Key Homework # 5

Problem #1  Consider a field...

The point of this problem and the next few is to look at a few examples of how fields may change over time. In the first part, the rule for how the field changes is:

\[
\frac{\Delta A}{\Delta t} = -kA
\]

By applying dimensional analysis, we see that k has dimensions of \(\frac{1}{\text{time}}\).

One important characteristic of this rule is that it is local. The left hand side of the equation tells us that we are talking about how much the field A changes, \(\Delta A\), over a time period \(\Delta t\). This is presented as a ratio of the two. The right hand side tells us that this ratio is proportional to a negative number, \(-k\), times \(A\) at that same place. What is \(A\)? It is the amount of field we have at a certain point. Remember, a field is defined at all points, so for some position \(x\) and some time \(t\) the field has a certain value. In other words, \(A\) is a function of \(x\) and \(t\). If we do a little algebra and multiply both sides of the equation by \(\Delta t\), we get:

\[
\Delta A = -kA \Delta t
\]

What are \(\Delta A\) and \(\Delta t\)? As we said above, they describe the kind of changes we're talking about; \(\Delta t\) is just the time interval we're looking at, and \(\Delta A\) is how much \(A\) changes over that time interval. This is true for any time interval. In other words:

\[
\Delta A = A(x, t_{\text{final}}) - A(x, t_{\text{initial}})
\]

\[
\Delta t = t_{\text{final}} - t_{\text{initial}}
\]

\[
A(x, t_{\text{final}}) - A(x, t_{\text{initial}}) = -kA(x, t_{\text{initial}})(t_{\text{final}} - t_{\text{initial}})
\]

or

\[
A(x, t_{\text{final}}) = A(x, t_{\text{initial}}) - kA(x, t_{\text{initial}})(t_{\text{final}} - t_{\text{initial}})
\]

So now we know how to apply this rule. If we know how much field \(A\) we have at \(x\) at a \(t_{\text{initial}}\), then we know how to find out how much field is there a short time later. We just apply the rule above. The key here is that this is all an approximation, so it’s important that we only use this rule for very small time intervals \(\Delta t\). In the graph below I have plotted the field when we start (at \(t_{\text{initial}}\)) and two views of the graph at two later times. The first dashed line is the field after one \(\Delta t\), the second dashed line is the field after a second \(\Delta t\). I have picked \(\Delta t\) so that \(k\Delta t\) is 0.1.
How do these three curves compare? Well according to our rule above, the amount by which the field changes over a small amount of time is given by a negative number \((-k)\) times how much field we have there at that time, \(A\). Places where the value of the field is positive, the number \(-kA\) is negative. In this region \(\Delta A\) is negative, the amount of field we have decreases. For the places where \(A\) is negative the increment is positive. Where the field \(A\) is bigger, the field changes more there over the short time interval. That is why the amount by which the curve \(\prime\) dropped \((\text{you can see it on the graph})\) looks bigger at \(x > 2\) than at the other \(x\). In fact the curve at every point falls an amount that is proportional to the size of the field there. This is like a bad investment, in every time interval the amount that the account decreases is proportional to the amount in the account, If we keep applying our rule for how the field changes we \(\prime\) ll see that the curve keeps getting lower and lower, closer to zero. Remember, however, that the amount of change is proportional to how much field we have when we apply our rule, so that the amount of change becomes less and less for successive \(\Delta t\)’s. Like the bad bank account, we never actually reach zero amount of field.

We do however, reach a negligible amount of field after some amount of time. The question asks you how long this takes. Since we didn’t give you any numbers to help your estimates, how can you answer this? We’re asking you for a time, a dimensionful quantity. Even though we didn’t give you the exact numbers for how much field you start with and how quickly it changes, we gave you a more general rule for how this happens. And in order to make a statement about how \(A\) changed over time, we need to include a constant \(k\), which has dimensions of \(\frac{1}{\text{time}}\). So even though you can’t give a numerical answer to this question, you can say that the only characteristic time associated with this system is \(\frac{1}{k}\). Why? Because that is the only thing that you can get out of what we gave you with the dimension you want. So your answer should be in terms of multiples of \(\frac{1}{k}\). In fact is you understand how interest works you can see how many half lives you have. Actually, if we were to write out the calculus equivalent of this rule we’d find out that after a time \(t\), the field has changed by a factor of \(e^{-kt}\). Thus, after a time interval of \(\frac{1}{k}\), the field has shrunk to about 0.37 times it original value.
■ Problem #2 Now suppose that we look...

