

Chapter 4

Introduction to Maxwell's Theory

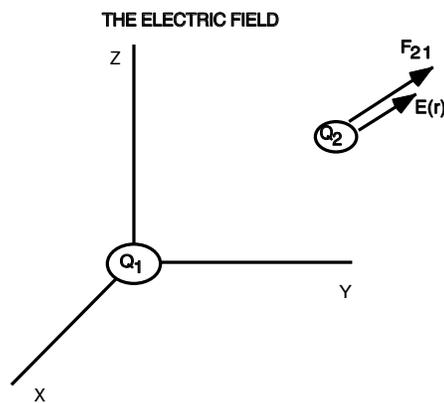


Figure 4.1: **The Electric Field and Electric Forces** Maxwell said that electric and magnetic forces were due to the presence of the electric and magnetic field. In this figure, the electric force on Q_2 is due to the presence of the field at its location, $\vec{F}_{21} = Q_2 \vec{E}(\vec{r})$. There is a similar relationship for the magnetic force.

Maxwell was interested in developing a mechanical basis for the phenomena associated with electricity and magnetism, [Maxwell 1861]. In those days and especially to Maxwell, a mechanical basis was one that could ultimately be traced to an underlying atomic constituent and simple force relationships. In addition to providing a mechanical “explanation”, he could also show that electric and magnetic phenomena were inter-related phenomena,

i. e. unifying the two force systems.

In his time, many of the basic ideas of the electric and magnetic force systems were known. The law for the electric interaction between charged particles had been articulated in the period 1785 and 1791 by Coulomb. The force law between magnets and the force between moving charges and magnets was known and even Faraday's Law about the relationship of changing magnetic environments and electric currents was known. In fact, Faraday had already begun to describe magnetic and electric phenomena in a field like language. What Maxwell sought was an underlying mechanical basis for all the phenomena associated with electricity and magnetism. Reducing electricity and magnetism to a mechanical basis meant that he was looking for something to push or pull but it had to do so locally. He could not believe that fundamental phenomena could take place as an action at distance phenomena like gravity was thought to be at the time. In order to have a thing which could push or pull locally, he hypothesized the existence of a rather rich structure for the vacuum of space, whirling vortices in an ether that produced the electric and magnetic force. These were to be his atoms. This was an obvious extension of the prevailing ideas used in fluid flow. Fluids seemed continuous systems that had smooth spatial properties and produced macroscopic forces but were composed of the underly atoms. Regardless, the flow mechanics was well described by a field dynamic. Thus not only did he seek a mechanical source for electric and magnetic phenomena, he developed a field theory basis for it. His picture of electric and magnetic forces was that they were mediated by fields, the electric, \vec{E} , and the magnetic, \vec{B} , fields. It was his basic idea that the correct description of electromagnetic phenomena required a locally causal dynamic. The idea was that not only did the charges generated the fields but the fields themselves responded to the local environment of the fields themselves. In addition, the forces experienced by the charges were because of the values of the fields at the place occupied by the charges, $\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$, where q is the charge in question and \vec{v} is its velocity.

In order to create the mechanical basis for the fields, Maxwell was forced to endowed the ether with the correct mechanical properties of inertia and size to replicate the success of the earlier laws but now in context of a local mechanical model. The underlying idea was simple. Let's look at the simplest of the cases, Coulomb's Law. The situation is shown in Figures 4.2 and 4.3. A force on a charged particle took place as a two step process. A charge Q_1 is placed in empty unexcited space. This charge excites the ether next to it by creating vortices at its location. These vortices in turn excite neighboring vortices until space is full of whirling vortices. Each vortex is in

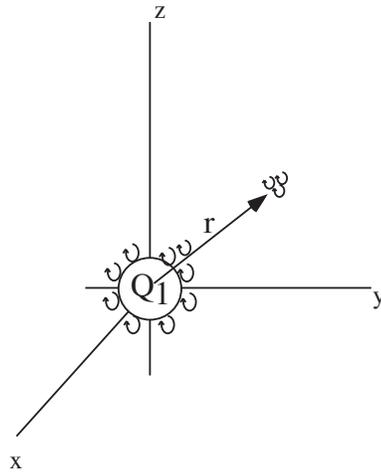


Figure 4.2: **Maxwell's Vortices** Maxwell pictured the electric force as emerging in two steps. First any charged particle would excite vortices in the ether at its location. These vortices would excite other vortices nearby and so forth until all of space would fill with whirling vortices. In a sense, the whirliness of the vortices at any place was a measure of the strength of the electric field at that point.

dynamic equilibrium with its neighbors. There is a 'thing', the whirliness, which is a measure of the electric field at that point.

