COMPLEX AMPLITUDES and IMPEDANCES

Consider a harmonic alternating current of frequency ω flowing through some circuit,

$$I(t) \;=\; I_0 \cos(\omega t + \phi_i). \tag{1}$$

If the circuit is linear, then the voltage across the circuit would also be harmonic of the same frequency omega,

$$V(t) \;=\; V_0 \cos(\omega t + \phi_v), \tag{2}$$

but if the circuit includes any capacitors or inductors, then the voltage and the current may have different initial phases, $\phi_v \neq \phi_i$. The best way to handle such harmonic currents and voltages — and the relations between them — is to combine their amplitudes and the initial phases into *complex amplitudes*

$$\hat{I} \;=\; I_0 \times e^{-i\phi_i} \quad \text{and} \quad \hat{V} \;=\; V_0 \times e^{-i\phi_v}. \tag{3}$$

Note that for any real phase φ the complex exponential $e^{-i\varphi}$ has real part

$$\operatorname{Re} e^{-i\varphi} \;=\; \cos\varphi. \tag{4}$$

Consequently, we may describe the time dependence of a harmonic current (1) as

$$\begin{aligned} I(t) \;&=\; I_0 \times \cos(\omega t + \phi_i) \;=\; \operatorname{Re}\bigl[I_0 \times e^{-i(\omega t + \phi_0)}\bigr] \\ &=\; \operatorname{Re}\bigl[I_0 \times e^{-i\phi_i} \times e^{-i\omega t}\bigr] \\ &=\; \operatorname{Re}\bigl[\hat{I} \times e^{-i\omega t}\bigr], \end{aligned} \tag{5}$$

and likewise for the harmonic voltage (2)

$$V(t) \;=\; V_0 \times \cos(\omega t + \phi_v) \;=\; \operatorname{Re}\bigl[\hat{V} \times e^{-i\omega t}\bigr]. \tag{6}$$

Moreover, for linear circuits comprised of resistors, capacitors, and inductors, the complex current amplitude and the complex voltage amplitude are proportional to each other,

$$\hat{V} \;=\; Z \times \hat{I} \tag{7}$$

where Z is the complex *impedance* of the circuit in question. In a moment, we shall learn how to calculate the impedances of various circuits.

But first, a point of notations: Somehow, the Physicists and the Electric Engineers ended up with different conventions for the complex amplitudes of harmonic quantities. In Physics convention, the imaginary unit $\sqrt{-1}$ is denoted i, and the harmonic time dependence is written as $\exp(-i\omega t)$, hence

$$I(t) \;=\; \mathrm{Re}\bigl(\hat{I}\times e^{-i\omega t}\bigr), \quad V(t) \;=\; \mathrm{Re}\bigl(\hat{V}\times e^{-i\omega t}\bigr). \tag{8}$$

Moreover, when focusing on linear quantities such as voltages, currents, or EM fields, the Physicists often make the Re operation — taking the real part of a complex number — implicit, thus writing

$$I(t) \;=\; \hat{I}\times e^{-i\omega t}, \quad V(t) \;=\; \hat{V}\times e^{-i\omega t}. \tag{9}$$

However, all such formulae should be understood as *the LHS is the real part of the complex RHS* rather than as simple identity between the two sides of an equation. In other words, eqs. (9) should be understood as short-hand notation for eqs. (8).

In the Electric Engineering convention, the imaginary unit $\sqrt{-1}$ is denoted j rather than i because i is often used for a current. Also, the harmonic time dependence is written as $\exp(+j\omega t)$, hence

$$I(t) \;=\; \mathrm{Re}\bigl(\hat{I}\times e^{+j\omega t}\bigr) \quad\text{and}\quad V(t) \;=\; \mathrm{Re}\bigl(\hat{V}\times e^{+j\omega t}\bigr) \tag{10}$$

for

$$\hat{I} \;=\; I_0\times e^{+j\phi_i} \quad\text{and}\quad \hat{V} \;=\; V_0\times e^{+j\phi_v}. \tag{11}$$

Fortunately, we may reconcile the two conventions by identifying

$$j \;=\; -i. \tag{12}$$

Calculating Impedances

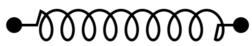
Let's learn how to calculate the impedances of various circuits and circuit elements.

