Chapter 13

Relativistic Dynamics

13.1 Relativistic Action

As stated in Section 4.4, all of dynamics is derived from the principle of least action. Thus it is our chore to find a suitable action to produce the dynamics of objects moving rapidly relative to us. For a starter, we will consider only the action that would be associated with point particles but even more simply freely moving particles. Later we can discuss the action for relativistic fields and actions that combine particles and fields, see Section **??**.

As we saw in Section 5.4, it is advantageous if the action possess the maximum amount of symmetry. This will produce the largest number of conserved quantities which in turn will simplify the analysis. In other words, in addition to having the usual symmetries of space and time translation, it would be nice to have the action be symmetric under Lorentz transformations. Remember that the classical actions are not symmetric under Galilean transformations but are invariant instead, see Section 5.4.4. Having an action that is symmetric under the Lorentz transformations will expand the set of conserved quantities available for the solution of dynamical problems. In the following sections, we will be more careful in our handling of the notation and remember that there are three spatial directions, i. e. the position is $\vec{x}$. Where it is unimportant for the interpretation, we will suppress the vector designation.

13.1.1 The Action for a Free Particle

In order to discover the action for rapidly moving particles, we should look at simple situations. For the free particle, we know what the natural trajectory in space time is – a straight line. We want this to be the trajectory with the least action. In addition, if we want the set of Lorentz transformations to be a symmetry for for this action, we should construct it from form invariants of the Lorentz Transformations, see Section 5.4.3, and Section 10.6. For timelike trajectories, the form invariant that characterizes the trajectory is the proper time. The action for the free particle should be dependent on the proper time and only on the proper time. The simplest possibility is that the action depend on the proper time linearly. Since action has the dimensions of an energy times a time, we have to multiply the proper time by something with the dimensions of an energy. Fortunately, the relevant dimensionful parameters are available. One is the mass of the particle. In fact, when you think about it this will be the definition of mass. Well, actually only that mass that is called the inertial mass. We will expand on this idea in Chapter 14 and below in Section 13.3. Also, since in the case of the free particle, we know that the trajectory is a straight time-like line. This is because there must exist a Lorentz observer who has the particle at rest in his/her frame. Since the straight trajectory is the longest worldline between two events, see Section 10.3, and we want the action to be a minimum for the naturally occuring trajectory, the action should be proportional to the negative of the proper time. In this way, the greatest proper time will correspond to the least action. The unique combination that we have been led to is

$$S(\vec{x}_0, t_0, \vec{x}_f, t_f; trajectory) = -mc^2 \tau_{(\vec{x}_0, t_0, \vec{x}_f, t_f; trajectory)} \quad (13.1)$$

$$= -mc^2 \sum_{trajectory, \vec{x}_0, t_0}^{\vec{x}_f, t_f} \Delta\tau_i \quad (13.2)$$

where the $\Delta\tau_i$ are the proper time intervals in each segment, see Figure 13.1.

This form is inappropriate for the interpretation of an action since it is not time sliced. To transform from segment slicing which is what we have in Equation 13.2 to time slicing, we use the fact that we can relate proper time intervals to coordinate time intervals as $\Delta\tau_i = \sqrt{(t_i - t_{i-1})^2 - \frac{(\vec{x}_i - \vec{x}_{I-1})^2}{c^2}}$. If we now factor out the $(\Delta t_i)^2$ and realize that the velocity in space time is the inverse slope to the trajectory, $\frac{\Delta\vec{x}_i}{\Delta t_i} \equiv \vec{v}_i$, or

$$S(\vec{x}_0, t_0, \vec{x}_f, t_f; trajectory) = -mc^2 \sum_{trajectory, \vec{x}_0, t_0}^{\vec{x}_f, t_f} \sqrt{(\Delta t_i)^2 - \frac{(\Delta\vec{x}_i)^2}{c^2}}$$

