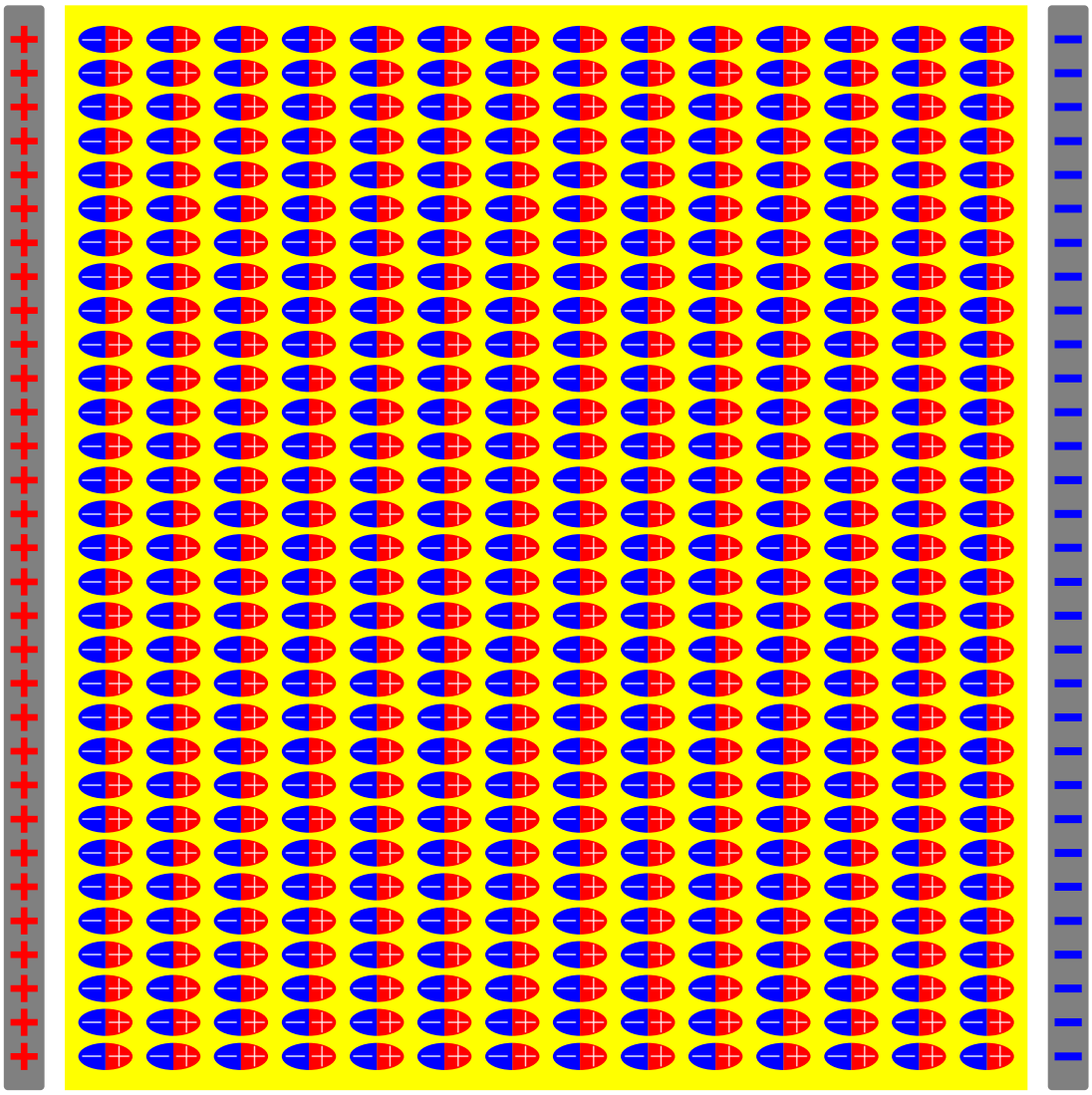
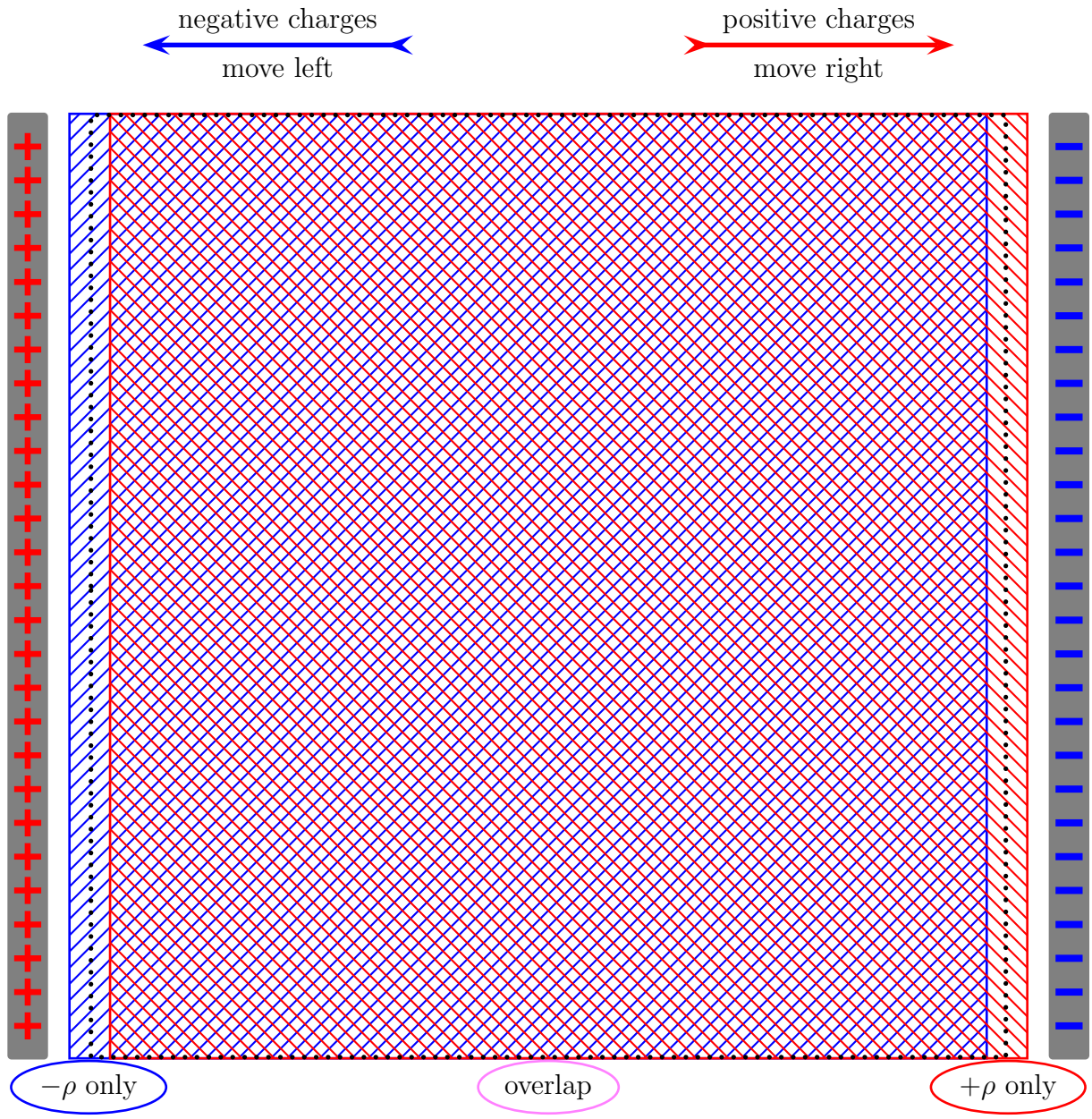
Dielectric Polarization, Bound Charges, and the Electric Displacement Field