1. For an **Ohmic resistor** of resistance R, the impedance is real

$$Z = R. \tag{13}$$

Indeed, for the resistor $V(t) = R \times I(t)$ at all times t, thus the same phase $\phi_v = \phi_i$ for the harmonic current and harmonic voltage while the amplitudes related as $V_0 = R \times I_0$, hence $\hat{V} = R \times \hat{I}$ and therefore $Z = R$.

2. For an ideal **inductor** of self-inductance L, the impedance is purely imaginary



$$Z = j\omega L. \tag{14}$$

To see this, note that a time-dependent current through an inductor leads to EMF

$$\mathcal{E}(t) = -L \times \frac{dI}{dt}\,. \tag{15}$$

The voltage across an ideal inductor is minus the EMF (since they are counted in opposite directions), thus

$$V(t) = +L \times \frac{dI}{dt}, \tag{16}$$

which for a harmonic current $I(t) = \mathrm{Re}\big(\hat{I} \times e^{j\omega t}\big)$ becomes

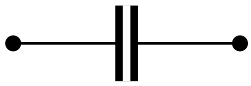
$$\begin{aligned} V(t) &= L \times \frac{d}{dt}\,\mathrm{Re}\big(\hat{I} \times e^{j\omega t}\big) = \mathrm{Re}\left(L \times \hat{I} \times \frac{d}{dt} e^{j\omega t}\right) \\ &= \mathrm{Re}\left(L \times \hat{I} \times j\omega e^{j\omega t}\right) = \mathrm{Re}\big(\hat{V} \times e^{j\omega t}\big) \\ \text{for} \quad \hat{V} &= L \times \hat{I} \times j\omega. \end{aligned} \tag{17}$$

In other words, for a harmonic current of complex amplitude $\hat{I}$ we have a harmonic voltage of complex amplitude

$$\hat{V} = j\omega L \times \hat{I} = Z \times \hat{I} \quad \text{for} \quad Z = j\omega L. \tag{18}$$

Note that the inductor's impedance increases with the frequency ω.

3. For a **capacitor** of capacitance C, the impedance is also purely imaginary, albeit of the opposite sign from the inductor's impedance:



$$Z \;=\; \frac{1.}{j\omega C} \;=\; \frac{-j}{\omega C}\,. \tag{19}$$

To see this, note that a voltage V across a capacitor comes with stored charge $Q = C \times V$, and if the voltage and hence the charge are time dependent, then there is a current through the capacitor — or rather, through the wires connected to the capacitor —

$$I(t) \;=\; \frac{dQ}{dt} \;=\; C \times \frac{dV}{dt}\,. \tag{20}$$

For a harmonic voltage $V(t) = \mathrm{Re}\bigl(\hat{V} \times e^{j\omega t}\bigr)$, the current becomes

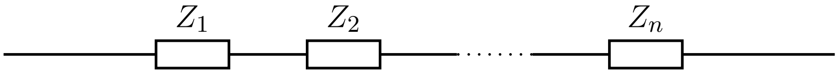
$$\begin{aligned} I(t) \;&=\; C \times \frac{d}{dt}\mathrm{Re}\bigl(\hat{V} \times e^{j\omega t}\bigr) \;=\; Re\left(C \times \hat{V} \times \frac{d}{dt}e^{j\omega t}\right) \\ &=\; \mathrm{Re}\left(C \times \hat{V} \times j\omega e^{j\omega t}\right) \;=\; \mathrm{Re}\bigl(\hat{I} \times e^{j\omega t}\bigr) \\ \text{for}\quad \hat{I} \;&=\; C \times \hat{V} \times j\omega. \end{aligned} \tag{21}$$

In other words, for a harmonic voltage of complex amplitude $\hat{V}$ we have a harmonic current of complex amplitude

$$\hat{I} \;=\; j\omega C \times \hat{V} \;=\; \frac{\hat{V}}{Z} \quad \text{for} \quad Z \;=\; \frac{1}{j\omega C}\,. \tag{22}$$

Note: unlike the inductor's, the capacitor's impedance decreases with frequency ω.

