Chapter 11

Kinematics of special relativity

11.1 Special Relativity

11.1.1 Principles of Relativity

Einstein postulated that there was still Galilean invariance, i. e. all uniformly moving observers had the same laws of physics; there was still no way to determine a velocity. The thing that they also agreed upon included Maxwell's equations and thus the speed of light. The problem then becomes one of defining lengths and times so that this can be done. From Section **??**, we realize that, instead of an arbitrary distance between scratches on a bar being the standard, distance can be defined from a velocity and a time. Thus, if we have a time such as the period of light from a particular atom, we can define lengths from the speed of light. If Maxwell's Equations are to be valid in all frames the speed of light , c, must be a universal constant. We will examine this concept later. We can use this so that we no longer have a fundamental unit of length. Lengths follow from this velocity and a standard to time. In other words, we use a time and c as our fundamental units and c is defined in such a way that we recover the usual meter. This change in the definition of length manifests itself in a good table of physical values by having the speed of light given as

$$c = 2.99792458 \times 10^8 \frac{\mathrm{m}}{\mathrm{sec}} \quad \text{(exact)}. \tag{11.1}$$

In other words, we can pick the value for c since it is the standard. It is chosen so that the distance that we called the meter is what it was before.

Said another way, the meter is the length of the path traveled by light in vacuum during the time interval of $\frac{1}{299,792,458}$ of a second.

Digression on Dimensions

In olden times, the basic measured quantities were a mass, a length and a time. The standards were arbitrary and chosen for convenience. We then chose to use standards that were stable, accessible, and easy to use: the kilogram, the meter, and the second. We realized though in Section **??** that we could use any set of algebraically independent combination of the three fundamental dimensional entities such as an energy, velocity, and a momentum. Then, you may ask, what could be more accessible and stable than the fundamental dimensional constants? The problem is to chose. There are lots of constants in physics that have dimension and could be called fundamental. One obvious example is the mass of an elementary particle like the electron. In some sense that is what was done when we chose the mass of the nucleus of carbon 12. Modern physicists would not choose this as a standard because we feel that we will calculate it in some future Theory of Everything. In fact, the hope is that the future theory will contain only the constants c, the speed of light, $\hbar$, Planck's constant divided by 2π, and G, Newton's constant in the gravitational force. These form an independent set that contain a length, mass, and a time. As indicated above, we already use c. With the increase in the precision with which we can measure $\hbar$, it will not be long before we replace the standard of mass with a standard based on $\hbar$. This will still leave time as the remaining old fashioned standard. The current standard is based on the frequency of a specific emission of the light from the cesium atom. Time can be measured with great precision and reproducibility and this is not likely to change. This is in contrast to G, Newton's constant, which because the gravitational force is so weak is difficult to measure with any precision.

Prior to Einstein's development of the Special Theory of Relativity, we had as the basis for our understanding of space time that:

1. There is no experiment that can detect a uniform state of motion.

   Another way to say this is that you are always at rest in your own rest frame. It also means that you can **not** talk about going at a certain speed. All you can talk about is how fast you are moving relative to some other thing.

   This and

2. Length and time scales are absolute. This is the statement that regardless of your motion clocks run the same and the definition of length is the same.

A direct result of these postulates is that the relationship for the coordinates for an events when observed by two uniformly moving observers with relative speed v along the x axis is Equations (10.1).

With Einstein, by requiring that Maxwell's equations are the same to all observers, these postulates have to change. The new postulates are:

1. There is no experiment that can detect a uniform state of motion.

   Galilean invariance is retained although the transformation rule, Equation (10.1), will have to be changed.

2. The speed of light is a universal constant.

   Although Einstein came to this conclusion from his work with Maxwell's equations, it is also a direct consequence of the Michelson Morley Experiment. The implications of this postulate are far reaching. Some are obvious. It implies that the speed of light is the same in all directions and it is the same value to all inertial observers with measuring instruments that are commensurable. Others more subtle.