When a new charge, Q_2 , is located at some distance, \vec{r} , from the first charge, it detects the level of excitement of the local vortices and thus feels a corresponding force. The force is proportional to the charge Q_2 at that place and the amount of whirliness or electric field at that point.

The mechanical properties of the ether and its vortices determine how the whirliness develops. This is set by the vortices inertia and size. These parameters for the mechanical properties of the vortices are then adjusted to accommodate Coulomb's Law.

In other words, Maxwell introduced local fields – a continuous quantity defined at all points in space and for all times – with a rule of dynamics to produce the electromagnetic forces. If an object experiences a force, there must be something at that place, the whirliness. In addition, the whirliness itself must be determined locally in both space and time. Let's go through the example of Coulomb's Law in a little more detail to see how this idea works.

The first problem is to reproduce the well known Coulomb's law of force

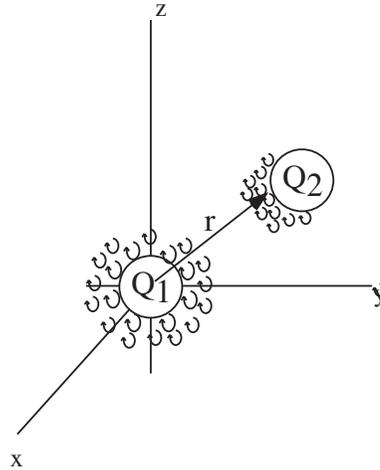


Figure 4.3: **Vortices and the Electric Force** When a charged particle, Q_2 , is positioned, the particle detects the local amount of whirliness in the vortices of the ether. This generates the electric force in proportion to the charge and amount of whirliness at its location. The local whirliness is \vec{E} at \vec{r} .

for static situations. Coulomb's Law is an action at a distance description of interaction,

$$\vec{F}_{21} = \frac{1}{4\pi\epsilon_0} \frac{Q_1 Q_2}{r_{12}^2} \frac{\vec{r}_{21}}{r_{12}} \quad (4.1)$$

where \vec{r}_{12} is the separation between the charges. In order to simplify the discussion, let's place charge Q_1 at the origin. Since the force on the charge Q_2 is supposed to be $Q_2 \vec{E}(\vec{r})$, where \vec{r} is now the position at which Q_2 is located. For this case, we can identify the electric field as

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{Q_1}{r^2} \frac{\vec{r}}{r} \quad (4.2)$$

around a spherically symmetric charge placed at the origin. You will reproduce the static Coulomb's Law results with the electric field if you can make a local rule about how $\vec{E}(\vec{r})$ develops that reproduces this result. It should be clear that the hard part will be to reproduce the inverse square fall off with distance in the strength of the field.

In some sense, it is really not correct to say that Q_1 is the source of this field. The field is not attached to the charge. At any point, there is a field only if there is a field or a charge in the neighborhood. The field at some

point, like all things, is to be determined locally. Maxwell used his whirling vortices of the ether to discover a rule for whirliness and how whirliness effected whirliness that recovers the characteristic the inverse square fall off with distance of Coulomb's Law. Like the stretched string, Section 2.2, in which the transverse position of a place on the string is determined by the transverse position of the neighbors to that place, similarly here, the idea is to find the rule on how the field arranges itself and forget about the whirlies. The following analysis reviews the process and becomes somewhat technical but the struggle to follow it is worth the effort.

Since the electric field is meant to produce a force, it must be a vector field, a directed quantity defined at every point in space and with a local rule for its construction. Basically you ask how much does the field change at a place because of what is there. For now, we are looking at a static case – no time change. But we can still ask about how the field varies as we change positions in space.

For a vector field such as the electric field since it is a vector field, you have a directed strength at each point in space and around each point you have directed strengths. At any point you can ask how much more “outpointy” these directed strengths become as you go from place to place. The analogy for our stretched string is that, at any place on the string, you can ask how “bendy” is the string. On a string, “bendiness” happens when that place differs from its neighbors. The string bends up when the place is lower than its neighbors and it bends down when it is higher. When there is no bend, that place on the string is at the average of its neighbors. In the static string it takes a force to maintain a bend in the string. Our case for the vector field case using “outpointiness” works in the same fashion. You can have “outpointiness” only if there are charges that are placed there, i. e. charge causes an outward directed field. Of course, we have to develop a definition, a measure, of “outpointy” and test it.