4. For a **serial circuit** of several elements or sub-circuits, the net impedance is the sum of sub-circuits' impedances,



$$Z_{\mathrm{net}} \;=\; Z_1 \;+\; Z_2 \;+\; \cdots \;+\; Z_n\,. \tag{23}$$

Indeed, in a serial circuit the same current $I(t)$ flows through all the element or sub-circuits, while their voltages add up,

$$V_{\rm net}(t) \;=\; V_1(t) \;+\; V_2(t) \;+\; \cdots \;+\; V_n(t). \tag{24}$$

For harmonic voltages, their complex amplitudes add up as complex numbers:

$$\begin{aligned} V_{\rm net}(t) \;&=\; \sum_{k=1}^{n} \mathrm{Re}\big(\hat{V}_k \times e^{j\omega t}\big) \;=\; \mathrm{Re}\left(\left(\sum_{k=1}^{n} \hat{V}_k\right) \times e^{j\omega t}\right) \\ &=\; \mathrm{Re}\big(\hat{V}_{\rm net} \times e^{j\omega t}\big) \\ \text{for}\quad \hat{V}_{\rm net} \;&=\; \sum_{k=1}^{n} \hat{V}_k\,, \end{aligned} \tag{25}$$

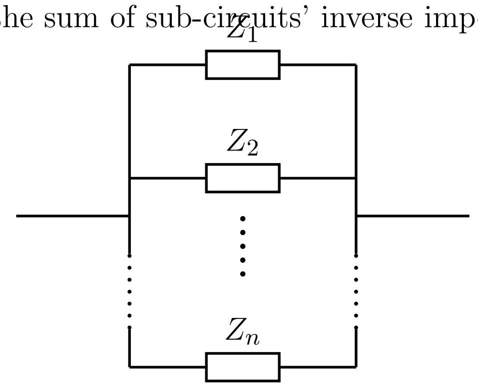
hence in terms of the common current amplitude $\hat{I}$ and the impedances Z_k,

$$\hat{V}_{\rm net} \;=\; \sum_{k=1}^{n} Z_k \times \hat{I} \;=\; \left(\sum_{k=1}^{n} Z_k\right) \times \hat{I} \tag{26}$$

and therefore

$$Z_{\rm net} \;=\; \sum_{k=1}^{n} Z_k\,. \tag{27}$$

5. For a **parallel circuit** of several elements or sub-circuits, the inverse net impedance is the sum of sub-circuits' inverse impedances,



$$\frac{1}{Z_{\rm net}} \;=\; \frac{1}{Z_1} \;+\; \frac{1}{Z_2} \;+\; \cdots \;+\; \frac{1}{Z_n}\,. \tag{28}$$

Indeed, this time we have the same voltage across every sub-circuit while their respective currents add up,

$$I_{\rm net} \;=\; \sum_{k} I_k(t)\,. \tag{29}$$

Consequently, for harmonic currents through each sub-circuit, the current amplitudes add up as complex numbers,

$$\hat{I}_{\rm net} \;=\; \sum_k \hat{I}_k\,, \tag{30}$$

or in terms of the common voltage amplitude $\hat{V}$ and the individual circuit impedances Z_k,

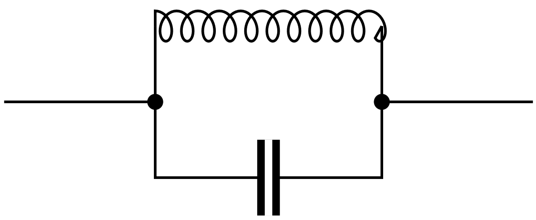
$$\hat{I}_{\rm net} \;=\; \sum_k \frac{\hat{V}}{Z_k} \;=\; \hat{V}\times\sum_k \frac{1}{Z_k}\,, \tag{31}$$

hence

$$\hat{I}_{\rm net} \;=\; \frac{\hat{V}}{Z_{\rm net}} \quad\text{for}\quad \frac{1}{Z_{\rm net}} \;=\; \sum_k \frac{1}{Z_k}\,. \tag{32}$$

EXAMPLE: LC CIRCUIT

Consider a parallel circuit made from an inductor L and a capacitor C,



For simplicity, consider an ideal inductor (without any Ohmic resistance) and an ideal capacitor; a more realistic circuit will be the subject of a homework problem. Thus, the capacitor has impedance $Z_C=1/j\omega C$, the inductor has impedance $Z_L=j\omega L$, and the whole parallel circuit has

$$\begin{aligned}\frac{1}{Z_{LC}} &= \frac{1}{Z_L} + \frac{1}{Z_C} \;=\; \frac{1}{j\omega L} \;+\; j\omega C\\ &= \frac{1-\omega^2 LC}{j\omega L} \;=\; \frac{(1/LC)-\omega^2}{(1/LC)}\times\frac{1}{j\omega L}\\ &= \frac{\omega_0^2-\omega^2}{\omega_0^2}\times\frac{1}{j\omega L}\end{aligned} \tag{33}$$