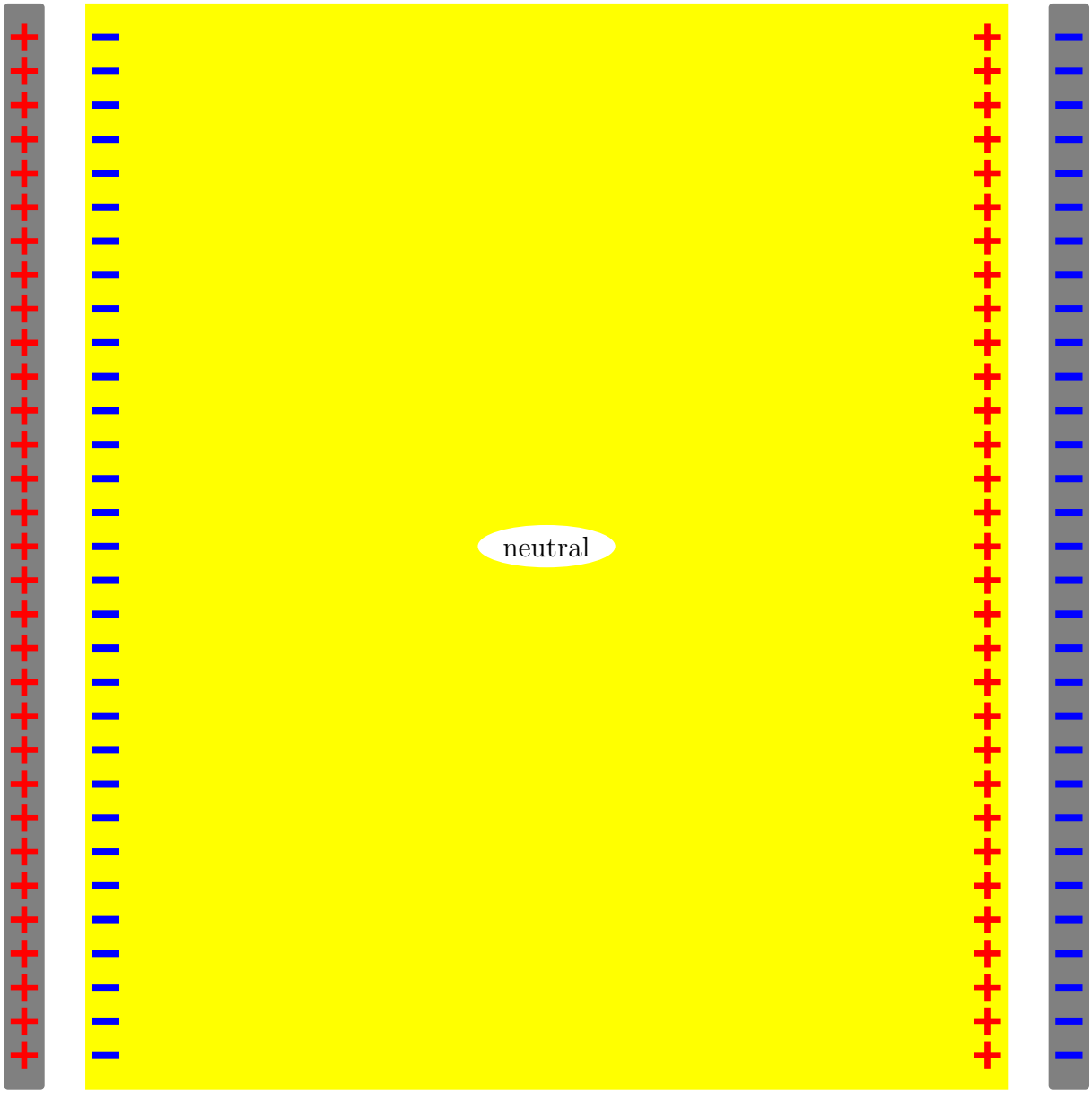
Any kind of matter is full of positive and negative electric charges. In a dielectric, these charges are *bound* — they cannot move separately from each other through any macroscopic distance, — so when an electric field is applied there is no net electric current. However, the field does push the positive charges just a tiny bit in the direction of **E** while the negative charges are pushed in the opposite directions. Consequently, the atoms and the molecules comprising the dielectric acquire tiny *electric dipole moments* in the direction of **E**.



