Chapter 1

Perturbation Theory for Non-Stationary States

1.1 Introduction

Examples of non-stationary states: scattering, decay of excited atom or molecule...

1.2 Time Evolution

In the Schrödinger picture, a quantum mechanical system in a pure state is characterized by a vector $\phi(t)$ that depends on the time. Equivalently, it can be described by the time-dependent statistical operator (or density operator) $W(t) = |\phi(t)\rangle\langle\phi(t)|$. The density operator can also describe states that are not pure; then $W(t)$ is a mixture of pure states, $W(t) = \Sigma_n w_n |\phi_n(t)\rangle\langle\phi_n(t)|$.

According to the basic postulates of Chap. XII, the Hamiltonian H of the system, assumed to be a self-adjoint, semi-bounded operator, is the generator of translations in time. With the Hamiltonian, one can construct the dynamical equations that govern the time evolution of the state. For a state vector $\phi(t)$, the dynamical equation is the Schrödinger equation:

$$i\hbar \frac{\partial \phi(t)}{\partial t} = H\phi(t). \tag{1.1a}$$

To solve this equation one must provide an initial condition, for example the state of the vector at $t = 0$

$$\phi(t = 0) = \phi_0. \tag{1.1b}$$

For a density operator, the von Neumann equation

$$\frac{\partial W(t)}{\partial t} = \frac{i}{\hbar}[H, W(t)] \tag{1.2a}$$

governs the time time evolution of an initial state

$$W(t=0) = W_0. \tag{1.2b}$$

However, neither the state vector and the density operator nor their time evolutions are observable quantities. The observable quantity is the quantum mechanical probability for measuring the observable Λ in the state $W(t)$:

$$\mathcal{P}(t) = \mathrm{Tr}(\Lambda_0 W(t)), \tag{1.3a}$$

or for the special case of a pure state $W(t) = |\phi(t)\rangle\langle\phi(t)|$ and a 'pure observable' (called a property) $\Lambda = |\psi\rangle\langle\psi|$:

$$\mathcal{P}(t) = |\langle\psi_0|\phi(t)\rangle|^2. \tag{1.3b}$$

The probabilities (1.3) have been written in the Schrödinger picture; they describe the probability of measuring and external, unchanging observable Λ in the time-evolving state $W(t)$.

Alternatively and equivalently, the time evolution of the probabilities can be calculated in the Heisenberg picture. The state W is considered fixed for all time and the observable $\Lambda = \Lambda(t)$ changes in time according to the dynamical law (differential equation). For properties, the differential equation is

$$i\hbar\frac{\partial\psi(t)}{\partial t} = -H\psi(t). \tag{1.4a}$$

with the initial condition

$$\psi(t=0) = \psi_0 \tag{1.4b}$$

For a general observable the equation

$$\frac{\partial\Lambda(t)}{\partial t} = -\frac{i}{\hbar}[H, \Lambda(t)] \tag{1.5a}$$

governs the time time evolution of an initial state

$$\Lambda(t=0) = \Lambda_0. \tag{1.5b}$$

Note that the only difference in the dynamical equations of the Schrödinger picture and the Heisenberg picture is a change in sign on the left hand side of (1.4a) and (1.5a). Then the probability is

$$\mathcal{P}(t) = \mathrm{Tr}(\Lambda(t)W_0) \tag{1.6a}$$

or for the special case of a fixed pure state ϕ and time-evolving property $\psi(t)$

$$\mathcal{P}(t) = |\langle\psi(t)|\phi_0\rangle|^2. \tag{1.6b}$$

Equations (1.3) and (1.6) give two ways to connect the states and observables to the physically observable probabilities. To find the theoretical expression for the probability as a function of time (1.3) in the Schrödinger picture, the differential equations for the states, (1.1a) or (1.2a), must be solved under the initial conditions (1.1b) or (1.2b). A similar calculation gives the time evolution in the Heisenberg picture. Solving the differential equations is called the quantum mechanical Cauchy problem.