Reversing our thinking. Since the way that light travels is determined from Maxwell's equations, we have to find the transformation law between inertial observers that will preserve Maxwell's equations. Another way to say this is that we know the correct transformations of space and time between inertial observers must be such that Maxwell's Equations are invariant. Actually, it is even more general than that. We will have a set of transformations that leave a certain velocity, the speed of light, invariant. This is the velocity that light travels at because Maxwell's equations do not have any additional dimensional fundamental constants other than the speed of light.

Later, Section 12.2.2, we will develop the set of transformations that will yield the same speed for light for all observers. For two observers with a relative speed v and choosing the positive x axis along the direction of relative motion between the second and the first observer, this set of transformations is called the Lorentz transformations and is:

$$x' = \gamma(x - \beta ct)$$

$$\begin{aligned} y' &= y \\ z' &= z \\ ct' &= \gamma(ct - \beta x) \end{aligned} \tag{11.2}$$

where

$$\beta = \frac{v}{c} \tag{11.3}$$

and

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{11.4}$$

For $\frac{v}{c} \ll 1$ these reduce to the Galilean transformations. We will derive them later, Section 12.2.2. For now, just realize that they exist.

11.2 Harry and Sally and Space Time Diagrams

11.2.1 Introduction

The idea will be to develop an understanding of the implications about the nature of space and time that are implied by our postulates about relativity. We will do this by looking at a simple case of two relatively moving observers, Harry and Sally, and their observations. At the same time we will develop a powerful graphical analysis that will allow us to understand different situations.

11.2.2 The Paradox of Harry and Sally

Harry and Sally are two inertial observers. Harry is moving toward Sally at a high rate of speed. He is equipped with a battery pack and plug that fits an outlet Sally is wearing and is connected to a light bulb that she has on her head. When he passes her the circuit is complete and he lights her light bulb.

A while later she writes to him. She says that she liked it when he went by and often looks out at the outgoing sphere of light that they generated together and remembers him fondly. She wishes that he was with her again at the center of that sphere of outgoing light.

He writes back that yes it was nice when he passed her but he has to inform her that he is at the center of the outgoing sphere of light and not her.

The paradox of this situation is that Harry and Sally are both correct. They both measure the light as traveling at the same speed, c. The speed of light for both of Harry and Sally is the same in all directions and thus they both see themselves as always at the center of the outgoing sphere of light. Since once they have parted, they are at different places this is a paradox. The resolution of this paradox will be at the heart of understanding relativity. In the following section we will resolve this paradox.

11.3 The Relativity of Simultaneity

In order to better understand the what is going on with Harry and Sally, let's look at another but similar situation. Consider two inertial observers. One is on a train standing in the center of one of the cars and the other is on the platform. The train is moving relative to the platform. At the instant that the train and platform observers coincide, a small firecracker explodes at their common position. There are photocells at each end of the rail car. The light from the firecracker travels to the ends of the car and triggers the two photocells. The observer on the train says that the events of triggering the photocells happen at the same time; that observer says that they are simultaneous. See Fig 11.1. The observer on the platform, on the other hand, says that the photocell in the back of the car fired before the photocell in the front of the car. See Fig 11.2. To that observer the events of the arrival of the light at the photocell were not simultaneous but the arrival of the light on the back of the car preceded the one on the front.

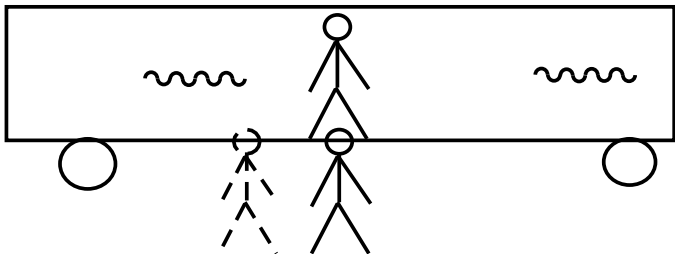

Figure 11.1: **Observer on a Moving Train:** In this case, the observer who is in the center of the car says that the light from the firecracker reaches the back of the car and the front of the car at the same time. The train is moving from left to right so we see the platform observer to the left of the original position, shown dashed, at a later time.