The measure of “outpointiness” is called the divergence and it is what you would have thought to define it as if you spent some time playing with the ideas of a vector field. At any point, find out how much the neighboring fields point away from where you are. That should indicate the “outpointiness”.

Since fluid flow is also a vector field it is worthwhile to think in terms of it. The vector field in this case is the velocity of the fluid. If at a point all the flow is uniform about you, you would not think of the field as becoming “out pointy”. On the other hand, if you were at a place like the drain, you would consider the surrounding flow to be “inpointy”, the opposite of “outpointy”. To be more quantitative, think of surrounding the place that

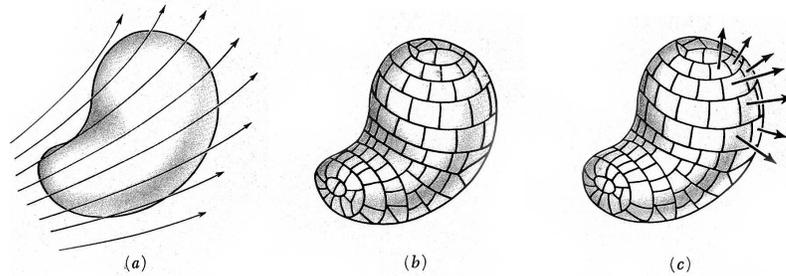


Figure 4.4: **Construction of the Divergence** To find the divergence or “outpointiness” of a vector field at a point, surround the point with a surface, step (a). Cover the surface with small elements of area so that to all intents and purposes they can be considered flat. Each element of surface will now have a normal vector. Find the magnitude of the vector field at the surface to is along the surface. Add these magnitudes for each element of surface and the total is the divergence or “outpointiness” of the vector field at the point surrounded by the surface. Then shrink the volume surrounded to a point. For a fluid, applied to the velocity field, this tells the amount of fluid that goes into a point. This series of steps is encoded in the first part of Equation 4.4 for the case of the electric field.

you are interested in and measuring how much stuff flows in or out. By enclosing the point of interest with a surface, we can measure the incoming fluid by assessing how much stuff comes into any element of area on the surrounding surface and then adding the contribution to each part. In other words, surround the point with a surface. Cover it with elements of area, postage stamps labeled $\Delta^2 s$. Each element of area has a normal vector, \hat{s} , that points either outward or inward, see Figure 4.4. Choosing the outward normal, we are defining “outpointiness”, the amount of the vector field that is along the outward normal; it is the “flow” through that element of area. Now do this for the each element of the entire surface and add up all the contribution from all the pieces, $\sum_{S \supset V}$. To reduce this analysis to a point, shrink the volume enclosed by the surrounding surface to zero, $\lim_{V \rightarrow 0}$. This same analysis holds for all vector fields. This construction at each point assesses the “outpointiness” of the neighborhood of the point and is called the divergence. Thus,

$$\text{Div}(\vec{\mathbf{E}})(\vec{r}) \equiv \lim_{V \rightarrow 0} \frac{\sum_{S \supset V} \vec{\mathbf{E}}(\vec{r}') \cdot (\Delta^2 s \hat{s})}{V} \equiv \lim_{V \rightarrow 0} \frac{\oint_{S \supset V} \vec{\mathbf{E}}(\vec{r}') \cdot \hat{s} d^2 s}{V} \quad (4.3)$$

$$= \lim_{V \rightarrow 0} \frac{1}{\epsilon_0} \frac{Q_{\text{inside } V}}{V} = \frac{1}{\epsilon_0} \rho(\vec{r}) \quad (4.4)$$

where the first part is a mathematical statement of what is stated above for the definition of the divergence but for the case of the electric field and the subsequent parts are the relationship with charge that is necessary to recover Coulomb's Law, i. e. electric charge is the source of "outpointiness" of the electric field. Note that the divergence of a vector field, $\text{Div}(\vec{\mathbf{E}})(\vec{r})$, is a scalar field. In cartesian coordinates, the divergence is simply written

$$\text{Div}(\vec{\mathbf{E}})(\vec{r}) = \frac{\partial E_x(\vec{r})}{\partial x} + \frac{\partial E_y(\vec{r})}{\partial y} + \frac{\partial E_z(\vec{r})}{\partial z}. \quad (4.5)$$