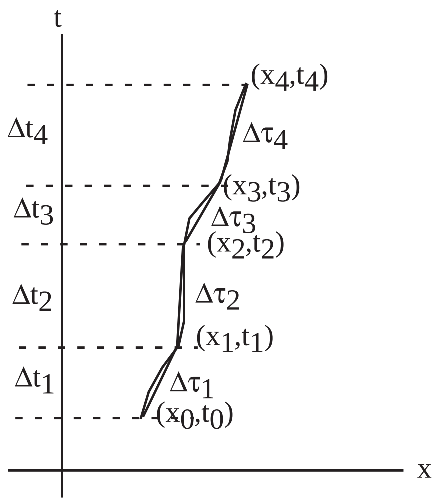

Figure 13.1: **Segmented Relativistic Action** The action for a relativistic particle is naturally expressed in terms of the proper time intervals as $-mc^2 \sum_{trajectory,\vec{x}_0,t_0}^{\vec{x}_f,t_f} \Delta\tau_i$, where the $\Delta\tau_i \equiv \sqrt{(t_i - t_{i-1})^2 - \frac{(x_i - x_{i-1})^2}{c^2}}$ are the proper time intervals in each segment . Actions are best interpreted in terms of coordinate time, $\Delta t_i = (t_i - t_{i-1})$.

$$
\begin{aligned}
&= -mc^2 \sum_{trajectory,\vec{x}_0,t_0}^{\vec{x}_f,t_f} \sqrt{1 - \frac{\frac{(\Delta\vec{x}_i)^2}{(\Delta t_i)^2}}{c^2}}\Delta t \\
&= -mc^2 \sum_{trajectory,\vec{x}_0,t_0}^{\vec{x}_f,t_f} \sqrt{1 - \frac{\vec{v_i}^2}{c^2}}\Delta t \qquad (13.3)
\end{aligned}
$$

With this time slicing, we identify the Lagrangian for the free particle as $L(\vec{v}, \vec{x}) = -mc^2\sqrt{1 - \frac{\vec{v}^2}{c^2}}$. We should compare this result with the classical lagrangian for the free particle, $L_{Class}(\vec{v}, \vec{x}) = m\frac{v^2}{2}$. Remember, $v \equiv |\vec{v}|$ and $v^2 = \vec{v}^2$.

In the limit the $\frac{v^2}{c^2} \ll 1$, $L(\vec{v}, \vec{x}) = -mc^2(1 - \frac{v^2}{2c^2} \cdots) = (m\frac{v^2}{2} - mc^2 \cdots)$. Thus, the relativistic lagrangian is the same as the classical lagrangian to within an additive constant. An added constant in the lagrangian adds a term in the action that is

$$
- \sum_{trajectory,\vec{x}_0,t_0}^{\vec{x}_f,t_f} mc^2\Delta t = -mc^2(t_f - t_0)
$$

which does not depend on the trajectory and thus does not effect the path selection process. Therefore, the physics is the same for these two lagrangians in the low velocity limit.

We can also calculate the relativistic free particle action between two events over the natural path since we already know that this is the straight line trajectory; that was how we decided what the action was. The action for the naturally occurring trajectory is

$$S(\vec{x}_0, t_0, \vec{x}_f, t_f; natural) = -mc^2\sqrt{1 - \frac{\frac{(\vec{x}_f - \vec{x}_0)^2}{(t_f - t_0)^2}}{c^2}}\Big(t_f - t_0\Big) \tag{13.4}$$

13.2 Energy and momentum of a single free particle

Using the fact that the spatial and temporal translations are a continuous symmetry for this action, we have energy and momentum conservation. Using Noether's theorem, Section **??**, the energy is the change in the action when the final time is shifted or

$$E = \frac{\delta S}{\delta t_f} \tag{13.5}$$

$$= \frac{mc^2}{\sqrt{1 - \frac{\frac{(\vec{x}_f - \vec{x}_0)^2}{(t_f - t_0)^2}}{c^2}}} \tag{13.6}$$

$$= \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{13.7}$$

since for the straight line trajectory, $\frac{\vec{x}_f - \vec{x}_0}{t_f - t_0} = \vec{v}$.

