This whole exam concerns a QFT comprising the following fields: A charged Dirac spinor $\Psi_C(x)$ (charge $= -e$), a neutral Dirac spinor $\Psi_N(x)$, a charged pseudo-scalar $\Phi(x)$ (charge $= -e$), and the electromagnetic field $A^\mu(x)$. The physical Lagrangian of the theory is

$$\begin{aligned}\mathcal{L} \;=\; & -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \;+\; D_\mu\Phi^* D^\mu\Phi \;-\; M_S^2\Phi^*\Phi \;+\; \overline{\Psi}_C(i\not{D}-M_C)\Psi_C \;+\; \overline{\Psi}_N(i\not{\partial}-M_N)\Psi_N \\ & -\; ig\Phi\times\overline{\Psi}_C\gamma^5\Psi_N \;-\; ig\Phi^*\times\overline{\Psi}_N\gamma^5\Psi_C \;-\; \tfrac{1}{4}\lambda\bigl(\Phi^*\Phi\bigr)^2.\end{aligned} \tag{1}$$

Note: Ψ_N is *electrically* neutral, but it's a Dirac spinor rather than Majorana spinor, so a particle is different from an antiparticle. Just like a neutron is different from an antineutron.

1. [13 points] First, a few simple questions:

   (a) Are there any *renormalizable* couplings one may add to the Lagrangian (1) without breaking any symmetries of the theory? If yes, spell out such couplings; otherwise, explain why they do not exist.

   (b) Write down the bare Lagrangian of the quantum field theory — including all the counterterms needed to cancel the divergences.

   (c) Write down all the relations between different counterterms' coefficients due to symmetries and/or Ward identities.

   (d) Spell out the Feynman rules of the counterterm perturbation theory. Or rather, spell out all propagators, physical vertices, and counterterm vertices of the theory, never mind the rest of the Feynman rules.

   Please use visually different lines for the neutral and the charged fermions' propagators. For example, black lines for the Ψ_N and red lines for the Ψ_C, or single lines for the Ψ_N and double lines for the Ψ_C. Likewise, use visually different lines — for example, dotted rather than solid — for the scalar propagators.

2. [42 points] Second, a lot of hard work: Calculate the UV-infinite parts of all the independent counterterms at the one-loop order of the perturbation theory. Do not bother calculating the UV-finite parts, even if they are IR-divergent — this would take you way too much time. To

save more time, use the relations you wrote down in part 1(c): If two or more counterterms are related by a symmetry and/or a Ward identity, calculate just one of those counterterms, whichever you think is simpler.

For each independent counterterm, start by drawing all the relevant Feynman diagrams. If a diagram was evaluated in class, in my notes, or in a homework, don't waste your and my time redoing the work, just quote the result and move on to the next diagram. For the remaining diagrams — and there will be plenty of those — use dimensional regularization and work hard. But if some part of the calculation is similar to what I wrote in the solutions or in the notes, don't reproduce my work but simply quote it and adapt it to your needs.

Many diagrams — especially those contributing to the divergence canceled by the δ^λ counterterm — are related by permutations of the external legs. Such symmetries can save you a lot of work, but please be careful counting similar diagrams. Remember that Φ is different from Φ^*, and Ψ_C is different from the Ψ_N. Consequently, the diagram counts and/or symmetry factors are likely to be different from what you have seen in class or in the homeworks.

Here are some useful formulae for the dimensionally regulated momentum integrals: For any logarithmic UV divergence,

$$\textbf{if}\ \frac{\mathcal{N}}{\mathcal{D}} \to \frac{C}{(k^2)^2}\ \text{for}\ k^2\to\infty,\quad \textbf{then}\ \int\limits_{\text{reg}} \frac{d^4k}{(2\pi)^4}\,\frac{\mathcal{N}}{\mathcal{D}} = \frac{iC}{16\pi^2}\times\frac{1}{\epsilon} + \text{finite.} \tag{2}$$