where

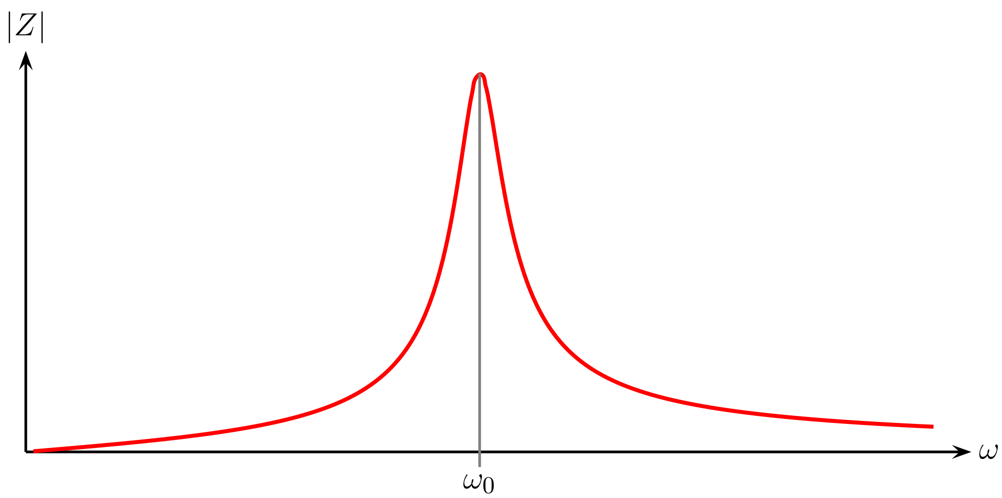
$$\omega_0 \;=\; \frac{1}{\sqrt{LC}} \tag{34}$$

is the resonant frequency of the LC circuit, hence

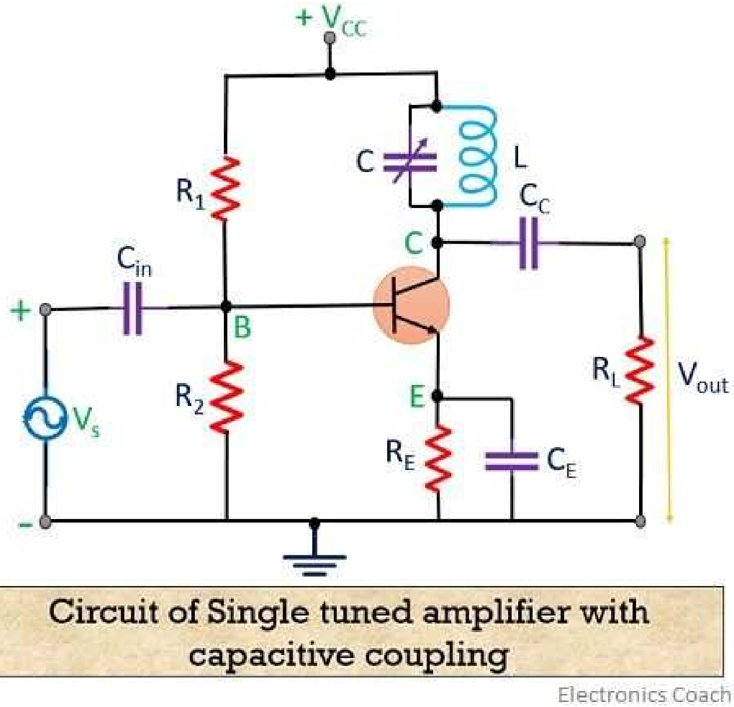
$$Z_{\rm LC} \;=\; j\omega L \times \frac{\omega_0^2}{\omega_0^2 - \omega^2}\,. \tag{35}$$

Note infinite impedance at the resonant frequency, $Z = \infty$ for $\omega = \omega_0$.