In the simplest case of a pure state and a property, the quantum mechanical probability has the form (1.3b) or (1.6b). If we choose to work in the Schrödinger representation, the differential equation (1.1a) must be solved. In order to do this, one must specify the boundary conditions. Assumptions must be made about the space to which the vectors $\phi(t)$ belong and about the 'initial' state ϕ_0, where $t = 0$ can be arbitrarily set.

In the standard mathematical theory of quantum mechanics set forth by von Neumann, one solves the quantum mechanical Cauchy problem in the Hilbert space $\mathcal{H}$. There is assumed to be a one-to-one correspondence between the pure physical states $|\phi)(\phi|$ and the one dimensional subspaces of the Hilbert space. The set of pure quantum physical states is the Hilbert space,

$$\{\phi\} = \mathcal{H}. \tag{1.7a}$$

The same assumption is made for the observables: properties correspond to one dimensional subspaces of the Hilbert space and the set of a all properties is the Hilbert space

$$\{\psi\} = \mathcal{H}. \tag{1.7b}$$

Using this mathematical idealization, the sets of possible pure states and properties are equivalent and equal the entire Hilbert space. More general states and observables are constructed from the pure states and properties. The action of a pure state $|\phi)$ on a property $|\psi)$, written before as $\langle\psi|\phi\rangle$, is then equivalent to the scalar product of two Hilbert space elements (ψ, ϕ). To reflect this, we have changed notation from the pointed brackets to the curved brackets—we used the curved brackets for vectors in $\mathcal{H}$.

Within the Hilbert space formulation, one solves the dynamical equation for the pure state (1.1a) by choosing an initial state $\phi_0 \in \mathcal{H}$. One assumes the Hamiltonian H is self-adjoint and semi-bounded and is continuous (bounded) on a domain $D(H)$ dense in $\mathcal{H}$. A theorem of Stone and von Neumann[1] then gives the solution to the quantum mechanical Cauchy problem in the Hilbert space. The solution is given by a one parameter group of (strongly continuous) unitary operators $U^\dagger(t)$ in $\mathcal{H}$:

$$\phi(t) = U^\dagger(t)\phi_0 \equiv e^{-iHt}\phi_0 \text{ for} -\infty < t < \infty, \tag{1.8}$$

where $\phi(t)$ depends continuously on the initial condition. The operators $U^\dagger(t)$ form a group:

$$U^\dagger(t+\tau) = U^\dagger(t)U^\dagger(\tau), \quad U^\dagger(-t) = (U^\dagger)^{-1}(t) = U(t). \tag{1.9}$$

[1]The theorem of Stone and von Neumann [???] defines the operators $U^\dagger(t) = e^{-iHt}$ (or $U(t) = e^{iHt}$) by Stone-von Neumann calculus. The operator e^{-iHt} is not defined by the exponential series $\sum_{n=0}^{\infty} \frac{t^n}{n!}(-iH)^n$ but the equality $U^\dagger(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!}(-iH)^n$ holds only on a dense subspace $\mathcal{A} \subset \mathcal{D} \subset \mathcal{H}$. The space $\mathcal{D}$ is the subspace of differentiable vectors and the space $\mathcal{A}$ is the subspace of analytic vectors.

The construction of these operators using the Stone-von Neumann calculus guarantees their differentiability and continuity

$$\frac{d^pU(t)}{dt^p}\phi = (-iH)^p U^\dagger(t)\phi = U^\dagger(t)(-iH)^p\phi \text{ for } \phi \in D(H). \tag{1.10}$$

A similar result holds for the solution of the dynamical equation (1.4a) in the Heisenberg picture for the time evolution of the property $\psi_0 \in \mathcal{H}$:

$$\psi(t) = U(t)\psi_0 = e^{iHt}\psi_0 \text{ for} -\infty < t < \infty. \tag{1.11}$$