And here are some quadratic UV divergences:

$$\int\limits_{\text{reg}} \frac{d^4\ell}{(2\pi)^4}\,\frac{A}{\ell^2-\Delta+i0} = (A\Delta)\times\frac{i}{16\pi^2}\times\frac{1}{\epsilon} + \text{finite,} \tag{3}$$

$$\int\limits_{\text{reg}} \frac{d^4\ell}{(2\pi)^4}\,\frac{A\ell^2+B}{[\ell^2-\Delta+i0]^2} = (2A\Delta+B)\times\frac{i}{16\pi^2}\times\frac{1}{\epsilon} + \text{finite,} \tag{4}$$

$$\int\limits_{\text{reg}} \frac{d^4\ell}{(2\pi)^4}\,\frac{A(\ell^2)^2+B\ell^2+C}{[\ell^2-\Delta+i0]^3} = (3A\Delta+B)\times\frac{i}{16\pi^2}\times\frac{1}{\epsilon} + \text{finite.} \tag{5}$$

Note: The on-shell physical amplitudes are gauge invariant, but the off-shell loop diagrams and the counterterms depend on the gauge you work in. To be consistent, you must use the same gauge in all the calculations. Even the diagrams you use to calculate different counterterms must be evaluated in the same gauge.

- ⋆ For extra credit (up to 10 points), allow for an arbitrary gauge parameter ξ and work out how (the infinite parts of) all the counterterms depend on ξ.

- Otherwise, stick to a particular value of ξ and perform all calculation in that gauge for all photon propagators in all diagrams. I recommend you use either Feynman gauge $\xi = 1$ or Landau gauge $\xi = 0$: The Feynman gauge makes for an easier numerator algebra for all the diagrams involving photon propagators; while in the Landau gauge many diagrams happen to vanish — or at least their UV divergences vanish — and one can see that before shifting the loop momenta or doing any integrals.

Advice: The hardest counterterm to calculate is δ_λ due to sheer number of one-loop diagrams contributing to the 4-scalar amplitude. The next hardest are δ_Z^ϕ and δ_M^ϕ for the charged scalar field; this time there are fewer diagrams, but they are harder to evaluate due to quadratic UV divergence (in 4D). Make sure to allocate plenty of time for calculating these counterterms, they take *a lot* of hard work.

3. [10 points] Third, calculate (to the one-loop order) the anomalous dimensions of all the fields and the beta–functions for all the running couplings of the theory — $e(E)$, $g(E)$, and $\lambda(E)$ — as well as the fermion masses $M_C(E)$ and $M_N(E)$. Assume high energies $E \gg$ all masses.

   After all the hard work you did in part (2), this part should be simple. Remember: at the one-loop level, the infinite part of a counterterm determines its dependence on the renormalization energy scale $E \gg$ masses according to

$$\delta \;=\; (\text{overall coefficient}) \times \left(\frac{1}{\epsilon} + \log\frac{\mu^2}{E^2} + \text{a numeric constant}\right). \tag{6}$$

   Note: while the anomalous dimensions of the charged fields may be gauge-dependent, all the beta-functions must be gauge invariant. So if you have calculated all the counterterms for a general gauge parameter ξ and then ended up with a ξ-dependent beta-function, you must have made a mistake somewhere (or perhaps several mistakes).

4. [35 points] Finally, consider the electromagnetic form factors $F_1(q^2)$ and $F_2(q^2)$ of the *neutral* fermion Ψ_N.

(a) First of all, explain the physical meaning of these form factors and what do they probe. Also, explain why in the $q^2 \to 0$ limit, the $F_1(0)$ form factor must vanish but it's OK to have $F_2(0) \neq 0$.

(b) Now let's start calculating the $F_1(q^2)$ and the $F_2(q^2)$. Draw *all* Feynman diagrams contributing to these form factors at the one-loop level and spell the resulting amplitudes as momentum integrals,

$$ie\Gamma^\mu(\text{diagram}) = \int \frac{d^4k}{(2\pi)^4}\,(\text{stuff}). \tag{7}$$

By the way, do any of the counterterms contribute at this level? Explain your answer.

(c) Show that the individual diagrams suffer from logarithmic UV divergences, but the divergences cancel out from the net 1PI amplitude $\Gamma^\mu_{1\,\text{loop}}$. If they do not, make sure you have not forgotten a diagram and double-check your signs.