For a real LC circuit where the inductor also has some small Ohmic resistance R, the impedance at the resonant frequency is finite but very large, and the resonance happens for a narrow but finite range of frequencies around the ω_0:



In the days of discrete electronics, the LC circuits were often used as loads of radio-frequency amplifiers, for example



Here the input voltage governs the current through the transistor and the LC load, so the output voltage is proportional to the load impedance at the signal's frequency. This gives a strong amplification of the signal at near-resonant frequencies, but suppresses other frequencies. Also, by tuning the capacitance C, one may tone the resonant frequency $\omega_0 = 1/\sqrt{LC}$ and hence select the desired signal from the incoming mess.

Power and other quadratic quantities

As long as we stick to linear relations between various currents and voltages, and/or electric and magnetic fields, we may use complex numbers for harmonic variables such as $I(t) = \hat{I}e^{-i\omega t}$, *etc.*, and eventually takes the real parts of all such variables at the very end of the calculation. This approach does not work for the non-linear quantities; nevertheless, we may use the complex amplitudes to calculate the *quadratic or bilinear quantities* such as power or energy density.

Let's start with the ordinary electric power. Suppose a harmonic current flows through some circuit and there is a harmonic voltage across it,

$$I(t) \;=\; I_0 \times \cos(\omega t + \phi_i), \quad V(t) \;=\; V_0 \times \cos(\omega t + \phi_v). \tag{36}$$

Then the instantaneous power consumed by that circuit is

$$\begin{aligned} P(t) &= I(t) \times V(t) = I_0 V_0 \times \cos(\omega t + \phi_i) \cos(\omega t + \phi_v) \\ &= I_0 V_0 \times \Bigl(\tfrac{1}{2} \cos(\phi_i - \phi_v) + \tfrac{1}{2} \cos(2\omega t + \phi_i + \phi_v)\Bigr). \end{aligned} \tag{37}$$

Time-averaging this power over an oscillation period (or even over a half a period), we get

$$\langle \cos(2\omega t + \phi_i + \phi_v) \rangle = 0 \tag{38}$$

(where $\langle \cdots \rangle$ denotes time-averaging), hence

$$\langle P \rangle = I_0 V_0 \times \tfrac{1}{2} \cos(\phi_i - \phi_v). \tag{39}$$

In terms of the complex amplitudes $\hat{I} = I_0 e^{-i\phi_i}$ and $\hat{V} = V_0 e^{-i\phi_v}$, we have

$$\mathrm{Re}\bigl(\hat{I}^* \times \hat{V}^*\bigr) = I_i V_0 \times \mathrm{Re}\bigl(e^{+i\phi_i} \times e^{-i\phi_v}\bigr) = I_0 V_0 \times \cos(\phi_i - \phi_v), \tag{40}$$

and likewise

$$\mathrm{Re}\bigl(\hat{I} \times \hat{V}^*\bigr) = I_0 V_0 \times \cos(\phi_i - \phi_v). \tag{41}$$

Consequently, eq. (39) for the time-averaged electric power becomes

$$\langle P \rangle = \tfrac{1}{2} \mathrm{Re}\bigl(\hat{I}^* \times \hat{V}\bigr) = \tfrac{1}{2} \mathrm{Re}\bigl(\hat{I} \times \hat{V}^*\bigr). \tag{42}$$

Similarly, for a harmonic electromagnetic wave

$$\begin{aligned} \mathbf{E}(\mathbf{r}, t) &= \mathrm{Re}\bigl(\hat{\mathbf{E}}(\mathbf{r}) e^{-i\omega t}\bigr), \\ \mathbf{H}(\mathbf{r}, t) &= \mathrm{Re}\bigl(\hat{\mathbf{H}}(\mathbf{r}) e^{-i\omega t}\bigr), \end{aligned} \tag{43}$$

the time-averaged power flow density — *i.e.*, the time-averaged Poynting vector — is

$$\langle \mathbf{S} \rangle = \tfrac{1}{2} \mathrm{Re}\bigl(\hat{\mathbf{E}}^* \times \hat{\mathbf{H}}\bigr) = \tfrac{1}{2} \mathrm{Re}\bigl(\hat{\mathbf{E}} \times \hat{\mathbf{H}}^*\bigr), \tag{44}$$

the time-averaged energy density is

$$\langle u\rangle \;=\; \frac{\epsilon\epsilon_0}{4}\left|\hat{\mathbf{E}}\right|^2 \;+\; \frac{\mu\mu_0}{4}\left|\hat{\mathbf{H}}\right|^2, \tag{45}$$

and the time-averaged stress-tensor is

$$\left\langle T^{ij}\right\rangle \;=\; \frac{\epsilon\epsilon_0}{2}\,\mathrm{Re}\big(E^{*i}E^j\big) \;+\; \frac{\mu\mu_0}{2}\,\mathrm{Re}\big(H^{*i}H^j\big) \;-\; \langle u\rangle\times\delta^{ij}\,. \tag{46}$$

We shall explore these formulae in a later class, once we get familiar with the electromagnetic waves.