In the second part you are given another rule. Specifically:

$$\frac{\Delta B}{\Delta t} = -k' \frac{\Delta B}{\Delta x}$$

This time, dimensional analysis shows that $k'$ has units of \(\frac{\text{length}}{\text{time}}\). Even though this rule looks different than the first rule, we go about applying it in the same manner. Different fields may change according to different rules, but the one thing that stays the same for all of them is that the rules are local. That is, how a field changes at a certain value of $x$ only depends on what is going on in the immediate vicinity of $x$, not at some far away point.

Thus, we apply this in exactly the same way as before, except that now the change in $B$ over a time interval $\Delta t$ is proportional to a negative number (-$k$) times the quantity $\frac{\Delta B}{\Delta x}$. What is this? It's just the slope of $B$ at that point. So after applying the same type of algebra as above, our rule is:

$$B(x, t_{\text{final}}) = B(x, t_{\text{initial}}) - k' (\text{Slope of } B \text{ at } x \text{ at } t_{\text{initial}})$$

Thus, if the slope of $B$ is negative at $x$, the net change is positive. If the slope is positive, the net change is negative. If we apply this rule to the field shown in the problem, the effect is that (roughly) the whole thing keeps its shape and moves off to the right. What is its speed? Using the same reasoning as in the first part we get that it speed is proportional to $k'$, since it has units of velocity. The following is $B(x)$ at $t=0, t=\Delta t$, and $t=2\Delta t$:

```
Plot[Exp[-(x + 2)^2] + \frac{1}{\pi} ArcTan[2 (x - 2)] + \frac{1}{2}', {x, -6, 6},
PlotRange -> All, AxesLabel -> "x", "B(x,0)"], AxesOrigin -> {0, 0}]
```

You should can see this most easily if you plot separately the slopes at each stage. For the first stage, you use the slope of $B(x)$ at the initial time. This is
Thus after a short time,

\[
\text{Plot}\left[\frac{2}{\pi \left(1 + 4 \left(-2 + x\right)^2\right)} - 2 e^{-(2-x)^2} \left(2 + x\right)\right], \{x, -6, 6\}, \\
\text{PlotRange} \to \text{All}, \text{AxesLabel} \to \{"x", "Slope of B(x,0)"\}
\]

Where for clarity I show the original field configuration dashed. We can see that the configuration moves to decreasing \(x\).

In subsequent steps it will continue to move over.

**Problem #3** *Now suppose that we look at two different...*

Now let's do this case,
\[
\frac{\Delta A}{\Delta t} = k' \frac{\Delta B}{\Delta x} \\
\frac{\Delta B}{\Delta t} = \frac{\Delta A}{\Delta x}
\]

Here \(k'\) has the dimensions of a velocity squared. The initial conditions are

\[A(x) \text{ at } t=0\]

\[B(x,0)\]

Slope of \(A(x)\) at \(t=0\)
For our increment we will take $\Delta t$ such that is $1/5$ and $k' = 1$. We take $1/5$ of the slope of B and subtract it from A and similarly subtract $1/5$ of the slope of A but since A is zero this is trivial.
Now increment this $A$ by $\frac{1}{5}$ of the slope of $B$ etc.
It does not look like B has changed but it has. It is flatter and broader than the original. The two traveling waves are starting to emerge.

**Problem #4** You are planning a trip...

The first issue is to decide how well you have to measure the distances. If you only need half hour precision in time, this is 32.5 miles on the superhighway and 12.5 miles on the back roads. This means that you have to measure the distances to only that same precision. A careful measurement of the length of the superhighway segment is that it is 4 cm long and thus a cm is 15 miles. Therefore we need the superhighway to only a precision of about 2 cm and the back roads to about 1 cm. At this precision we can darn near just eyeball the thing. Taking out my trusty nanostick, very roughly rectifying, and measuring the distances to 1/10 of a cm. I get that the all superhighway dominated route is 4 cm of superhighway and 1.4+1.8 cm of back roads. Thus the travel time is $4 \times \frac{15}{65} + 3.2 \times \frac{15}{25}$ or about 3 hours. Doing the same for the all back roads route, I get a length of 1.4+1.6+1.4+1.8 cm or 6.2 $\times \frac{15}{25}$ or about 4 hours. You do the other routes similarly. If we now need 5 minute precision, this is 1/6 of the previous case and thus 1/3 cm on the superhighway and 1/6 cm on the back roads. This is pretty precise and to be confident at this level we really need 1/10 cm or better. Rerectifying the paths and measuring to 0.05 cm, I get $3.8 \times \frac{60}{3.8} \times \frac{1}{65} + 3.25 \times \frac{60}{3.8} \times \frac{1}{25}$ hours or 2.9 hours and $6.20 \times \frac{60}{3.8} \times \frac{1}{25}$ hours or 3.9 hours.
Problem #5: If you had a world in which the electric and...