To see the the net effect of all these dipole moments on the macroscopic scale, imagine smearing all the positive bound charges into a large uniform charge density $+\rho$ and likewise all the negative bound charges into uniform charge density $-\rho$. Without the electric field, these densities overlap each other over the whole dielectric, so the net charge density cancels out. But when we turn on the field, the positive density moves a tiny bit in the direction of **E** while the negative density moves in the opposite direction:



As the result of this move, the bulk of the dielectric — where the positive and negative charges continue to overlap each other — remains electrically neutral. But in a thin surface layer on the left side of the dielectric there are only negative charges while in a similar layer on the right side there are only positive charges. Altogether, *the dielectric's surfaces acquire non-zero net densities σ_b of bound charges* as shown on the picture below:



Now let's turn from pictures to math and calculate the net surface density of the bound charges in terms of the polarization

$$\mathbf{P} \;=\; \frac{\text{net dipole moment}}{\text{volume}}\,. \tag{1}$$

In general, the polarization is not uniform but varies on the macroscopic scale, maybe because the dielectric is non-uniform, maybe because it's subject to a non-uniform electric field, maybe both. In any case, mathematically the non-uniform $\mathbf{P}(\mathbf{r})$ acts as a *macroscopic field*. So let's calculate the electric potential $V(\mathbf{r})$ due to a general polarization field, and then re-interpret the result in terms of the net density of bound charges.

Let's start with the potential of a single ideal dipole,

$$V(\mathbf{r}) \;=\; \frac{1}{4\pi\epsilon_0}\,\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} \;=\; -\frac{1}{4\pi\epsilon_0}\,\mathbf{p}\cdot\nabla\left(\frac{1}{r}\right), \tag{2}$$

where the second equality is just the mathematical identity, $\nabla(-1/r) = \hat{\mathbf{r}}/r^2$. More generally, for a dipole $\mathbf{p}$ located at some point $\mathbf{r}_0$, the potential is

$$V(\mathbf{r}) \;=\; -\frac{1}{4\pi\epsilon_0}\,\mathbf{p}\cdot\nabla_{\mathbf{r}}\frac{1}{|\mathbf{r}-\mathbf{r}_0|} \tag{3}$$

where the the subscript of $\nabla_{\mathbf{r}}$ indicates that the gradient is taken with respect to the 3 coordinates of the $\mathbf{r}$ vector rather than of the $\mathbf{r}_0$.

Eq. (3) easily generalizes to the potential of several dipole moments, and hence to the potential of a continuous density of the dipole moment, thus

$$V(\mathbf{r}) \;=\; -\frac{1}{4\pi\epsilon_0}\iiint_{\text{dielectric}} d^3\text{Vol}\;\mathbf{P}(\mathbf{r}')\cdot\nabla_{\mathbf{r}}\frac{1}{|\mathbf{r}-\mathbf{r}'|}\,. \tag{4}$$

Note that the gradient here is taken with respect to the $\mathbf{r}$ vector — the point where we measure the potential — rather than the with respect to the integration variable $\mathbf{r}' = (x', y', z')$.

However, since $1/|\mathbf{r}-\mathbf{r}'|$ depends only on the difference between the two position vectors, we may trade the gradient WRT $\mathbf{r}$ for the (minus) gradient WRT $\mathbf{r}'$ using

$$\nabla_{\mathbf{r}} \frac{1}{|\mathbf{r}-\mathbf{r}'|} \;=\; -\nabla_{\mathbf{r}'} \frac{1}{|\mathbf{r}-\mathbf{r}'|}\,, \tag{5}$$

thus

$$V(\mathbf{r}) \;=\; +\frac{1}{4\pi\epsilon_0} \iiint\limits_{\text{dielectric}} d^3\text{Vol}\; \mathbf{P}(\mathbf{r}') \cdot \nabla_{\mathbf{r}'} \frac{1}{|\mathbf{r}-\mathbf{r}'|}\,. \tag{6}$$

At this point, the gradient acts on the integration variable, so we may integrate by parts using

$$\mathbf{P}(\mathbf{r}') \cdot \nabla_{\mathbf{r}'} \frac{1}{|\mathbf{r}-\mathbf{r}'|} \;=\; \nabla_{\mathbf{r}'} \cdot \left(\frac{\mathbf{P}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \right) \;-\; \big(\nabla \cdot \mathbf{P}(\mathbf{r}')\big) \frac{1}{|\mathbf{r}-\mathbf{r}'|} \tag{7}$$

and the Gauss theorem:

$$\begin{aligned} V(\mathbf{r}) \;&=\; +\frac{1}{4\pi\epsilon_0} \iiint\limits_{\text{dielectric}} d^3\text{Vol}\; \nabla_{\mathbf{r}'} \cdot \left(\frac{\mathbf{P}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \right) \;-\; \frac{1}{4\pi\epsilon_0} \iiint\limits_{\text{dielectric}} d^3\text{Vol}\; \big(\nabla \cdot \mathbf{P}(\mathbf{r}')\big) \frac{1}{|\mathbf{r}-\mathbf{r}'|} \\ &=\; +\frac{1}{4\pi\epsilon_0} \iint\limits_{\substack{\text{dielectric} \\ \text{surface}}} \frac{\mathbf{P}(\mathbf{r}') \cdot \mathbf{d^2A}}{|\mathbf{r}-\mathbf{r}'|} \;-\; \frac{1}{4\pi\epsilon_0} \iiint\limits_{\text{dielectric}} d^3\text{Vol}\; \big(\nabla \cdot \mathbf{P}(\mathbf{r}')\big) \frac{1}{|\mathbf{r}-\mathbf{r}'|}\,. \end{aligned} \tag{8}$$