Meanwhile, let's go back to the electric power (42). The current and the voltage amplitudes are related to each other as $\hat{V}=\hat{I}\times Z$ where Z is the circuit's impedance. Consequently,

$$\mathrm{Re}\big(\hat{I}^*\times\hat{V}\big) \;=\; \mathrm{Re}\big(\hat{I}^*\times\hat{I}\times Z\big) \;=\; \left|\hat{I}\right|^2\times\mathrm{Re}(Z) \tag{47}$$

and hence

$$\langle P\rangle \;=\; \tfrac{1}{2}\left|\hat{I}\right|^2\times\mathrm{Re}(Z). \tag{48}$$

At the same time,

$$\mathrm{Re}\big(\hat{V}^*\times\hat{I}\big) \;=\; \mathrm{Re}\left(\hat{V}^*\times\frac{\hat{V}}{Z}\right) \;=\; \left|\hat{V}|\right|^2\times\mathrm{Re}\,\frac{1}{Z} \tag{49}$$

and hence

$$\langle P\rangle \;=\; \tfrac{1}{2}\left|\hat{V}\right|^2\times\mathrm{Re}\,\frac{1}{Z}\,. \tag{50}$$

But please note that for a complex impedance Z,

$$\mathrm{Re}\,\frac{1}{Z} \;\neq\; \frac{1}{\mathrm{Re}(Z)}\,. \tag{51}$$

Instead,

$$\mathrm{Re}\,\frac{1}{Z} \;=\; \frac{\mathrm{Re}(Z)}{|Z|^2} \;=\; \frac{\mathrm{Re}(Z)}{\mathrm{Re}^2(Z)+\mathrm{Im}^2(Z)} \;\leq\; \frac{1}{\mathrm{Re}(Z)}\,, \tag{52}$$

unless Z happens to be real.

Finally, a comment about electric power in Electric Engineering notations. In Engineering, one usually specifies the strength of an AC voltage or current not by their amplitudes but by its RMS (root-mean-square) values

$$V_{\rm rms} \;=\; \sqrt{\langle V^2\rangle} \quad\text{and}\quad I_{\rm rms} \;=\; \sqrt{\langle I^2\rangle}. \tag{53}$$

For harmonic voltages and currents,

$$\bigl\langle\cos^2(\omega t+\phi)\bigr\rangle \;=\; \tfrac{1}{2}, \tag{54}$$

hence

$$V_{\rm rms} \;=\; \sqrt{\tfrac{1}{2}}\times V_0\,, \quad I_{\rm rms} \;=\; \sqrt{\tfrac{1}{2}}\times I_0\,. \tag{55}$$

For example, the AC voltage in American houses is 120 V (although it's often called 110 V because it used to be 110 V many years ago), meaning $V_{\rm rms}=120$ V; the amplitude of this voltage is $V_0=\sqrt{2}\times 120\ \text{V}\approx 170$ V.

In terms of the RMS voltage and current, the (time-averaged) electric power is

$$\langle P\rangle \;=\; I_{\rm rms}\times V_{\rm rms}\times\cos(\Delta\phi) \quad\text{without the factor } \tfrac{1}{2}. \tag{56}$$

The $\Delta\phi=\phi_i-\phi_v$ in this formula is the phase difference between the voltage and the current. Note: if you pay a big electric bill, you should try to minimize this phase difference and keep $\cos(\Delta\Phi)$ as close to 1 as you can. The reason for this is that the electric utility charges you not for the Watt-hours

$$W \;=\; \int P\,dt \;=\; \int V_{\rm rms}I_{\rm rms}\cos(\Delta\phi)\,dt \tag{57}$$

you actually get to use but for the Volt-Ampere-hours

$$\overline{W} \;=\; \int V_{\rm rms}I_{\rm rms}\,dt \tag{58}$$

they have to deliver, so if the electric equipment you use has $\cos(\Delta\phi)<1$, you are paying for more electric power than you get to use!