For the more complicated states $W = \sum_n w_n|\phi_n)(\phi_n|$ and observables $\Lambda = \sum_n \lambda_n|\psi_n)(\psi_n|$ one obtains the same result: the solution of (1.2) is

$$W(t) = e^{-iHt}W_0e^{iHt} \text{ for } -\infty < t < \infty \tag{1.12}$$

and the solution of (1.5) is

$$\Lambda(t) = e^{iHt}\Lambda_0e^{-iHt} \text{ for } -\infty < t < \infty. \tag{1.13}$$

Notice that the solutions of the quantum mechanical Cauchy problem in the Hilbert space (HS) are all valid for all t and the operators $U(t)$ (or $U^\dagger(t)$) form a group. Therefore, the conclusion is that with the HS hypothesis (1.7) that $\phi \in \mathcal{H}$ and $\psi \in \mathcal{H}$, time evolution of states and observables *must* be time symmetric, reversible group evolution. The are *no* states in the HS quantum mechanics of closed systems (systems which obey (1.1) or (1.2)) which have *asymmetric* (irreversible) time evolution. The evolution of the quantum mechanical observables is also time symmetric. The observables $\Lambda(t-t_0)$ can evolve relative to the state $W(t_0)$ by an arbitrary amount $t - t_0 \geq 0$ as well as $t - t_0 \leq 0$. For every $U(t_2 - t_1)$ that time translates the observable $\psi(t_1)$ to the observable $\psi(t_2)$, there also exists a time translation $U(t_1 - t_2) = U^{-1}(t_2 - t_1)$ that reverses the transformation.

Time symmetric dynamics for the states and observables is a mathematical consequence of the HS hypothesis (1.7). This conclusion is not compelling from an empirical point of view as can be seen if we consider the quantum mechanical probability $\mathcal{P}(t)$ to detect the observable $\Lambda(t)$ or $\psi(t)$ in the state W_0 or ϕ_0. These probabilities are measured as the ratio of $N(t)$, the detector counts in the interval between $t = t_0 = 0$, and $N = N(\infty)$, the detector counts in all time:

$$\mathcal{P}(t) \approx \frac{N(t)}{N}. \tag{1.14}$$

With appropriate representation for the state W_0 and for the time-evolving observable $\Lambda(t)$, the experimental probability measurement can be compared to (1.6), $\mathcal{P}(t) = \mathrm{Tr}(\Lambda(t)W_0)$.

An obvious expression of causality is that a state W_0 has to be first prepared *before* an observable $\Lambda(t)$ can be measured in it. If one calls $t = t_0 = 0$ the time at which the preparation of the state W is completed, then the time translation of $\Lambda(t)$ makes physical sense only for $t \geq 0$; no measurement is possible in a state which does not yet exist.

Experimentally, $N(t)$ makes sense for $t > t_0$, the detector is registering measurement of the prepared state, but for $t < t_0$ the detector should not measure anything. Any counts of the detector before the preparation apparatus has created the state would be disregarded as noise. Thus, $\mathrm{Tr}(\Lambda(t)W_0)$ will have an experimental counterpart for $t \geq t_0$ but not for $t < t_0$. The experimental data for $\mathcal{P}(t)$ would give

$$\mathcal{P}(t) = \mathrm{Tr}(\Lambda(t)W_0) \approx \frac{N(t)}{N} \text{ for } t \geq t_0 \tag{1.15a}$$

and

$$\mathcal{P}(t) = \mathrm{Tr}(\Lambda(t)W_0) \approx 0 \text{ for } t < t_0. \tag{1.15b}$$

Often, such as for stationary states or time independent observables, it does not matter at which time $\mathcal{P}(t) = \mathrm{Tr}(\Lambda(t)W_0) = \mathrm{Tr}(\Lambda_0 W(t))$ is calculated. But to make it a general principle that time evolution of the observable $\Lambda(t)$ or state $W(t)$ must go in both direction would ignore empirical facts: $\mathcal{P}(t)$ only has physical meaning for $t \geq 0$. The time evolution of the states (1.12) and the observable (1.13) predicted using the HS hypothesis have not and cannot be tested for $t - t_0 < 0$.