Note that both terms on the bottom line here look like Coulomb potentials of continuous charges. Specifically, the second term looks like the Coulomb potential of the volume charge density

$$\rho_b(\mathbf{r}') \;=\; -\nabla \cdot \mathbf{P}(\mathbf{r}'), \tag{9}$$

while the first term looks like the Coulomb potential of the surface charge density

$$\sigma_b(\mathbf{r}') \;=\; \mathbf{P}(\mathbf{r}') \cdot \mathbf{n} \tag{10}$$

where $\mathbf{n}$ is the unit vector normal to the dielectric's surface at the point $\mathbf{r}'$, hence

$$\mathbf{P} \cdot \mathbf{d^2A} \;=\; (\mathbf{P} \cdot \mathbf{n})\, d^2A \;=\; \sigma_b\, d^2A. \tag{11}$$

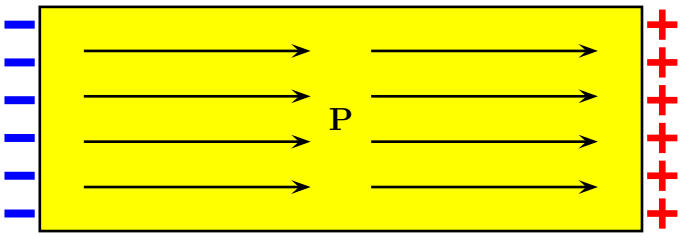
Indeed, in terms of the $\rho_b(\mathbf{r})$ and $\sigma_b(\mathbf{r})$, eq. (8) becomes

$$\begin{aligned} V(\mathbf{r}) &= \underset{\substack{\text{dielectric}\\ \text{surface}}}{\iint} \frac{\sigma_b(\mathbf{r}')\, d^2A}{4\pi\epsilon_0\,|\mathbf{r}-\mathbf{r}'|} + \underset{\substack{\text{dielectric}\\ \text{volume}}}{\iiint} \frac{\rho_b(\mathbf{r}')\, d^3\text{Vol}}{4\pi\epsilon_0\,|\mathbf{r}-\mathbf{r}'|} \\ &= V(\mathbf{r})[\text{surface charge } \sigma_b(\mathbf{r}')] \;+\; V(\mathbf{r})[\text{volume charge } \rho_b(\mathbf{r}')]. \end{aligned} \tag{12}$$

Physically, we identify the $\sigma_b = \mathbf{P}\cdot\mathbf{n}$ as the *net surface density of the bound charges* and the $\rho_b = -\nabla\cdot\mathbf{P}$ as the *net volume density of the bound charges.* Note: for general non-uniform polarizations $\mathbf{P}(\mathbf{r})$, the positive and the negative bound charge densities may mis-cancel not only on the surface of a dielectric but also inside its volume. However, for the uniform polarization there are no net volume bound charges but only the surface bound charges.

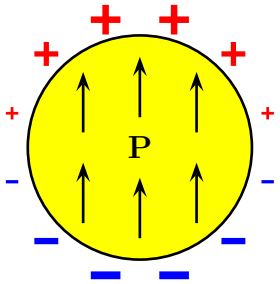
Consider a few examples:

- Dielectric cylinder, uniform $\mathbf{P}$ parallel to the axis:



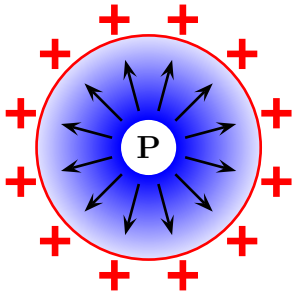
$$\begin{aligned} \sigma_b^{\text{left cap}} &= -P, \\ \sigma_b^{\text{right cap}} &= +P, \\ \sigma_b^{\text{sidewall}} &= 0, \\ \rho_b^{\text{bulk}} &= 0. \end{aligned} \tag{13}$$

- Dielectric sphere, uniform $\mathbf{P}$ in z direction:



$$\begin{aligned} \sigma_b(\theta,\phi) &= P\cos\theta, \\ \rho_b^{\text{bulk}} &= 0. \end{aligned} \tag{14}$$

- Dielectric cylinder, non-uniform polarization $\mathbf{P} = P\hat{\mathbf{s}}$ (constant magnitude in radial direction):



$$\begin{aligned} \sigma_b^{\text{sidewall}} &= +P, \\ \sigma_b^{\text{endcaps}} &= 0, \\ \rho_b^{\text{bulk}} &= -\frac{P}{s}, \\ Q_b^{\text{net}} &= 0. \end{aligned} \tag{15}$$

In fact, for any dielectric geometry and any polarization field, uniform or not, the net bound charge is zero since the bound charges cannot in or out of the dielectric. Indeed,

$$\begin{aligned} Q_b^{\text{net}} &= \underset{\substack{\text{dielectric}\\\text{surface}}}{\iint} \sigma_b \, d^2A \; + \underset{\substack{\text{dielectric}\\\text{volume}}}{\iiint} \rho_b \, d^3\text{Vol} \\ &= \underset{\substack{\text{dielectric}\\\text{surface}}}{\iint} (\mathbf{P}\cdot\mathbf{n}) \, d^2A \; - \underset{\substack{\text{dielectric}\\\text{volume}}}{\iiint} (\nabla\cdot\mathbf{P}) \, d^3\text{Vol} \\ &= \underset{\substack{\text{dielectric}\\\text{surface}}}{\iint} \mathbf{P}\cdot\mathbf{d^2A} \; - \underset{\substack{\text{dielectric}\\\text{volume}}}{\iiint} (\nabla\cdot\mathbf{P}) \, d^3\text{Vol} \\ &= 0 \quad \text{by Gauss theorem.} \end{aligned} \tag{16}$$