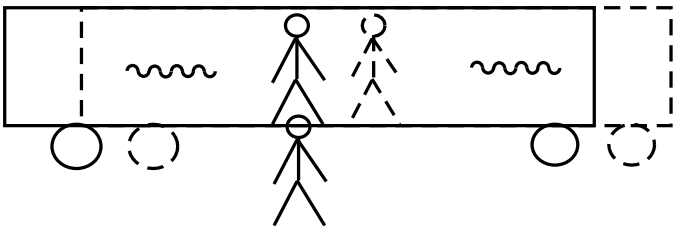

Figure 11.2: **Observer on the Platform:** In this case, at a later time, the observer who is on the platform sees the car move to the right. Since the speed of light is the same in the right and left directions, the light traveling toward the back of the car goes a shorter distance and, thus, arrives at the back of the car before the light that is sent to the front of the car. The events of the arrival of the light at the back and the front of the car are not simultaneous to the platform observer.

In summary, because of the constancy of the speed of light, we must conclude that the two spatially separated events that are simultaneous to one observer will not be simultaneous to a relatively moving observer.

11.3.1 Harry and Sally's Movements in a Diagram

To understand what is going on with Harry and Sally, we will analyze the situation graphically. For simplicity of analysis and presentation, we will work in only one space dimension. Later, when we derive the Lorentz transformations, three spatial dimensions will be used, see Section 12.2.2.

If we assign a coordinate system to Sally, we obtain the following description of what is going on. First, let's clarify some notation. In an ordinary graph, for instance plotting the xy plane, the line labeled the x axis is really the set of places that have coordinate y take the value zero or, better said, the x axis is better thought of as the $y = 0$ line. Similarly, the y axis is better thought of as the $x = 0$ line.

In space-time, we will draw the time axis vertically and the position or x axis horizontally. Again, you should think of the time axis as the place that is $x = 0$ for all times and the x axis as the time $t = 0$ for all places.

If we draw what is happening in a system based on Sally's observations, see Figure 11.3, we will place Sally's time axis, her $x_s = 0$ line, vertically. Her x axis, the $t_s = 0$ line, will be horizontal. Harry is going by her at a relative speed of v. Therefore, the set of events that is Harry is a line with

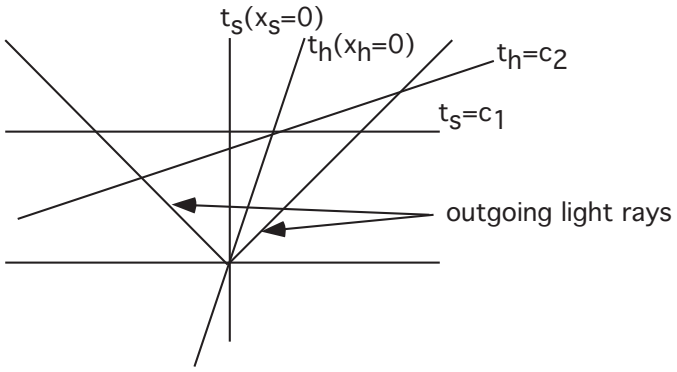

Figure 11.3: **Sally's Space-time Diagram:** Sally's space-time description of her meeting with Harry. Sally's time axis is vertical and her space axis is horizontal. Events at some time t according to Sally are horizontal lines such as $t_s = c_1$. Harry is the line $t = \frac{1}{v}x$. The events that are simultaneous to Harry are a line slopped at $\frac{v}{c^2}$ such as $t_h = c_2$. See Figure 11.5 and the following text for details. The light rays generated at their meeting at the event $(0, 0)$ are the lines $t = \pm\frac{1}{c}$.