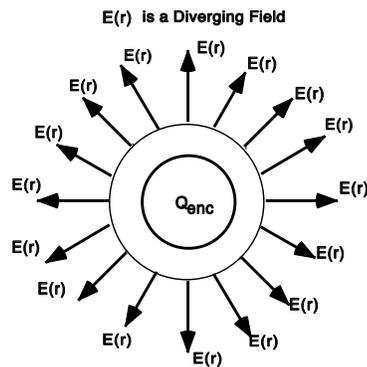


Figure 4.5: "Outpointiness of the Electric Field" A characteristic property of the electric field is that charge is the source of "outpointiness". This is the idea that the electric field points away from nearby positive charges and toward nearby negative charges. This last example being negative outpointiness.

Notice that this law, Equation 4.4, says that for a static electric field there is divergence of the field only where there is charge. Yet the picture that we all have of the static electric field around an isolated point charge is a diverging field, the electric field points outward from the origin everywhere, see Figure 4.5. How do we reconcile this?

Consider a point away from the isolated point charge. If a surface such as that shown in Figure 4.4 is constructed area at nearer the charge is smaller whereas the area more distant is larger. In fact, the areas are in the ratio of the distances squared. Thus the field strength and the areas combine so that the net "outpointiness", actually inpointiness, of the nearer surface

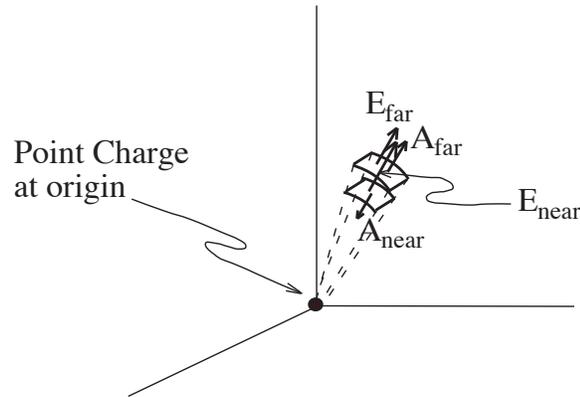


Figure 4.6: **Divergence Outside of Charge** A characteristic property of the electric field is that charge is the source of “outpointiness”. This is the idea that the electric field points away from nearby positive charges and toward nearby negative charges. This last example being negative outpointyness.

balances the outpointiness of the far surface and the net is zero. Thus it is because the divergence is zero at places other than the charge that the field strength falls off with distance as $\frac{1}{r^2}$.

Another property that a vector field can manifest is a rotation or curl. The curl of a vector field is itself a vector field. Again you develop a definition and test it. The first step is to pick a small plane area surrounding the point of interest, $s\hat{s}$, where again \hat{s} is the unit normal to the area. Remember that area is a directed quantity and thus a vector. The area is bounded by a closed path $\vec{r}'(z), 0 \leq z \leq 1, \vec{r}'(z) \subset s$. The component of the curl along the \hat{s} direction at the point is the amount of the field circulating with the tangents to the path $\vec{r}'(z)$. In other words, the idea is to follow a closed path around the point and see how much of the vector field follows the path. Varying all directions for \hat{s} the value of the curl at the point p is maximum obtained and the curl is directed in that direction. Another way to find the value of the vector field that is curl of a vector field is to establish three independent orthogonal directions for the areas, a coordinate frame, and the obtained values are then the three components of the vector field. In the static case, the electric field does not have curl.

$$\vec{\text{Curl}}(\vec{\mathbf{E}}(\vec{r})) \cdot \hat{s} \equiv \lim_{s \rightarrow 0} \frac{\sum_{\vec{r}' \supset S} \vec{\mathbf{E}}(\vec{r}') \cdot \frac{\Delta \vec{r}'}{\Delta z} \Delta z}{s} \quad (4.6)$$

$$\equiv \lim_{s \rightarrow 0} \frac{\oint_{\vec{r}' \supset S} \vec{\mathbf{E}}(\vec{r}') \cdot \frac{d\vec{r}'}{dz} dz}{s} = 0. \quad (4.7)$$

The curl in cartesian coordinates takes the simple form

$$\vec{\mathbf{Curl}}(\vec{\mathbf{E}})(\vec{r}) = \left(\frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y} \right) \hat{x} + \left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} \right) \hat{y} + \left(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} \right) \hat{z} \quad (4.8)$$