Gauss Law and the Electric Displacement Field

Besides the bound charges due to polarization, a dielectric material may also contain some extra charges which just happen to be there. For a lack of a better term, such extra charges are called *free charges*, just to contrast them from the bound charged due to polarization. Anyway, the net macroscopic electric field does not care for the origin of the electric

charges or how we call them but only for the net electric charge of whatever origin,

$$\rho_{\text{net}}(\mathbf{r}) \;=\; \rho_{\text{free}}(\mathbf{r}) \;+\; \rho_{\text{bound}}(\mathbf{r}), \quad \sigma_{\text{net}}(\mathbf{r}) \;=\; \sigma_{\text{free}}(\mathbf{r}) \;+\; \sigma_{\text{bound}}(\mathbf{r}), \quad \textit{etc.} \tag{17}$$

In particular, the Gauss Law in differential form says

$$\begin{aligned} \epsilon_0 \nabla \cdot \mathbf{E}(\mathbf{r}) \;=\; \rho_{\text{net}}(\mathbf{r}) \;+\; \sigma_{\text{net}} \delta(\text{coordinate} \perp \text{surface}) \\ +\; \text{more } \delta\text{-function terms due to linear and point charges.} \end{aligned} \tag{18}$$

For simplicity, let's ignore for a moment the surface, line, and point charges and focus on just the volume charge density. This gives us

$$\epsilon_0 \nabla \cdot \mathbf{E} \;=\; \rho_{\text{net}} \;=\; \rho_{\text{free}} \;+\; \rho_{\text{bound}} \;=\; \rho_{\text{free}} \;-\; \nabla \cdot \mathbf{P}\,, \tag{19}$$

which we may rewrite as

$$\nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) \;=\; \rho_{\text{free}}\,. \tag{20}$$

In light of this formula, the combination

$$\mathbf{D}(\mathbf{r}) \;=\; \epsilon_0 \mathbf{E}(\mathbf{r}) \;+\; \mathbf{P}(\mathbf{r}) \tag{21}$$

called the *electric displacement field* obeys the Gauss Law involving only the free charges but not the bound charges,

$$\nabla \cdot \mathbf{D}(\mathbf{r}) \;=\; \rho_{\text{free}}\,. \tag{22}$$

⋆ A point of terminology: in contrast to "the electric *displacement* field" $\mathbf{D}$, the $\mathbf{E}$ is called "the electric *tension* field". But usually, $\mathbf{E}$ is simply called "the electric field" while $\mathbf{D}$ is called "the displacement field".

Outside the dielectric there is no polarization, so we simply set $\mathbf{D}(\mathbf{r}) = \epsilon_0 \mathbf{E}(\mathbf{r})$. Consequently, at the dielectric's boundary, the $\mathbf{D}$ field has two sources of discontinuity: (1) the abrupt disappearance of the $\mathbf{P}$ term, and (2) discontinuity of the $\mathbf{E}$ field due to the net surface charge density. If we focus on the component of the $\mathbf{D}$ vector $\perp$ to the boundary, we get

$$\mathrm{disc}(D_\perp) \;=\; -P_\perp^{\text{inside}} + \epsilon_0\,\mathrm{disc}(E_\perp) \;=\; -\mathbf{P}\cdot\mathbf{n} \;+\; \sigma_{\text{net}} \;=\; -\sigma_{\text{bound}} \;+\; \sigma_{\text{net}} \;=\; \sigma_{\text{free}}\,. \quad (23)$$

Again, only the free surface charge affects this discontinuity, the bound surface charge cancels out between the $\mathbf{P}$ and the $\epsilon_0\mathbf{E}$ terms.

In terms of the $\mathbf{D}$ field's divergence, the discontinuity (20) becomes

$$(\nabla\cdot\mathbf{D})_{\text{singular}} \;=\; \mathrm{disc}(D_\perp)\,\delta(\text{coordinate}\perp\text{surface}) \;=\; \sigma_{\text{free}}\,\delta(\text{coordinate}\perp\text{surface}). \quad (24)$$

Similar δ-function terms would obtain at free surface charges stuck inside the dielectric or outside it, and if we also allow for line or point free charges, there would also be $\delta^{(2)}$ and $\delta^{(3)}$ on the RHS. Altogether, for a most general geometry of free charges the complete Gauss Law for the $\mathbf{D}$ field becomes

$$\begin{aligned}\nabla\cdot\mathbf{D}(\mathbf{r}) \;=\; \rho_{\text{free}}(\mathbf{r}) \;&+\; \sigma_{\text{free}}\,\delta(\text{coordinate}\perp\text{surface}) \\ &+\; \text{more } \delta\text{-function terms due to } \textit{free} \text{ linear and point charges,}\end{aligned} \quad (25)$$

but only the free charges appear on the RHS here. All contributions of the bound charges cancel against the $\nabla\cdot\mathbf{P}$ term hiding inside the $\nabla\cdot\mathbf{E}$ on the LHS.

Therefore, the integral form of the Gauss Law for the $\mathbf{D}$ field is

$$\iint\limits_{\substack{\text{complete}\\ \text{surface } \mathcal{S}}} \mathbf{D}\cdot\mathbf{d^2A} \;=\; Q_{\text{free}}[\text{surrounded by } \mathcal{S}] \quad (26)$$

where the RHS includes all possible configurations of free charges surrounded by the surface $\mathcal{S}$, but only the free charges. By the way, the surface $\mathcal{S}$ in this formula can be any complete surface — it can lie wholly completely the dielectric, or completely outside it, or even cross the dielectric's surface — but the Gauss Law will work in any case, as long as we properly account for the net free charge inside $\mathcal{S}$.

