Velocity and Rapidity

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1 Problem

A truck is moving with a speed u in the lab frame. A cannon, mounted on the truck in the forward direction, fires a cannon ball with a speed v in the truck's rest frame. What is the velocity v' of the cannon ball in the lab frame? Assume the cannon ball can reach speeds up to (but not equal to) the speed of light c. Furthermore, treat this as a 1-dimensional problem.

2 Solution

2.1 Velocity Addition in Classical Physics

For small enough speeds, $u \ll c$ and $v \ll c$, we can use the classical velocity addition law:

$$v' = v + u. \tag{2.1}$$

The linearity of (2.1) makes it easy to switch between reference frames.

2.2 Velocity Addition in Relativistic Physics

In special relativity, the velocity addition law is not linear any more:

$$v' = \frac{v + u}{1 + \frac{vu}{c^2}}. \tag{2.2}$$

Remarks:

- Note that (2.2) reduces to (2.1) for $c \to \infty$.
- From now on we will use the natural unit system: $c = 1$.

2.3 Rapidity

The non-linearity of (2.2) makes it less straightforward to switch between reference frames in special relativity. This can be avoided by using the concept of rapidity y, defined as

$$y_v = \frac{1}{2}\ln\left(\frac{1+v}{1-v}\right), \tag{2.3}$$

where $v = v/c \equiv \beta$ is dimensionless in natural units. Multiplying both the numerator and denominator inside the logarithm with a factor γm allows us to express rapidity in terms of energy and momentum instead of velocities

$$y = \frac{1}{2}\ln\left(\frac{E+p}{E-p}\right), \tag{2.4}$$

where we used $E = \gamma m$ and $p = \gamma m v$.

The rapidity addition formula reads

$$y_{v'} = y_v + y_u. \tag{2.5}$$

Using (2.3) we can show that (2.5) holds true:

$$\begin{aligned}
&\frac{1}{2}\ln\left(\frac{1+v'}{1-v'}\right) = \frac{1}{2}\ln\left(\frac{1+v}{1-v}\right) + \frac{1}{2}\ln\left(\frac{1+u}{1-u}\right) \\
\Leftrightarrow &\ln\left(\frac{1+v'}{1-v'}\right) = \ln\left(\frac{1+v}{1-v} \times \frac{1+u}{1-u}\right) \\
\Leftrightarrow &\frac{1+v'}{1-v'} = \frac{1+v}{1-v} \times \frac{1+u}{1-u} \\
\Leftrightarrow &1+v' = \frac{1+v}{1-v} \times \frac{1+u}{1-u} \times \left(1-v'\right) \\
\Leftrightarrow &v' = \frac{\frac{1+v}{1-v} \times \frac{1+u}{1-u} - 1}{\frac{1+v}{1-v} \times \frac{1+u}{1-u} + 1} \\
\Leftrightarrow &v' = \frac{(1+v)(1+u) - (1-v)(1-u)}{(1+v)(1+u) + (1-v)(1-u)} \\
\Leftrightarrow &v' = \frac{1+v+u+vu-1-vu+v+u}{1+v+u+vu+1+vu-v-u} \\
\Leftrightarrow &v' = \frac{v+u}{1+vu},
\end{aligned}$$

which is the same result as (2.2) for $c = 1$. Unlike the velocity addition formula, the rapidity addition formula is linear in special relativity, thus making it a useful tool to switch between reference frames.