On the other hand, the magnetic field does curl. The magnetic field is the force experienced by a moving charged particle.

$$\vec{\mathbf{F}}_{\text{mag}} = Q\vec{v} \times \vec{\mathbf{B}}(\vec{r}). \quad (4.9)$$

The magnetic field lines tend to wrap around their sources, the currents.

$$\vec{\mathbf{Curl}}(\vec{\mathbf{B}})(\vec{r}) \cdot \hat{s} \equiv \lim_{s \rightarrow 0} \frac{\sum_{\vec{r}' \supset S} \vec{\mathbf{B}}(\vec{r}') \cdot \frac{\Delta \vec{r}'}{\Delta z} \Delta z}{s} \quad (4.10)$$

$$\equiv \lim_{s \rightarrow 0} \frac{\oint_{\vec{r}' \supset S} \vec{\mathbf{B}}(\vec{r}') \cdot \frac{d\vec{r}'}{dz} dz}{s} \quad (4.11)$$

$$= \lim_{s \rightarrow 0} \frac{1}{\mu_0} \frac{\mathbf{i}_{\text{through } s}}{s} = \frac{1}{\mu_0} \vec{\mathbf{j}}, \quad (4.12)$$

and does not diverge

$$\text{Div}(\vec{\mathbf{B}}(\vec{r})) = 0. \quad (4.13)$$

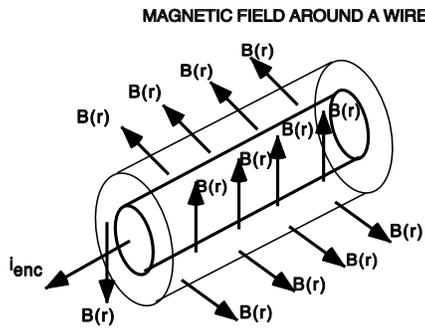


Figure 4.7: **The Curl of the Magnetic Field** In contrast to the electric field, the magnetic field wraps around or curls around its sources, the currents in the problem.

Note that we have not added a time dependence. These are all static situations.

Maxwell insisted that the field was not established everywhere at once. It was made up of whirling vortices that pushed on each other. The rate at which the vortices could push was set by the parameters of the static theory. By endowing these whirling vortices with the correct properties to reproduce the laws of static electricity and magnetism, he found how to add a local set of rules for the time evolution of the fields. These are the full set of Maxwell's equations including time dependence:

$$\text{Div}(\vec{\mathbf{E}}(\vec{r}, t)) = \frac{1}{\epsilon_0} \rho(\vec{r}, t) \quad (4.14)$$

$$\vec{\text{Curl}}(\vec{\mathbf{E}}(\vec{r}, t)) = \frac{\partial \vec{\mathbf{B}}}{\partial t}(\vec{r}, t) \quad (4.15)$$

$$\text{Div}(\vec{\mathbf{B}}(\vec{r}, t)) = 0 \quad (4.16)$$

$$\vec{\text{Curl}}(\vec{\mathbf{B}}(\vec{r}, t)) = \mu_0 \vec{\mathbf{j}}(\vec{r}, t) - \mu_0 \epsilon_0 \frac{\partial \vec{\mathbf{E}}}{\partial t}(\vec{r}, t) \quad (4.17)$$

This is the standard format for these equations. For a discussion of the field dynamics, it is important to realize that only two, actually six since each is a vector equation, of these equations are a dynamic, Equations 4.15, and 4.17. The other two equations, Equations 4.14 and 4.16, are what are called constraint equations; they control the pattern of the field but not the temporal evolution. It is apparent that the electromagnetic field is a much more complex field than the stretched string whose dynamic is Equation 2.59. The vector nature of the field, the existence of constraints, and the sources, $\rho(\vec{r}, t)$ and $\vec{\mathbf{j}}(\vec{r}, t)$, obviously complicate the situation. In Section 2.3.1, we added external forces to the dynamic of the string and as noted there in other field theories these forces are called sources. This is generally an unfortunate change in nomenclature since there is a misleading implication that without sources there would be no field. Here as in the developments for the stretched string, we will discuss the electromagnetic field first without the presence of $\rho(\vec{r}, t)$ and $\vec{\mathbf{j}}(\vec{r}, t)$, see Section 6. Rearranging and omitting the sources, the dynamical equations for the evolution of the electromagnetic field become

$$\frac{\partial \vec{\mathbf{E}}}{\partial t}(\vec{r}, t) = -\frac{1}{\mu_0 \epsilon_0} \vec{\text{Curl}}(\vec{\mathbf{B}}(\vec{r}, t)) \quad (4.18)$$

$$\frac{\partial \vec{\mathbf{B}}}{\partial t}(\vec{r}, t) = \vec{\text{Curl}}(\vec{\mathbf{E}}(\vec{r}, t)) \quad (4.19)$$