slope $\frac{1}{v}$. Don't forget, we are drawing the time axis vertically and slope is rise divided by run. Now, this set of events is what Harry would call his $x_h = 0$ line. In other words, if we choose the event of their coincidence as the origin event, $(0, 0)$, the equation of Harry's time axis on Sally's coordinate system is

$$t = \frac{1}{v}x. \tag{11.5}$$

Of course, this is because we chose $t_s = 0$ as the time for the event when they were together. We choose this as $t_h = 0$ for Harry also. They both label the event of coincidence as $(0, 0)$. At $t_{s,h} = 0$, a light pulse emerges at $x_{s,h} = 0$ and moves away from both of them at the speed of light. On Sally's coordinate system, these events are two lines through the event $(0, 0)$ with slope $\pm\frac{1}{c}$. At some time later, $t_s = c_1$, Sally determines that she is at the center of the outgoing pulses of light and that Harry is not at the center, which is always at her place, $x_s = 0$, but instead he is at $x = v(t_s = c_1) > 0$.

We can just as well draw all of this from Harry's point of view, see Figure 11.4. Harry is an inertial observer also. Now it is Harry's time axis that is vertical. Sally's time axis is now a straight line slopped at $-\frac{1}{v}$. We have picked the positive x direction to be the same for Harry and Sally. Thus, she is moving to negative position values in reference to Harry. Events at

some time t to Harry are horizontal lines on this coordinate system and again at any time $t_h = C$ that Harry looks out he is at the center of the outgoing pulses of light and Sally is at the place labeled by $x = -v(t_h = C) < 0$.

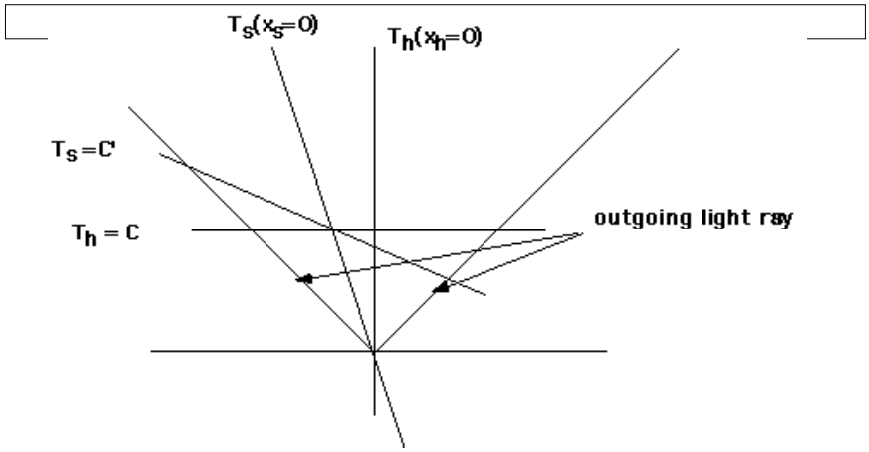

Figure 11.4: **Harry's space-time diagram:** Harry's space-time description of his meeting with Sally. In this case, Harry's time axis is vertical and Sally's is slopped $-\frac{1}{v}$. If at anytime,$t_h = C$, Harry describes the situation, he is at the center of the outgoing light pulses. She is always seen as being off center at some negative x.

Both Harry and Sally are inertial observers. There is no experimental way to distinguish them and, therefore, neither of them is to be preferred. How do we resolve this conflict?

Let's return to Sally's description of what is going on. From Section 11.3, we realize that events that are simultaneous to Sally will not be simultaneous to Harry and visa versa. In order to understand the situation, we can endow Harry with two rods of equal length, one in front, leading, and one in back, trailing. From the discussion of Section 11.3, we can now find how events that are simultaneous to Harry appear on Sally's diagram. The ends of the rods are carried along with Harry and the events that are the ends of the rods have the equations $x = vt - L_0$ for the back and $x = vt + L_0$, where L_0 is a measure of the lengths of the rods. From the situation of the boxcar in Section 11.3, we realize that the event that has the back rod coincident with the back going light ray and the event that has the front rod coincident with the forward traveling light ray are simultaneous to Harry. These lines will intersect the light lines at $(\frac{cL_0}{c-v}, \frac{L_0}{c-v})$ for the front rod and $(-\frac{cL_0}{c+v}, \frac{L_0}{c+v})$.

