Homework Set #6
Introduction to Modern Physics
Due on Thursday, March 8, 2007

Reading: My notes Chapter #5
              Feynman Handout

1. We know that the naturally occurring trajectory for a particle of mass m when there are
no forces acting is a straight line. For the case that the particle does not move this is easy.
Show it. For the case in which the particle connects two events with a spatial and temporal
separation it is more interesting. Consider a case in which the initial event is at (0, 0) and
the final event is at (xf, tf). By considering as the world of all possible trajectories, the once
kinked trajectory in which the kink event is (xf + a, tf), show that the straight line path is
the least action. Note that the parameter a can play the role of path label. Make a drawing
of the trajectories. What does this problem say about Galilean invariance?

2. In Figure 1 is a possible trajectory for an object in motion in the near vicinity of the earth.
The Lagrangian is \( L(v, x) = \frac{mv^2}{2} - mgx \), where \( x \) is the height. We are approximating all
paths with the set of paths that are twice kinked. It is thus spanned by the two parameter
space \((x_1, x_2)\). Give a convincing argument that the least action path must be among the
paths in which \( x_1 = x_2 \). In terms of the one remaining variable \( x_1 \), find the least action path.

3. In order to understand what Feynman is saying in Chapter 19 of his lectures, handed out in
class, we will look at a simplified situation. Look at the problem of the chalk toss. Consider
the case of the chalk returning to the same height that it started from in time T. Divide the
interval into five time slices. Using a kinked set of lines, the path is specified by the positions
at the time slices, \( x_1, x_2, x_3, x_4 \). The least path is a set of these values, \( x_{10}, x_{20}, x_{30}, x_{40} \).
Now look at a nearby path, \( x_{10}, x_{20} + \epsilon, x_{30}, x_{40} \). Now use \( \epsilon \) small to show that the requirement
for the action for the two paths to be the same to the first order in \( \epsilon \) is the discrete version
of \( a = -g \) in the time slices around \( x_2 \). (Hint \( a \) is the change in velocity per unit time.) The
expression “to the first order in \( \epsilon \)” means you can throw away anything that has an \( \epsilon^2 \) in it.
The argument for that is that if \( \epsilon \) is small, then \( \epsilon^2 \) is much smaller. Obviously, we could have
put the kink at any of the time slices so that we have shown that \( a = -g \) in an time slice.
4. In the previous problem, we show that, in a time sliced action analysis for the natural trajectory in the up down direction in the near vicinity of the earth, the change in velocity between any two time slices is \(-g\) times the size of the time slice. Well we actually only showed it for the central part of the a four sliced piece but we can see how this can be extended to the general case. Now consider the simple problem of a piece of chalk released at a height \(h\) of one meter, not tossed. On the graph below, Figure 2, using this rule find the time it takes to fall. From our experience with dropped chalk, we realizing that it takes about \(\frac{1}{2}\) second for the chalk to fall 1 meter, therefore if we pick time slices on the graph of \(\frac{1}{10}\) sec for the intervals of the chart, there will be about four or five units until the chalk hits the ground. In an action analysis, we set the events at the end and beginning of the path. What changes in the problem if we want to consider the motion for the chalk for a start at height 1 meter but we want the chalk to arrive at the ground in \(\frac{3}{4}\) sec. We cannot mess around with \(g\). In terms of the action, why does nature do this?

![Figure 2: Figure for problem # 4](image)

5. In class, we have spent a great deal of time discussing the chalk toss problem. The action is

\[
S(\text{Path}) = \sum_{\text{Path,seg.}} \left( \frac{mv^2}{2} - mgx \right) \Delta t
\]

where \(x\) is measured in the up/down direction from some reference height, say the floor.

(a) Consider the case of my doing a series of chalk tosses standing on the floor and on a chair. Does it make make a difference? What transformation must be made to the variables of the floor-referenced chalk-tossing action in order to describe paths starting from the chair? Is this transformation a symmetry? Is it an invariance? Is there a conserved quantity associated with this transformation?

(b) Similarly, what transformation must be made to describe experiments beginning at different times? Is this transformation a symmetry? An Invariance? Is there a conserved quantity?

(c) Suppose we now choose to change the scale of heights that we use. The transformation in
this case is to replace all the heights by new heights, \( x' = \lambda x \). Does this system possess a symmetry or an invariance for rescaling the heights? Is there a conserved quantity?

(d) Does it possess a symmetry or an invariance for rescaling the time, \( t' = \beta t \)?

(e) Is there a combination of height and time rescaling that is a symmetry or an invariance and if so what is it? Is there a conserved quantity associated with this case?

(f) Is there an example of a discrete symmetry or invariance in this problem and what is it?

(g) The natural path for an initial event at the origin and the final event at the same position at a time \( T \) later is \( x_{\text{nat}}(t) = -\frac{g}{2} t (t - T) \), see my notes for the derivation. For this case, rescale the height and show that it is not a natural trajectory. Pick the value of the rescaling to be 2 and show the two trajectories on the same space time plot. Why is the new trajectory not the naturally occurring trajectory?

(h) Show that a rescaling in \( t \) is also not a natural trajectory.

(i) Show that the suitable time and height rescaling is a natural path. Plot on the same space time diagram the original and the new trajectory. Show the trajectory if you had just rescaled the time.

(j) What is the conserved quantity associated with rescaling?

6. Suppose we lived in a world where every place had the same potential energy but the potential energy of every place increased linearly in time. This is a one spatial dimension world. What is the action? Let’s try to find the least action path by looking at the once kinked trajectories connecting the event \((0,0)\) and \((x_f, T_f)\). Place the kink along the line \( t = \frac{T_f}{2} \) and label the trajectories with the distance from the event \((\frac{x_f}{2}, \frac{T_f}{2})\) along that line. What trajectory is the least action trajectory? Is this action time translation symmetric, invariant, or neither? For space translation, is it symmetric, invariant, or neither? Is it symmetric or invariant or neither under length scale changes? Is it symmetric or invariant or neither under time scale changes? Is there a combination of time and space scaling for which it is symmetric or invariant? Show that it is Galilean invariant. Is there an energy or momentum? Why or why not.

7. The gravitational potential energy between two bodies of mass \( m_1 \) and \( m_2 \) is \(-G\frac{m_1 m_2}{r_{12}}\). What is the Newton’s Law form of the equations describing this interaction between two bodies. Discuss the symmetries of this system in this form. Discuss the invariances? What is the action for this two body system? Name all the symmetries and invariances of this system. Indicate whether or not the symmetry or invariance is continuous or discrete and what the name of the conserved quantity is if that is appropriate. Identify all the form invariants and tell for which set of transformations they hold.

Home experiment #6: A certain venerable game show is played as follows: A contestant is shown three doors by the host. Behind each of two of the doors is a goat, and behind the third is a new car. The contestant chooses a door, and in response the host opens one of the other doors, revealing a goat. (The host knows where everything is, so there’s no chance he’ll unveil the car by mistake.) The contestant then has the chance to change his mind and pick the other door, if he
so chooses. Then the door of his choice is opened; if it hides the car, then he wins it and takes it home.

For awhile there were two widely discussed theories of how to best play the game. One asserted that you should always switch doors, while the other said that it didn’t matter if you switched or not, because whichever door you chose was still one out of the original three so you had a one in three chance of success regardless of your choice.

Your assignment this week, should you choose to accept it (and you do), is to decide which theory is correct by experiment. Play the game with a friend using, for example, different kinds of coins hidden under cups, and play 15 rounds using each of the above strategies. Total points: 15