Therefore, we shall solve the quantum mechanical Cauchy problem again, but this time we shall not impose HS boundary conditions. As seen above, imposing time symmetric boundary conditions (as all HS boundary conditions must be) on time symmetric dynamical equations will not reproduce only the testable results. To represent the time asymmetry, we shall modify the requirement that the space of solutions of the dynamical equations is the Hilbert space to include a larger class of solutions. In particular, we shall solve the quantum mechanical Cauchy problem in spaces that allow for semigroup time evolution. The spaces will include solutions which evolve from an initial condition at $t = 0$ only for times $t > 0$. However, we will retain everything of the mathematical formalism of quantum mechanics except for the boundary conditions

We begin by looking for a state $F^-(t)$ that solves the dynamical equation (1.1a) with semigroup evolution for $t > 0$ only, where we identify $t = 0$ as the time at which the state was prepared in a state F_0^- and as the time at which an observable ψ^- can be detected in the state. Since no semigroup solutions exist in the HS, we choose $F^-(t) \in \Phi_+^\times$, where $\Phi_+^\times$ is a larger space than the HS, $\mathcal{H} \subset \Phi_+^\times$.

To solve the quantum mechanical Cauchy problem, we then must find the semigroup solution of

$$i\hbar \frac{\partial F^-(t)}{\partial t} = H_+^\times F^-(t) \text{ for } t \geq 0 \tag{1.16a}$$

with the initial data

$$F^-(t = 0) = F_0^- \in \Phi_+^\times \supset \mathcal{H}. \tag{1.16b}$$

For the Hamiltonian we must use the operator $H_+^\times$, the extension of the operator $H^\dagger = H$ in $\mathcal{H}$ to the space $\Phi_+^\times$ ($\Phi_+^\times$ will be constructed such that $H_+^\times \supset H^\dagger$ is unique).

The solutions will be utilized as in standard quantum mechanics to calculate the probabilities (1.3b) in the Schrödinger picture,

$$\mathcal{P}(t) = |\langle\psi^-|F^-(t)\rangle|^2, \tag{1.17}$$

where $\langle\psi^-|F^-(t)\rangle$ defines the value of an antilinear functional F^- on the element ψ^-, $\langle\psi^-|F^-(t)\rangle = F^-(\psi)$. This is an extension of the HS scalar product $(\psi|\phi)$, $\phi, \psi \in \mathcal{H}$, to the larger space $\Phi_+^\times$. The domain of the antilinear functional F^- is not the entire Hilbert space, but is a dense subspace of $\mathcal{H}$ we call $\Phi_+ \subset \mathcal{H}$. (The origin of the sign disparity of the notation for the state F^- and the spaces Φ_+ and $\Phi_+^\times$ will be discussed later).

Thus to describe asymmetric time evolution we need a triplet of spaces (called a Gelfand triplet or rigged Hilbert space (RHS))

$$\Phi_+ \subset \mathcal{H} \subset \Phi_+^\times. \tag{1.18}$$

By choosing the larger space $\Phi_+^\times$ to incorporate the time asymmetric boundary conditions and semigroup evolution, we also must restrict the space of physical observables for our calculations to make sense. This restriction of observables to the space Φ_+ is not a limitation of the theory. As we discuss below, this space Φ_+ only contains observables with the properties we expect of them and excludes elements of the Hilbert space which could not possibly correspond to an observable defined by a detector.