The slope of the line connecting these two events is

$$\text{slope} = \frac{\frac{L_0}{c-v} - \frac{L_0}{c+v}}{\frac{cL_0}{c-v} + \frac{cL_0}{c+v}} = \frac{v}{c^2}. \tag{11.6}$$

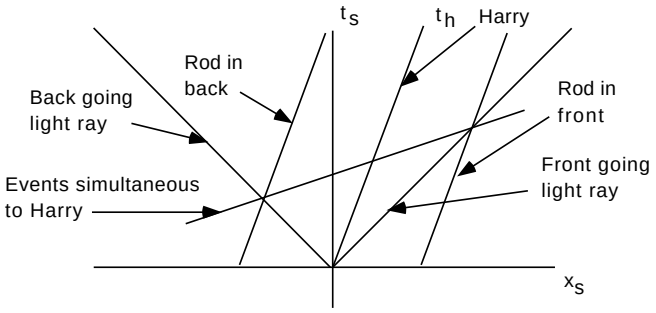

Figure 11.5: **Harry's Lines of Simultaneity:** The figure shows Harry's lines of simultaneity on Sally's diagram. On Harry's diagram these lines would be horizontal. To develop the lines of simultaneity, Harry carries equal length rods in front and in back of his position. In Sally's interpretation of this set up, Harry is at the center of an interval like the boxcar in Figure 11.2. The event that is the coincidence of the forward going light ray and the front rod and the event that is the coincidence of the back going light ray and the back rod are not simultaneous to Sally but are simultaneous to Harry. The lines connecting these events are the lines of simultaneity to Harry and have a slope of $\frac{v}{c^2}$.

It should be clear that, if Harry had been carrying a set of equal spaced confederates with synchronized clocks, the set of events that are the simultaneous reading at some time $t_h = C$ of these clocks will be a line with slope $\frac{v}{c^2}$ Realizing the lines of constant t to any observer are lines of simultaneity, we note that Harry's lines of $t_h = C$ appear on Sally's diagram as lines with slope $\frac{v}{c^2}$, see Figure 11.3. Similarly, Sally's lines of simultaneity, i. e. $t_s = c_1$, on Harry's diagram appear with slope $\frac{-v}{c^2}$ since she has a relative velocity of $-v$, see Figure 11.4. In particular, the events on Sally's diagram that represent Harry's x_h axis, his $t_h = 0$ line, is a line passing through the event $(0, 0)$ with slope $\frac{v}{c^2}$. Thus, we we can now resolve the paradox of Harry and Sally. They are both right. They are both at the center of the outgoing sphere of light. They have different definitions of simultaneity, i. e. where the light is at some time t on their respective clocks. This is an important point and at the heart of many of the paradoxes associated with the Special

Theory of Relativity. More importantly for our present needs, we see that we can construct a coordinate system for Harry on Sally's diagram. On Sally's diagram the coordinate axis for a Harry are no longer orthogonal.

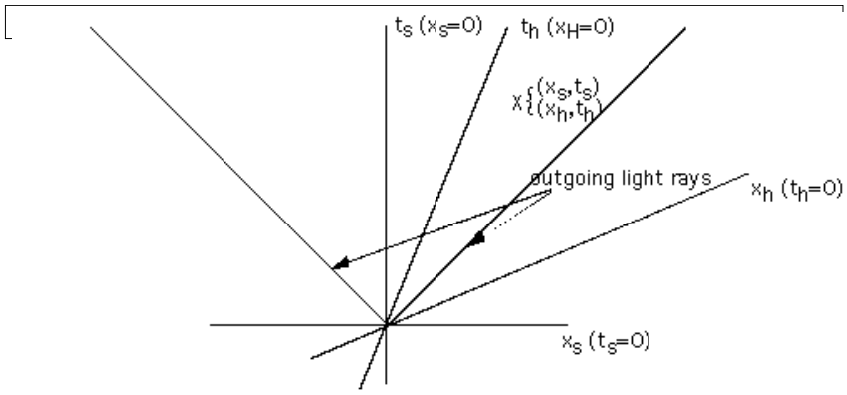

Figure 11.6: **Construction of coordinate axis for a relatively moving observer:** Harry and Sally have a relative velocity, v, with Harry moving to increasing x to Sally. They both agree to label the event of their coincidence as $(0,0)$. His time axis, his $x_h = 0$ line, is a straight line through the origin with slope $\frac{1}{v}$ and his x axis, his $t_h = 0$ line, also passes through the origin but has slope $\frac{v}{c^2}$.