Similarly, the momentum is the change in the action when you translate the final position.

$$\vec{p} = \frac{\delta S}{\delta \vec{x}_f} \tag{13.8}$$

$$= \frac{m\frac{(\vec{x}_f - \vec{x}_0)}{(t_f - t_0)}}{\sqrt{1 - \frac{\frac{(\vec{x}_f - \vec{x}_0)^2}{(t_f - t_0)^2}}{c^2}}} \tag{13.9}$$

$$= \frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}} \tag{13.10}$$

Note that, for a massive particle observed in its rest frame, $\vec{v} = \vec{0}$, the momentum is zero and the energy is mc^2. Thus this energy is not necessarily an energy of motion like the classical kinetic energy, see Section 13.4. On the other hand, note that, for a massive particle that is moving relative to some frame at a relative speed $\vec{v}$, the energy and momentum are dependent on the relative motion of the observer. This is how it was in old fashioned classical physics; the momentum was $\vec{p} = m\vec{v}$ and the energy of motion or kinetic energy was $KE = \frac{mv^2}{2}$ and the values of the momentum and energy depended on the velocity with which the particle is observed. The difference in this case is that there is still and energy term even in the case of zero relative motion.

What is this energy? This energy is the energy in the famous formula $E = mc^2$. We now realize that this formula is not completely correct as written. It is more properly written as

$$E_{v=0} = mc^2. \tag{13.11}$$

For a system that is basically not moving relative to an observer, i. e. a commover, this is the energy that is necessary to form the system; its rest energy, Section 13.9. This will make more sense when we talk about many particle systems, Section 13.8.

In that regard, please note that by dividing Equation 13.10 by Equation 13.7,

$$\frac{\vec{p}c^2}{E} = \vec{v}. \tag{13.12}$$

For a single particle, with mass m, this is an interesting observation and provides another way to measure the relative velocity of a particle. For the multi-particle case, Section 13.8, it will become the definition of the system velocity.

We should also note that with these formula's for the energy an momentum, we have a new interpretation of c. It is a conversion factor from momentum to energy units and even to mass units. For example, for Equation 13.12 to be true $pc \stackrel{\text{dim}}{=} E$ or, for Equation 13.11, $E \stackrel{\text{dim}}{=} mc^2$. The most common way that this conversion is seen is when you see momenta expressed in $\frac{\text{MeV}}{c}$. MeV is an energy unit, the energy scale of nuclear reactions. It is 10^6 eV where the eV is the energy that an electron gains by moving through a voltage of one volt and is the energy scale of chemical reactions, see Section 1.4.2, or an eV is 1.6×10^{-19}Joules. A momentum of $1\ \frac{\text{MeV}}{c} = 1.6 \times 10^{-13}\ \frac{\text{Joules}}{c} = \frac{1.6\times10^{-13}}{3\times10^{8}}\ \frac{\text{kg m}}{\text{sec}} = 5.3 \times 10^{-22}\ \frac{\text{kg m}}{\text{sec}}$. Sometimes it is even worse than this. The factors of c will be suppressed as when you see someone write that the mass of the electron is 0.51 MeV. What the more careful person means is $0.51\ \frac{\text{MeV}}{c^2} = 9.1 \times 10^{-31}$ kg.

13.3 Mass

In formulating the appropriate action for the relativistic particle, we needed to include the mass of the particle, see Section 13.1.1. It was indicated that this is the inertial mass, the mass that resists changes in the state of motion. For a free particle, the more the trajectory deviates from a straight trajectory, the more action that it costs, the proper time is shorter, and the mass is the weighting factor; the larger the mass, the higher the action for the same deviation in the trajectory from the straight trajectory.

We also saw that to the commover, the energy of the system is simply related to the mass, $E_{v=0} = mc^2$. This mass was interpreted as the energy needed to create the system, Section 13.2. This will be better understood when we discuss multi-particle states, Section 13.8.