This triplet of spaces, called a Gelfand triplet or rigged Hilbert space (RHS), is endowed with very special properties on which we will elaborate further below. For now, it is enough to say that because of these properties we can find semigroup solutions of (1.16) analogous to the unitary Hilbert space solutions (1.8):

$$F^-(t) = U_+^\times(t)F_0^- \equiv e_+^{-iH_+^\times t}F_0^- \text{ for } t \geq 0, \tag{1.19}$$

where the last equality is a definition (see footnote 1).

The operator $U_+^\times(t)$ is the unique extension of the HS operator $U^\dagger(t) = e^{-iHt}$ to the space $\Phi_+^\times$ for $t \geq 0$. It can also be thought of as the uniquely defined conjugate operator to $U_+(t)$, the restriction of the HS operator $U(t) = e^{iHt}$ to the space Φ_+, symbolically

$$U_+(t) = U(t)|_{\Phi_+}. \tag{1.20}$$

The operator $U_+^\times(t)$ is only defined for $t \geq 0$ and forms a semigroup. Since the operator $U(t)$ of which the operator $U_+(t)$ is the restriction is defined for all t, in principle $U_+(t)$ also is defined for $-\infty < t < \infty$. However, only for $t \geq 0$ is $U_+(t)$ a continuous operator on the space Φ_+, where continuous means continuous with respect to the topology of Φ_+:

$$U_+(t) : \Phi_+ \to \Phi_+ \text{ only for } t \geq 0. \tag{1.21}$$

For $t < 0$, $U_+(t)$ transforms out of the space Φ_+. Since we only want to consider continuous transformations, we complete the previous definition (1.20) of the restriction of $U(t)$ with the necessary condition:

$$U_+(t) = U(t)|_{\Phi_+} \text{ only for } t \geq 0. \tag{1.22}$$

Thus, corresponding to the the triplet of spaces above (1.18), there is a triplet of time evolution operators,

$$U_+(t),\ U^\dagger(t) \subset U_+^\times(t),\ \text{for } t \geq 0, \tag{1.23}$$

and also a triplet of Hamiltonians[2],

$$H_+ = H|_{\Phi_+} \subset H = H^\dagger \subset H_+^\times. \tag{1.25}$$

(Note that the relation $U_+(t) \subset U^\dagger(t)$ is not true. This is a consequence of the antilinearity of the scalar product in HS and the extension of this product to the triplet of spaces (1.18).

Now that we have established that a semigroup solution to the quantum mechanical Cauchy problem posed in (1.16) exists (although we have not proved it yet in a rigorous sense), we return to the probabilities. The probability $\mathcal{P}(t)$ will now also only be defined for $t \geq 0$, and the empirical problems that led us to seek a semigroup solution have been solved. Additionally, we can shift from the Schrödinger probability to the Heisenberg probability to define the time evolution of the observable $\psi^- \in \Phi_+$. The time dependence of the probability amplitude can be shifted to ψ^- and what results we define as the time evolution of the observable:

$$\langle \psi_0^- | U_+^\times(t) F^- \rangle = \langle U_+(t)\psi^- | F_0^- \rangle \equiv \langle \psi^-(t) | F_0^- \rangle \text{ for } t \geq 0, \tag{1.26a}$$

or removing the arbitrary $F^- \in \Phi_+^\times$,

$$\psi^-(t) = U_+(t)\psi_0^- \text{ for } t \geq 0 \text{ only}, \tag{1.26b}$$

though $U_+(t)\psi^-$ is an element of $\mathcal{H}$ (not not necessarily Φ_+) for $-\infty < t < \infty$.