Identifying $\vec{\mathbf{E}}(\vec{r}, t)$ with the displacement field of the string, $y(x, t)$, and $\vec{\mathbf{B}}(\vec{r}, t)$ with the velocity field of the string, $v(x, t)$, we see that the electromagnetic dynamic is more complex but similar in structure.

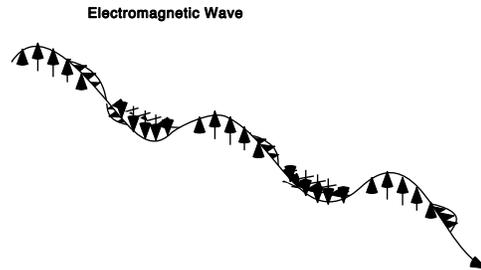


Figure 4.8: **The Field Configuration for Light** Light is a traveling wave solution of Maxwell's Equations and is composed of propagating combination of electric and magnetic fields. The direction of flow of energy and momentum is along the normal to the plane of the oscillating electric and magnetic field vectors. In the figure the upward arrows represent the electric field and the perpendicular arrows are the magnetic field.

An important feature of the electromagnetic field that can be seen from the equations above is that, if you have an electric field in a localized region of space, finite somewhere but zero elsewhere like the pulse in the stretched string, the electric field will have a curl. Thus even if there are no charges or currents, this curl is the source of a developing magnetic field, Equation 4.19. This is like the case in the string where the spatial variation of the displacement field induces a velocity field, Equation 2.59. As the new magnetic field grows which will also be localized and thus curled, it produces a reduction in the original electric field, Equation 4.18. Thus the original field will start to reduce and there will be a growing magnetic field. This magnetic field will in turn change and produce a electric field. The relationship of the magnetic and electric fields is much like that of the velocity and displacement of the stretched string which produces traveling pulses, Section 2.2. In fact, using Equations 4.18 and 4.19, in a region without charges or currents, the vacuum, you find that the electric and magnetic fields are a wavelike system and that a field configuration such as that shown in Figure 4.8 produces a traveling wave that travels in the plane perpendicular to the plane of $\vec{\mathbf{E}}(\vec{r}, t)$ and $\vec{\mathbf{B}}(\vec{r}, t)$ with a speed

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \quad (4.20)$$

which dimensionally is a speed and the only dimensional factor in the dynamic. This is the same result that Maxwell discovered with his whirlies. Putting the values of μ_0 and ϵ_0 this is the speed of light. If it walks like a duck and quacks like a duck, it is a duck and thus Maxwell concluded that

light is the traveling wave solutions to the equations of electromagnetism. Actually, we now reverse the process and assign a value to the speed of light and use the measured definition of the second to allow the definition of the meter and define μ_0 and ϵ_0 to be consistent with Equation 4.20, see [Gleeson Modern Physics].

It is important to realize that like in the stretched string which has only a transverse displacement and transverse velocity, the $\vec{\mathbf{E}}(\vec{r}, t)$ and $\vec{\mathbf{B}}(\vec{r}, t)$ fields are not traveling but only the disturbance – changes in the field configuration are. It is also important to realize that the velocity of the disturbance does not depend on the field configuration. It only depends on the dynamic of the field, Equation 4.20. Another way that this is often stated is that the velocity of propagation is a function only of the medium. Since the electromagnetic field operates in the vacuum of space, it is the properties of the vacuum that determine the speed with which light propagates. A difference for the electromagnetic travelers from the travelers of the field of the stretched string is that in the string any distortion will produce simply related travelers but for the electromagnetic field there are configurations of the field that do not have simply related travelers.

We now understand what the amplitude that was invented by Young and Fresnel to describe the phenomena of the interference and diffraction of light, see my notes [Gleeson Modern Physics]. The amplitude is the traveling waves solutions of the electric and magnetic fields.