Now let's do the case,

\[
\begin{align*}
\frac{\Delta E}{\Delta t} &= k' \frac{\Delta B}{\Delta x} \\
\frac{\Delta B}{\Delta t} &= \frac{\Delta E}{\Delta x}
\end{align*}
\]

Here \( k' = \mu_0 e_0 \) and has the dimensions of a velocity squared. The initial conditions are

![Graph of E(x) at t=0](image1)

![Graph of B(x) at t=0](image2)

![Slope of E(x) at t=0](image3)
For our increment we will take $\Delta t$ such that is $\frac{1}{5}$ and $\frac{1}{\mu_0 c_0} = 1$. We take $\frac{1}{5}$ of the slope of $B$ and add it to $E$ and similarly add $\frac{1}{5}$ of the slope of $E$ and add it to $B$ but since $E$ is zero this is trivial.
Now increment this new $E$ by $\frac{1}{5}$ of the slope of $B$ etc.
It does not look like E has changed but it has. I show the original case as dashed. The current field two lumps. The two traveling waves are starting to emerge. Each of half the height of the original and one traveling to the left and one traveling to the right with velocity $\sqrt{\frac{1}{\mu e_0}}$.

**Problem #6: Consider the system made of five...**

Since there are 5 masses there are 5 normal modes. The initial transverse displacements that will excite them and in the order of lower to higher frequency is
I have also shown as a dashed line the shape of the vibrating string starting the opposite way in the corresponding mode. In the case of the fundamental the system is using only two springs and is vibrating all 5 masses. I would guess that the frequency is \( \sqrt{\frac{2k}{5m}} \). The higher modes can be gotten from the frequencies of the stretched string. The frequencies scale as the inverse of the wavelength of the mode. Thus they are in the ratio of 1 : 2 : 3 : 4 : 5.

**Problem #7:** *Consider a stretched string between.....*

Disturbances in the string travel at a velocity of \( \pm \sqrt{\frac{L}{\rho}} \). The initial velocity distribution along the string is zero. In the figure below, are the left travelers that come to the barrier at \( x=0 \). The dashed figure is the initial disturbance and the left travelers make a continuous wave. These are shown solid. The left travelers velocity field is shown in the short dashed curve.

The right travelers traveling toward the barrier at \( L=1 \) and the initial disturbance and right traveler velocity field are
In a time $\frac{1}{12} L \sqrt{\frac{e}{T}}$ each right traveler advances by $\frac{1}{12}$ of a unit and the left travelers advance $-\frac{1}{12}$ of a unit. The original configuration is shown dashed.

Similarly the right travelers

Adding these two in the interval 0 to 1
Now we advance by a time $\frac{1}{6} L \sqrt{\frac{c}{T}}$. Each right traveler advances by $\frac{1}{6}$ of a unit and the left travelers advance $-\frac{1}{6}$ of a unit.

Now the displacement field is zero everywhere and the velocity field is at a maximum.

Advance by a time $\frac{1}{4} L \sqrt{\frac{c}{T}}$. 

Here the velocity field and the displacement field fall on each other.

Advance by a time \( \frac{1}{3} L \sqrt{\frac{\nu}{\gamma}} \).

Here the velocity field is zero everywhere and the displacement field is the reverse of the initial field.

Advance by a time \( \frac{1}{2} L \sqrt{\frac{\nu}{\gamma}} \).

Now the velocity field is the same as the original displacement field and the displacement field is zero everywhere.

Advance by a time \( L \sqrt{\frac{\nu}{\gamma}} \).
Now the displacement field is the opposite of the original and the velocity field is zero everywhere.