In place of (1.16) we can also seek a semigroup solution for $t < 0$. To do this we must again solve the quantum mechanical Cauchy problem but with different boundary conditions than either $\mathcal{H}$ or $\Phi_+^\times$. This construction will be parallel to the positive semigroup; we begin with the Schrödinger equation for the observables:

$$i\hbar \frac{\partial F^+(t)}{\partial t} = -H_-^\times F^+(t) \tag{1.27a}$$

[2]In $\mathcal{H}$, the Hamiltonian H is the HS derivative of $U(t)$ at $t = 0$ (the generator of the transformation),

$$iH = \frac{d}{dt} U(t)|_{t=0}, \tag{1.24a}$$

or more precisely

$$iH\psi = \overset{\tau_{\mathcal{H}}-\lim}{t \to 0} \frac{U(t) - U(0)}{t} \psi, \tag{1.24b}$$

where $\tau_{\mathcal{H}} - \lim$ means the limit is taken with respect to the HS topology and $\psi \in \mathcal{H}$. In the same way, the Hamiltonians H_+ and $H_+^\times$ are the derivatives of $U_+(t)$ and $U_+^\times(t)$ except that the derivative has to be taken with respect to the topologies of Φ_+ and $\Phi_+^\times$.

with the initial condition

$$F^+(t=0) = F_0^+ \in \Phi_-^\times \supset \mathcal{H}, \tag{1.27b}$$

where $\Phi_-^\times$ is larger space of solutions (and not the same as $\Phi_+^\times$). The quantum mechanical probabilities should again be given by the probabilities in the standard HS theory (1.6)

$$\mathcal{P}(t) = |\langle F^+(t)|\phi_0^+\rangle|^2, \tag{1.28}$$

where $\langle\phi^+|F^+(t)\rangle$ now defines the value of an antilinear functional F^+ at the element ϕ^+, $F^+(\phi^+) = \langle\phi^+|F^+(t)\rangle$, which is an extension of the HS scalar product $(\phi,\psi) = \psi(\phi)$. The antilinear functional F^+ is only defined for a dense subspace of $\mathcal{H}$, i.e. $\phi^+ \in \Phi_- \subset \mathcal{H}$. This leads to another triplet of spaces, another RHS, with the same Hilbert space $\mathcal{H}$ as in (1.18):

$$\Phi_- \subset \mathcal{H} \subset \Phi_-^\times. \tag{1.29}$$

The time evolution of the functional $F^+(t)$ is now given by

$$F^+(t) = U_-^\circ(t)F_0^+ \equiv e^{iH_-^\times t}F_0^+ \text{ for } t \le 0. \tag{1.30}$$

Again $U_-^\circ(t)$ is the uniquely defined conjugate of the operator

$$U_-^\dagger(t) = U^\dagger(t)|_{\Phi_-} = e^{-iHt}|_{\Phi_-}. \tag{1.31}$$

Thus we have for the triplet of spaces (1.29) the triplet of Hamiltonians

$$H_- = H|_{\Phi_-} \subset H = H^\dagger \subset H_-^\times \tag{1.32}$$

and the triplet of time evolution operators

$$U_-^\dagger(t),\ U(t) \subset U_-^\circ(t) \text{ for } t \le 0. \tag{1.33}$$

The time evolution of the state vectors $\phi^+ \in \Phi_-$ in the probabilities (1.28) can therefore be defined for $t \le 0$ only by

$$\langle\phi_0^+|U_-^\circ(t)|F_0^+\rangle = \langle U_-^\dagger(t)\phi_0^+|F_0^+\rangle \equiv \langle\phi^+(t)|F_0^+\rangle,\ t \le 0, \tag{1.34a}$$

or omitting the arbitrary F_0^+

$$\phi(t) = U_-^\dagger(t)\phi_0^+ \text{ for } t \le 0 \text{ only}. \tag{1.34b}$$

We have amended our notation of the states ϕ and observable ψ by superscripts. The pure state vectors is called ϕ^+ and the property (or observable vector) is called $|\psi^-)(\psi^-|$ (or ψ^-). This indicates that these vectors are not only in $\mathcal{H}$, but are elements of more specific spaces,

$$\phi^+ \in \Phi_- \subset \mathcal{H} \text{ and } \psi^- \in \Phi_+ \subset \mathcal{H}. \tag{1.35}$$