- Example: consider a solid dielectric ball or radius R with a free charge q stuck at its center. We know polarization field $\mathbf{P}(\mathbf{r}) = P(r)\hat{\mathbf{r}}$ is spherically symmetric, but we don't know its radial profile.

  In this case, the combination of spherical symmetry and the Gauss Law (26) immediately gives us the displacement field $\mathbf{D}$ both inside and outside the dielectric. Indeed, by spherical symmetry $\mathbf{D}(\mathbf{r}) = D(r)\hat{\mathbf{r}}$, hence for $\mathcal{S}$ being a sphere of any radius r

$$\iint_{\mathcal{S}} \mathbf{D} \cdot \mathbf{d^2 A} \;=\; D(r) \times 4\pi r^2. \tag{27}$$

  However, the only free charge inside any such sphere is the point charge q at the center, thus

$$D(r) \;=\; \frac{q}{4\pi r^2} \quad\Longrightarrow\quad \mathbf{D}(\mathbf{r}) \;=\; \frac{q\,\hat{\mathbf{r}}}{4\pi r^2}\,, \tag{28}$$

  both inside and outside the dielectric ball.

  As to the electric tension field $\mathbf{E}$, eq. (28) immediately tells us that outside the ball,

$$\mathbf{E}(\mathbf{r}) \;=\; \frac{\mathbf{D}(\mathbf{r})}{\epsilon_0} \;=\; \frac{q\,\hat{\mathbf{r}}}{4\pi\epsilon_0\, r^2} \tag{29}$$

  regardless of the polarization's radial profile $P(r)$. However, inside the dielectric ball we may only tell that

$$\mathbf{E}(\mathbf{r}) \;=\; \frac{1}{\epsilon_0}\left(\frac{q}{4\pi r^2} \;-\; P(r)\right)\hat{\mathbf{r}}, \tag{30}$$

  but we would need another equation relating the polarization to the electric field in order to completely determine the $\mathbf{E}(\mathbf{r})$.

Warning: While in highly symmetric cases — like the above example — we may calculate the $\mathbf{D}$ field just from the Gauss Law, more general geometries do not allow such shortcuts.

Even the Coulomb-like integrals over the free charges like

$$\mathbf{D}(\mathbf{r}) \stackrel{??}{=} -\nabla \iiint \frac{\rho_{\text{free}}(\mathbf{r}')\, d^3\text{Vol}}{4\pi|\mathbf{r}-\mathbf{r}'|} \tag{31}$$

do not work for general geometries of dielectrics and free charges. In fact, in general

$$\mathbf{D}(\mathbf{r}) \neq -\nabla(\text{anything}) \tag{32}$$

unlike $\mathbf{E}(\mathbf{r}) = -\nabla V(\mathbf{r})$. Indeed, while the electrostatic tension field always obeys $\nabla \times \mathbf{E} = 0$, for the displacement field we have

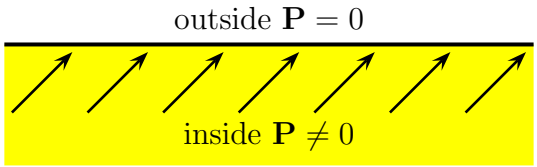
$$\nabla \times \mathbf{D} = \epsilon_0 \nabla \times \mathbf{E} + \nabla \times \mathbf{P} = \nabla \times \mathbf{P}, \quad \text{which does not need to vanish.} \tag{33}$$

This issue becomes particularly acute at the dielectric's boundary. We know that despite the surface charge density of the bound charges, the potential and the tangential components of the electric tension field must be continuous across the boundary,

$$V(\text{just outside}) = V(\text{just inside}), \tag{34}$$

$$\mathbf{E}^{\parallel}(\text{just outside}) = \mathbf{E}^{\parallel}(\text{just inside}). \tag{35}$$

On the other hand, the polarization field $\mathbf{P}$ abruptly drops to zero at the boundary, so if $\mathbf{P}$ just inside the dielectric is directed at some general angle to the surface, then both its tangential and normal components suffer discontinuities,



$$\begin{aligned} \mathbf{P}^{\parallel}(\text{just inside}) &\neq \mathbf{P}^{\parallel}(\text{outside}) = 0, \\ P^{\perp}(\text{just inside}) &\neq P^{\perp}(\text{outside}) = 0. \end{aligned} \tag{36}$$

Combining eqs (35) and (36), we find that in general *the tangential components of the displacement field are discontinuous across the dielectric's boundary,*

$$\mathbf{D}^{\parallel}(\text{just outside}) \neq \mathbf{D}^{\parallel}(\text{just inside}). \tag{37}$$

As to the normal components of the tension and the displacement fields, the $E^{\perp}$ is generally discontinuous across the boundary due to surface bound charges, but as we saw in eq. (23),

in the absence of free charges

$$D^{\perp}(\text{just outside}) \;=\; D^{\perp}(\text{just inside}). \tag{38}$$

SUMMARY.

So here is the summary of equations governing the electrostatic tension and displacement fields $\mathbf{E}(\mathbf{r})$ and $\mathbf{D}(\mathbf{r})$ in and around dielectric materials.