(d) Now comes the hard work of evaluating the diagrams: Introduce the Feynman parameters, shift the loop momenta, simplify the numerators, reorganize them according to the form factors F_1 and F_2, and finally integrate over the loop momenta. Throughout the calculation, keep the fermionic external legs on-shell — including the $\bar{u}(p')\Gamma^\mu(p',p)u(p)$ context of the amplitude, — but remember that $M_N \neq M_C$. Indeed, allow for general masses M_N, M_C, and M_S and any off-shell q^2.

Some stages of this calculation are going to be similar to what I did in my notes on the electron's form factors or in the solutions to homework#18, so quote my results and adapt them to your situation instead of redoing my work. Here is another useful formula for calculating the momentum integrals in $D = 4 - 2\epsilon$ dimensions:

$$\int \frac{\mu^{4-D}\,d^D\ell}{(2\pi)^D}\,\frac{A\ell^2 + B}{[\ell^2 - \Delta + i0]^3} = \frac{i}{16\pi^2}\left(\frac{4\pi\mu^2}{\Delta}\right)^{\epsilon}\Gamma(\epsilon) \times \left[\left(1 - \frac{\epsilon}{2}\right)A - \frac{\epsilon}{2}\frac{B}{\Delta}\right]. \tag{8}$$

At the end of this part of the problem, you should get formulae of the form

$$^{\text{neutral}}F_1(q^2) = \frac{g^2}{16\pi^2}\sum_i^{\text{diagrams}}\int d(FP)\,\mathcal{F}_1^{(i)}(q^2; FP; \text{masses}; \epsilon), \tag{9}$$

$$^{\text{neutral}}F_2(q^2) = \frac{g^2}{16\pi^2}\sum_i^{\text{diagrams}}\int d(FP)\,\mathcal{F}_2^{(i)}(q^2; FP; \text{masses}; \epsilon), \tag{10}$$

where FP denote the Feynman parameters.

Your task is to work out the functions $\mathcal{F}_1^{(i)}$ and $\mathcal{F}_2^{(i)}$ for each diagram.

(e) Next, use the results of part (d) to calculate the magnetic dipole moment of the neutral fermion. For general masses, you should get

$$\mathbf{m} = \frac{g^2 e\mathbf{S}}{16\pi^2}\,\mathcal{H}(M_N, M_S, M_C) \tag{11}$$

where $\mathcal{H}$(masses) obtains from an integral of a rational function. To simplify the integral, assume $M_C = M_S$ while $M_N \ll M_C$: Show that for such masses

$$\mathcal{H} \approx \frac{1}{M_C} \times \text{a non-zero number you should calculate,} \tag{12}$$

or in other words

$$g^{\text{gyromagnetic}}(\Psi_N) = (\text{a number}) \times \frac{g^2 M_N}{M_C}\,. \tag{13}$$

(f) Now consider the electric form factor $F_1(q^2)$ and verify that it duly vanishes for $q^2 = 0$. For this part of the problem, keep the masses general and do *not* take the $\epsilon \to 0$ limit.

Hint: For $q^2 = 0$, the numerators and the denominators of the diagrams depend on only one Feynman parameter z. Moreover, the Δ's in the denominators of two diagrams are related to each other as $\Delta_1(z_1) = \Delta_2(z_2)$ for $z_2 = 1 - z_1$. Check this relation, then use it to bring the net $F_1(0)$ to the form

$$F_1(0) = \frac{g^2}{16\pi^2} \times (4\pi\mu^2)^\epsilon \Gamma(\epsilon) \times \int_0^1 \frac{d}{dz}\left(\frac{z(1-z)}{[\Delta(z)]^\epsilon}\right) dz, \tag{14}$$

where the integral indeed evaluates to zero.

(g) Finally, calculate the $F_1(q^2)$ form-factor in the limit of very large $(-q^2) \gg$ all masses^2.

Hint: simply take the limit all masses $\to 0$, fixed $-q^2 \neq 0$ of the functions $\mathcal{F}_1^{(i)}$ in eq. (9), then sum up the diagrams and integrate over the Feynman parameters.

Your result should be non-zero. Note that by analyticity of the F_1 as a function of q^2, this means that for generic values of q^2 the electric form factor does not vanish. Instead, the F_1 vanishes only for $q^2 = 0$ (and perhaps also for some other discrete values of q^2).