The more general state W and the more general observable Λ are mixtures of $|\phi_n^+)(\phi_n^+|$ and $|\psi_n^-)(\psi_n^-|$:

$$W = \sum_n w_n|\phi_n^+)(\phi_n^+| \text{ with } \phi_n^+ \in \Phi_- \tag{1.36a}$$

$$\Lambda = \sum_n \lambda_n|\psi_n^-)(\psi_n^-| \text{ with } \psi_n^- \in \Phi_+. \tag{1.36b}$$

The odd thing about this notation is that we label the vectors by the superscript $^\pm$ and the space by the subscript $_\mp$. The reason for this disparity is the different origins of the two notations. For the vectors, we use the notation that is almost universal in scattering theory: the in-states are labeled by $+$ and the 'out-states' (the observables) are labeled by $-$. This same notation is used for the generalized kets of scattering theory such as the Dirac kets $|E^\pm\rangle$, familiar as the solutions of the Lippmann-Schwinger equation. However, the convention for the spaces $\Phi_\mp$ comes from mathematicians. Later we will show the connection between the spaces Φ_- of the in-states and Φ_+ of the 'out-observables' and the Hardy class spaces of the the lower- and upper-half complex planes, written $\mathcal{H}_-$ and $\mathcal{H}_+$.

In summary, we choose different dense subspaces of $\mathcal{H}$ for the set of states $\{\phi^+\}$ and for the set of observables $\{\psi^-\}$ in place of the HS hypothesis (1.7). This mathematical distinction between the states and observables we call the RHS hypothesis:

> The set of prepared states, which are experimentally defined by a preparation apparatus and a preparation procedure (e.g. an accelerator) are mathematically described by Φ_-, a dense subspace of $\mathcal{H}$:
>
> $$\text{pure state defined by a preparation procedure } \{\phi^+\} \doteq \Phi_-. \tag{1.37}$$
>
> The the set of detected observables, which are experimentally defined by a registration apparatus and a registration procedure (e.g. a detector) are mathematically described by Φ_+, a dense subspace of $\mathcal{H}$:
>
> $$\text{properties defined by a registration procedure } \{\psi^-\} \doteq \Phi_+. \tag{1.38}$$

Thus for each quantum mechanical (e.g. scattering) system there are two RHS's of the same Hilbert space, the triplet (1.29) with the physical interpretation of prepared states and the triplet (1.18) with the physical interpretation of detected observables. The two subspace Φ_- and Φ_+ are not disjoint, mathematically $\Phi_- \cap \Phi_+ \neq \emptyset$. In addition to the vectors $\phi^+ \in \Phi_- \subset \mathcal{H}$ and $\psi^- \in \Phi_+ \subset \mathcal{H}$ defined by the experimental apparatus, the RHS formulation also provides elements $F^\pm \in \Phi_\mp^\times$ which are outside of the HS. Well-known examples of these are the Lippmann-Schwinger Dirac kets $|E^\pm\rangle$, whose definition and nature will be discussed below.

With these two distinct RHS's for states and observables and their time evolution given by (1.34b) and (1.26b), the causality conditioned mention above (and formulated in (1.15)) is incorporated into the mathematical theory. This mathematical formulation of causality emerges from the intuitive properties of the macroscopic preparation and registration apparatuses. This theory-independent statement of causality we call the preparation→registration arrow of time. Let $t = t_0$ (which we identify for each single microphysical system with the mathematical time $t = 0$ of the Cauchy problem (1.16)) denote the point in time before which the preparation is complete and after which the registration begins. Then:

Time translation of the registration apparatus relative to the preparation apparatus only makes sense by an amount $t \geq t_0$. (1.39)

This means it makes no sense to turn on a detector at a time $t < t_0$ before the accelerator prepares the state; any detector counts for $t < t_0$ would not be attributed to a property of a state prepared by the accelerator but would only signify noise. This intuitively obvious statement of a privileged direction o time obtains a precise mathematical formulation within the RHS.