**Problem #8:** Consider a stretched string....

The configuration of the string at t=0 is shown as a solid line below

This leads to two traveling peaks one moving toward increasing $x$, a right traveler, and one moving toward decreasing $x$, a left traveler, and each of these has an infinite set of image waves caused by the fixed ends of the strings. The net effect of the images is that there is a continuous wave of right and left travelers such as show below for the right travelers that will move into the interval at later times.
There is a similar set of left travelers.

It is now a simple matter to advance the right travelers to the right and the left travelers to the left and add the signals. After \( \frac{L}{8} \sqrt{\frac{\mu}{T}} \), the right traveler is
and the left traveler is

![Graph showing velocity of the left traveler over time.]

adding in the interval

![Graph showing velocity of the right traveler over time.]

For these waves it is easy to see how the string is moving. In the travelers, the wave is moving in one direction with one speed, $\sqrt{\frac{T}{\rho}}$, for the right movers and $-\sqrt{\frac{T}{\rho}}$ for the left movers, and the velocity of the string can only be up or down, the velocity of the string is the slope times the negative of the sideways velocity. You add velocities from the two travelers to get the velocity of the string. Thus for the right travelers, the velocity of the string is
and for the left traveler,

Combining these for $\frac{1}{8} \sqrt{\frac{\theta}{\pi}}$,.
Doing the subsequent steps, for $\frac{L}{4} \sqrt{\frac{E}{T}}$, the height of the string is

![Graph showing the height of the string for $\frac{L}{4} \sqrt{\frac{E}{T}}$]

and the velocity is

![Graph showing the velocity of the string for $\frac{L}{4} \sqrt{\frac{E}{T}}$]

For $\frac{L}{2} \sqrt{\frac{E}{T}}$, the height is

![Graph showing the height of the string for $\frac{L}{2} \sqrt{\frac{E}{T}}$]
and the velocity is

```
-1
-0.5
0.0
0.5
1
```

the same as the original velocities.

For \( L \sqrt{\frac{\rho}{T}} \),

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or back to the original shape. The velocities will also be the same as the original case, all zero.

The same analysis using the fundamental and first harmonic proceeds by picking a best fit shape. You can do this by eye or rigorously but we will eyeball it.
or

about as good as you can do with only two harmonics.

Now the motion proceeds from the oscillation of the two standing waves. The frequency of the fundamental is $\frac{\sqrt{\frac{T}{L}}}{L}$ (remember that $L$ is twice the distance between the wall and the bracket) and the frequency of the first harmonic is $2 \cdot \frac{\sqrt{\frac{T}{L}}}{L}$. Thus in time $\frac{L}{8} \sqrt{\frac{\rho}{T}}$, the fundamental has advanced by one eighth of a cycle and the first harmonic by one quarter of a cycle. The height of the fundamental is zero everywhere and the height of the first harmonic is the opposite of the initial configuration or
The velocity is also easy to get since the velocity of each part of the standing wave is that of an oscillator that started the same way. If the start of the oscillator was stretched and at rest, and thus each standing wave has a velocity that is the initial height times $-\sin(2\pi ft)$ where $f$ is the frequency of the standing wave. Thus the velocity is

For the time $\frac{L}{4} \sqrt{\frac{\rho}{T}}$, the height is
and the velocity is

For the time $\frac{L}{2} \sqrt{\frac{g}{h}}$, the height is

and the velocity is zero everywhere
The last step gets us back to where we started. The other case operates the same but is more interesting.

**Home Experiment #5: Get a small elastic ball....**

First think of an elastic ball and a wall. The ball bounces off the wall at the same speed that it strikes it.

This is before and

this is after the collision. In the frame in which the ball is at rest initially and the wall is moving,

and

Notice the wall just keeps moving. The wall is like the heavy ball. It just keeps moving. In our case the situation is that the two balls are dropped together. The big ball hits the ground first and is headed up with a speed \( v \), it bounces elastically off the ground. The small ball is falling at a speed \( v \). In the frame of the big ball, the little ball is moving at speed \( 2v \). It bounces elastically. Thus in this frame, after the collision, it is moving up at speed \( 2v \). Transforming back to the ground
frame, the little ball is moving up at speed 3v. Since the rise of the ball is proportional to \( v^2 \), it rises 9 times as high as it does if it just hits the ground.