The events that constitute where someone is at any time t are called the person's world line. This is what we called their trajectory in our earlier analysis of action, see Chapter **??**. For a uniformly moving observer like Harry, his world line is a straight line and is also his time axis. Since uniformly moving observers are inertial, we see that all inertial observers appear as straight lines. For non-inertial objects the world line is curved.

On Sally's coordinate system, Harry's space axis, his locus of events that are simultaneous with $t = 0$ to him, has slope $\frac{v}{c^2}$. This is also a general result. For any two relatively moving inertial observers, if one is chosen with the time axis vertical, the other observers lines of simultaneity will appear with slope $\frac{v}{c^2}$ where v is their relative velocity. In other words, the equation for Harry's x axis on Sally's coordinate system is

$$t = \frac{v}{c^2}x \tag{11.7}$$

From the above discussion, it should be clear that any event that will be labeled by a place and a time by Harry and Sally will have different labels for any particular event except the origin event, $(0,0)$, see Figure 11.6. In fact

as discussed in Section 11.1, these labels for the same event are connected by the set of equations that are called the Lorentz transformations, see Equations 11.2. If we choose the x axis along the same direction as the relative motion and if Harry carries an identical clock to Sally and has the same definition of length, these are

$$\begin{aligned} x_h &= \gamma(x_s - \beta c t_s) \\ y_h &= y_s \\ z_h &= z_s \\ ct_h &= \gamma(ct_s - \beta x_s) \end{aligned} \tag{11.8}$$

where $\gamma \equiv \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ and $\beta \equiv \frac{v}{c}$.

In order to derive these equations, we will need to discuss more carefully this idea of identical clocks and the definition of length. We will do this in the next chapter. For now we can note several features of these equations. For example, if Harry carries an identical clock to Sally, then the events that are the ticks of his clock occur on his world line, his t axis or $x_h = 0$ line, at equal intervals, $t_h = n\Delta t_h$, but these equations will require that the intervals are spaced more than Sally's. This effect is called time dilation, see Section 12.3.1. We can get the amount of the dilation from the Lorentz transformations. The coordinates of any one of these ticks according to Harry is $(0, n\Delta t_h)_h$, where n labels the tick. These same events are recorded by Sally as $(nv\Delta t_s, n\Delta t_s)_s$. Remember that all the events on Harry's time axis take the coordinate form $(vt, t)_s$ to Sally. Plugging this into the Lorentz transformations:

$$\begin{aligned} 0 &= \gamma(nv\Delta t_s - \beta cn\Delta t_s) \\ nc\Delta t_h &= \gamma(cn\Delta t_s - \beta nv\Delta t_s) \end{aligned}$$

which implies $nc\Delta t_h = \gamma(1-\beta^2)(nc\Delta t_s) \Longrightarrow c\Delta t_h = \frac{c\Delta t_s}{\gamma}$ or

$$\gamma \Delta t_h = \Delta t_s. \tag{11.9}$$

Since $\gamma < 1$, Sally says the Harry's clock runs slow compared to her clock. By the way because of the equivalence of inertial observers, Harry will also conclude that Sally's clock runs slow compared to his.

In addition, an identical length carried by Harry is shorter to Sally, see Section 12.3.2. Here we measure the length by asking where the ends of the rods are at the same time. We will defer the derivation of the length

contraction formula that section and only quote the result here. If Harry is carrying a rod of length L_0, Sally will say that the length of the rod is

$$L_s = \gamma L_0. \tag{11.10}$$

All of these derivations require that we know the Lorentz transformations. Let's start over and carefully construct the coordinates and then derive the Lorentz transformations from our rules for constructing the coordinates.