In the middle of a dielectric or outside it:

$$\nabla \cdot \mathbf{D} \;=\; \rho_{\text{free}}\,, \tag{39}$$

$$\nabla \times \mathbf{E} = \; 0. \tag{40}$$

On the other hand,

$$\nabla \cdot \mathbf{E} \;=\; (1/\epsilon_0)\rho_{\text{net}}\,, \quad \text{which is often unknown,} \tag{41}$$

$$\nabla \times \mathbf{D} \;=\; \text{unknown.} \tag{42}$$

At the outer boundary of a dielectric, or at the boundary between two different dielectrics:

- V, $\mathbf{E}^{\parallel}$, and $D^{\perp}$ are continuous,
- but $E^{\perp}$ and $\mathbf{D}^{\parallel}$ are discontinuous.

To complete this equation system, we need a relation between the $\mathbf{E}$ and the $\mathbf{D}$ fields at the same point $\mathbf{r}$, and that relation depends very much on the dielectric material in question. For some material, such relation can be rather complicated, or even history-dependent. Fortunately, for most common dielectrics the relation between the $\mathbf{E}$ and $\mathbf{D}$ fields is *linear* (unless the fields become too strong). So in this class, we shall henceforth focus on such *linear dielectrics*.

Linear Dielectrics

In linear dielectrics, the polarization $\mathbf{P}$ is proportional to the macroscopic electric field $\mathbf{E}$,

$$\mathbf{P} \;=\; \chi\epsilon_0\,\mathbf{E} \tag{43}$$

where χ is the *susceptibility* of the particular dielectric material; the ϵ_0 factor is included in eq. (43) to make the susceptibility χ dimensionless. For gases at standard conditions $\chi \sim$ (a few) $\times\, 10^{-4}$, for many common liquid and solid insulators $\chi \sim$ (a few) to (few tens), but some special ceramics (titanates, *etc.*) have $\chi \sim$ a few thousands. For some anisotropic crystals the susceptibility is different along different crystalline axes, hence

$$P_i \;=\; \epsilon_0 \sum_{j=x,y,z} \chi_{i,j} E_j \,, \tag{44}$$

but for this class we shall assume isotropic susceptibility as in eq. (43).

In terms of the displacement field $\mathbf{D}$, eq. (43) means

$$\mathbf{D} \;=\; \epsilon_0\mathbf{E} \;+\; \chi\epsilon_0\mathbf{E} \;=\; (\chi+1)\epsilon_0\mathbf{E} \;=\; \epsilon\epsilon_0\mathbf{E} \tag{45}$$

where the coefficient

$$\epsilon \;=\; \chi \;+\; 1 \tag{46}$$

is called the *dielectric constant* or the *relative permittivity* of the dielectric in question.

- A point of terminology: the dimensionless factor $\epsilon = \chi + 1$ is called the *relative* permittivity while the overall D/E ratio $\epsilon \times \epsilon_0$ is called the *absolute* permittivity of the dielectric medium. In this context, the ϵ_0 factor of the MKSA unit system is called the *dielectric permittivity of the vacuum* or simply *vacuum permittivity.*

⋆ A table of dielectric constants of many common materials can be found at https://www.kabusa.com/Dilectric-Constants.pdf [note (mis)spelling!].

Now consider a few free charges stuck inside a uniform linear dielectric. For uniform the D/E ratio $\epsilon\epsilon_0 = \text{const}$, the Gauss Law

$$\nabla \cdot \mathbf{D} \;=\; \rho_{\text{free}} \tag{47}$$

translates to

$$\nabla \cdot \mathbf{E} \;=\; \frac{1}{\epsilon\epsilon_0}\, \nabla \cdot \mathbf{D} \;=\; \frac{\rho_{\text{free}}}{\epsilon\epsilon_0} \tag{48}$$

and hence

$$\rho_{\text{net}}(\mathbf{r}) \;=\; \epsilon_0 \nabla \cdot \mathbf{E} \;=\; \frac{1}{\epsilon}\, \rho_{\text{free}}(\mathbf{r}) \quad \Longrightarrow \quad \rho_{\text{bound}} \;=\; -\frac{\epsilon - 1}{\epsilon}\, \rho_{\text{free}}\,. \tag{49}$$

Thus, in the bulk of a uniform linear dielectric, the bound charge shadows the free charge, so the net charge is simply $(1/\epsilon) \times$ the free charge.

For example, consider a point free charge Q stuck inside a large piece of uniform dielectric. As long as we are much closer to that charge than to the boundary, the electric field is simply $(1/\epsilon)$ of the usual Coulomb field of Q,

$$\mathbf{E} \;=\; \frac{Q}{4\pi\epsilon\epsilon_0}\, \frac{\hat{\mathbf{r}}}{r^2}\,. \tag{50}$$

Consequently, the Coulomb force between two point free charges inside a uniform dielectric is simply $(1/\epsilon)$ of the Coulomb force in the vacuum,

$$\mathbf{F}_{12} \;=\; \frac{Q_1 Q_2}{4\pi\epsilon\epsilon_0}\, \frac{\hat{\mathbf{r}}_{12}}{r_{12}^2}\,. \tag{51}$$

In particular, the force between two ions in water — which has $\epsilon \approx 80$ — is 80 times weaker that the force between similar ions at similar distance in the vacuum.

