_2\right]$$

$$q_1 = \frac{-\sqrt{2}}{\varepsilon - \alpha t}\sin(\omega_1 t + \gamma) \qquad p_1 = \cdots$$

$$q_2 = \frac{-1}{\varepsilon - \alpha t}\sin(2\omega_1 t + \gamma) \qquad p_2 = \cdots$$

Nonlinearly Unstable!

Genericity:

- Bifurcation to $\omega = \pm\omega_r \pm i\omega_i$
  Krein's thm.
- Cherry arises upon averaging - pert. theory

Vlasov - Poisson as Hamiltonian System

$\dot{z} = J_c \nabla H$ $\qquad$ $[f, g] = \frac{\partial f}{\partial z^i} J_c^{ij} \frac{\partial f}{\partial z^j}$

Noncanonical Hamiltonian Mech. (S. Lie)

$\dot{z} = J \nabla H$

(i) $[f, g] = -[g, f]$

(ii) $[f, [g, h]] + \circlearrowleft = 0$

C. $\det J \neq 0$ $\quad$ Darboux $\longrightarrow$

$\downarrow$

N.C. $\det J = 0$

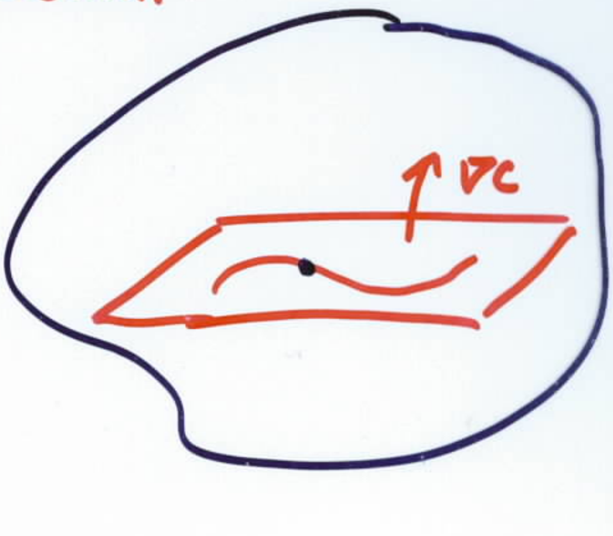
$$J = \left[\begin{array}{cc|c} 0 & I & 0 \\ -I & 0 & \\ \hline 0 & & 0 \end{array}\right]$$

Hamiltonian is not unique!

$J \nabla C = 0$ $\quad$ casimir

Liapunov stability:

$\frac{\partial}{\partial z^i} (H + C) = 0$



$\delta^2 (H + C)$ def ?

NEM can arise.

Vlasov-Poisson is ∞ dim. version

$$z \longrightarrow f(x, v, t)$$

$$\frac{\partial}{\partial z^i} \longrightarrow \frac{\delta}{\delta f}$$

$$J \longrightarrow \text{operator (P.d.e.)}$$

$$H[f] = \int \frac{v^2}{2} f \, dx\, dv + \int \frac{E[f]^2}{2} dx$$

$$C[f] = \int \mathfrak{Z}(f) \, dx\, dv$$

$$\frac{\delta H}{\delta f} = \frac{v^2}{2} + \phi \qquad \frac{\delta C}{\delta f} = \mathfrak{Z}'$$

$$\frac{\partial f}{\partial t} = \{f, H[f]\} = -v \frac{\partial f}{\partial x} + E[f] \frac{\partial f}{\partial v}$$

$$\{F, G\} = \int \frac{\delta F}{\delta f} \, \theta \, \frac{\delta G}{\delta f} \, dx\, dv$$

J

Formal Liapunov Stability

Old Argument (Kruskal, Oberman ... 50's)

$$F = H + C$$

$$\frac{\delta F}{\delta f} = 0 \Rightarrow \mathcal{F}'(F_0) + \mathcal{E} = 0$$

monotonic $\Rightarrow$ $F_0 = (\mathcal{F}')^{-1}(-\mathcal{E})$

only gives monotonic equil.

$$\delta^2 F = \frac{1}{2}\int \mathcal{F}''(F_0)\,\delta f^2 + \frac{1}{2}\int \delta E^2$$

$$\mathcal{F}''\frac{\partial F_0}{\partial \mathcal{E}} = -1 \Rightarrow$$

$$\delta^2 F = -\frac{1}{2}\int \frac{(\delta f)^2}{\partial F_0/\partial \mathcal{E}} + \frac{1}{2}\int (\delta E)^2\,dx$$

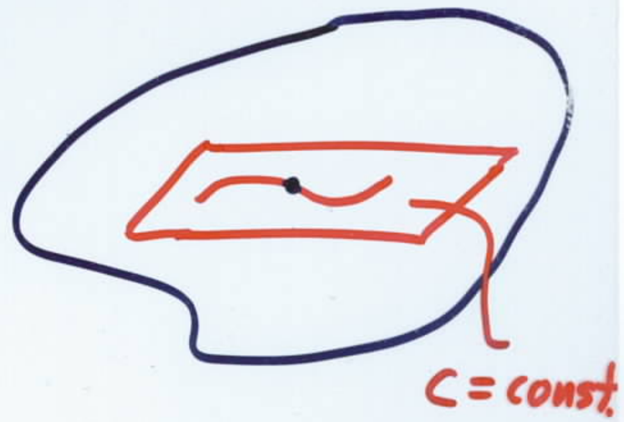
$$\frac{\partial F_0}{\partial \mathcal{E}} < 0 \Rightarrow \text{stable ?}$$

What if $\partial F_0/\partial \mathcal{E} = 0$?

What is the energy ? $\delta^2 F$?

Energy of Nonmonotonic Equilibria (Pfirsch)

Only allow variations that keep on $C = \text{const.}$



$$\delta C[f; \delta f] = \int \frac{\partial J}{\partial f}(f)\,\delta f$$

Phase space conserving Variations

$$\delta f = [g, F_0] = \frac{\partial g}{\partial x}\frac{\partial F_0}{\partial v} - \frac{\partial g}{\partial v}\frac{\partial F_0}{\partial x}$$

$\delta^2 H$ requires $\delta^{(2)} f$ 2nd order

$$\delta^2 f = \frac{1}{2}[g, [g, F_0]]$$

$$\delta H = 0 \Rightarrow [F_0, \varepsilon] = 0$$

$$\Rightarrow F_0(\varepsilon) \text{ arb.}$$

$$\delta^2 H = \int -\frac{1}{2}\frac{\partial F_0}{\partial \varepsilon}[g, \varepsilon]^2\, dx\, dv + \frac{1}{2}\int \delta E^2\, dx$$

Conjectured equilibria have NEM's!