But please note that the simple relations (49) between the free charge, the bound charge, and the net charge works only in the bulk of the dielectric. Locally, the bound charge *screens* part of the free charge, but globally the bound charge cannot leave the dielectric; instead, it goes to the dielectric's outer boundary. To see how this works, let's go back to the example

of a solid dielectric ball with a free charge q stuck at the ball's center. Earlier we used Gauss Law and spherical symmetry to show that

$$\mathbf{D}(\mathbf{r}) \;=\; \frac{q\,\hat{\mathbf{r}}}{4\pi r^2} \tag{52}$$

both inside and outside the dielectric ball. For the linear dielectric, this displacement field translates into the tension field

$$\begin{aligned} \mathbf{E}(\mathbf{r}) \;&=\; \frac{q}{4\pi\epsilon_0}\,\frac{\hat{\mathbf{r}}}{r^2} \quad \text{outside the ball,} \\ \mathbf{E}(\mathbf{r}) \;&=\; \frac{q}{4\pi\epsilon\epsilon_0}\,\frac{\hat{\mathbf{r}}}{r^2} \quad \text{inside the ball.} \end{aligned} \tag{53}$$

The field inside the ball corresponds to a point net charge at the center

$$Q_{\text{net}}[\text{center}] \;=\; \frac{q}{\epsilon} \tag{54}$$

in perfect agreement with eq. (49). Physically, the net charge is reduced from q to q/ϵ by the bound point charge

$$Q^{\text{net}}_{\text{bound}}[\text{center}] \;=\; Q_{\text{net}}[\text{center}] \;-\; q \;=\; -\frac{\epsilon-1}{\epsilon}\times q \tag{55}$$

which partially screens the free charge q.

Now consider the ball's surface at $r = R$ where the electric field (53) has a discontinuity, which indicates the surface charge density

$$\sigma \;=\; \epsilon_0\,\text{disc}(E_r) \;=\; \frac{q}{4\pi R^2}\left(1 \,-\, \frac{1}{\epsilon}\right). \tag{56}$$

Physically, this is the surface bound charge stemming from the polarization

$$\begin{aligned} \mathbf{P} \;&=\; \mathbf{D} \,-\, \epsilon_0\mathbf{E} \;=\; \mathbf{D}\left(1 \,-\, \frac{1}{\epsilon}\right) \;=\; \frac{\epsilon-1}{\epsilon}\,\frac{q\hat{\mathbf{r}}}{4\pi r^2}\,, \\ &\Downarrow \\ \sigma_b \;&=\; \mathbf{P}\cdot\mathbf{n} \;=\; P_r \;=\; \frac{\epsilon-1}{\epsilon}\,\frac{q}{4\pi R^2}\,. \end{aligned} \tag{57}$$

The net bound charge at the surface amounts to

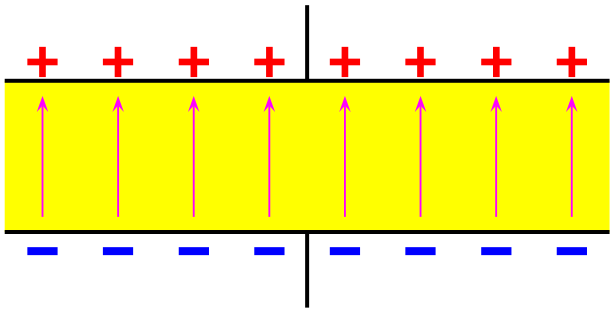
$$Q^{\text{net}}_{\text{bound}}[\text{surface}] \;=\; 4\pi R^2 \times \sigma_b \;=\; \frac{\epsilon - 1}{\epsilon} \times q, \tag{58}$$

which is precisely the opposite of the bound charge (55) at the ball's center,

$$Q^{\text{net}}_{\text{bound}}[\text{surface}] \;=\; \frac{\epsilon - 1}{\epsilon} \times q \;=\; -Q^{\text{net}}_{\text{bound}}[\text{center}]. \tag{59}$$

In less symmetric situations, the bound charges in the dielectric's bulk follow from the free charges according to eq. (49), but the bound charges on the dielectric's surface are much harder to figure out. My next set of notes gives two examples of such *dielectric boundary problems:* (1) a dielectric ball in external electric field, and (2) a point charge above a thick dielectric slab.

But for the moment, let me consider a much simpler setup: a parallel-plate capacitor filled with a uniform dielectric,



Let d be the distance between the plates and A the plates' area. Given the voltage V between the capacitor plates, the electric field inside the capacitor is approximately uniform

$$E \;=\; \frac{V}{d} \tag{60}$$

in the direction from the negative plate to the positive plate, and hence inside the dielectric

$$D \;=\; \epsilon\epsilon_0 E \;=\; \epsilon\epsilon_0 \times \frac{V}{d}\,. \tag{61}$$

At the boundaries of the dielectric there are both bound charges due to polarization and free charges on the capacitor plates. However, the displacement field cares only about the

free charges, so to maintain the field D inside the capacitor, we need the plates to have free charge densities

$$\sigma_{\text{free}} \;=\; \pm D. \tag{62}$$

That is

$$\sigma_{\text{top plate}} \;=\; +D, \quad \sigma_{\text{bottom plate}} \;=\; -D. \tag{63}$$

Altogether, the plates have net charges $\pm Q$ where

$$Q \;=\; A \times \sigma_{\text{free}} \;=\; A \times D \;=\; A \times \epsilon\epsilon_0 \times \frac{V}{d}\,. \tag{64}$$

In other words, $Q = C \times V$ where the capacitance C is

$$C \;=\; \epsilon\epsilon_0\,\frac{A}{d}\,. \tag{65}$$

Thus, comparing this capacitor to a capacitor with the same plate geometry but without the dielectric, we find that *filling a capacitor with dielectric increases its capacitance by the factor of* ϵ,

$$C[\text{with dielectric}] \;=\; \epsilon \times C[\text{without dielectric}]. \tag{66}$$

The same rule applies to all capacitor geometries — parallel plates, coaxial cylinders, concentric spheres, whatever — as long as the space between the plates is completely filled up with a dielectric. For example, for coaxial cylindrical plates of length L and radii a and b,

$$C \;=\; \epsilon \times \frac{2\pi\epsilon_0\,L}{\ln(b/a)}\,, \tag{67}$$

for concentric spheres of radii a and b,

$$C \;=\; \epsilon \times \frac{4\pi\epsilon_0\,ab}{b-a}\,, \tag{68}$$

etc., *etc.* However, the capacitors that are only partially filled with a dielectric are more complicated, and you should see a few relatively simple examples in your homework.