Note that the following combination of the energy and momentum does not have the velocity of the particle in it.

$$E^2 - \vec{p}^2c^2 = \left(\frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}\right)^2 - \left(\frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}}\right)^2 c^2 \qquad (13.13)$$

$$= (mc^2)^2 \qquad (13.14)$$

Even though E and p have different values depending on the relative motion. No matter what your relative motion to the particle is, the combination $E^2 - p^2c^2$ is the same and is m^2c^4. In fact, since p and E are the dynamical entities, they are the things that are generally measured in an experiment on elementary particles. You measure the momentum, p, by seeing how the particle is effected by a known force and the energy, E, by a direct energy transfer measurement. This is how the mass of elementary particles is actually measured. You independently measure E and p and then form the combination $E^2 - p^2c^2$ to determine the mass. Although mass is not the only identifying characteristic of an elementary particle, it is the most important one. The miracle of this operation is that you discover that with all the experiments that we have performed measuring a tremendous range of p and E that the masses observed are always in a small group of fixed values, the masses of the elementary particles. All electrons have a mass of 9.1×10^{-31} kg, all protons have a mass of 1.7×10^{-27} kg and so on. This even works for systems that we know are composite. All carbon twelve nuclei, composed of six protons and six neutron, have a mass of 1.9932×10^{-26}kg, which is **not** the mass of the six protons and six neutrons when measured carefully. The care that must be maintained if the mass difference is to be detected is the reason for the high precision. The mass of a hydrogen atom which is a proton and an electron is ???? and, again, is **not** the mass of an proton and an electron when measured very carefully. We will discuss this problem when we discuss multi-particle states, Section 13.8.

13.4 Kinetic Energy of a Single Particle

In non-relativistic physics the kinetic energy is zero if you are moving with the particle. It is in this sense that we define the kinetic energy for a relativistic particle as the energy of motion and thus the energy above the rest energy.

$$KE \equiv E - E_{v=0} \tag{13.15}$$

$$= mc^2\left\{\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1\right\} \tag{13.16}$$

If $\frac{v^2}{c^2} \ll 1$, using $(1+x)^n \approx 1 + nx \cdots$ for $x \ll 1$.

$$KE = mc^2\left\{\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1\right\} \tag{13.17}$$

$$\approx mc^2\{1 + \frac{v^2}{2c^2} \cdots - 1\} \tag{13.18}$$

$$= \frac{mv^2}{2} + \cdots \tag{13.19}$$

Thus, in the small $\frac{v}{c}$ limit, we recover the usual kinetic energy of classical physics.

Later, in the section on applications, Section 13.10 Item 1, we will discuss how in interactions of particles, in particular nuclei, energy is conserved, mass is reduced, and kinetic energy is produced.

13.5 Transformations of Momentum and Energy

In Section 13.3, we discovered that the combination $\frac{E^2}{c^4} - \frac{p^2}{c^2} = m^2$. In other words, the combination of variables $\frac{E^2}{c^4} - \frac{p^2}{c^2}$ although both E and p depend on the relative velocity v, does not depend on the relative velocity. Since the relative velocity is one of the ways to label the Lorentz transformations, the inference is that the energy and momentum combine to form a form invariant for the Lorentz transformations.

From the definitions of the energy, Equation 13.5, and the definition of the momentum, Equation 13.8, and the fact that the action, S, was made to be symmetric under the Lorentz transformations, Section 13.1.1, you can show that the momentum, $\vec{p}$, and the energy, actually $\frac{E}{c^2}$, transform like the position, $\vec{x}$ and the time, t, see Section **??** Equations **??**:

$$\vec{p'} = \frac{\vec{p} - \vec{v}\frac{E}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{13.20}$$

$$\frac{E'}{c^2} = \frac{\frac{E}{c^2} - \frac{v}{c^2}p}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{13.21}$$

For instance, as we saw in Section 13.2, a particle moving at a speed v, $p = \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}}$ and $E = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}$, is seen to be at rest, $p = 0$ and $E = mc^2$, by an observer moving at v relative to the original observer. In the active view of transformations, Section **??**, we would say that the transformation brings the particle to its rest frame.

The same set of ideas can be reversed. To the observer that is at rest with respect to the particle, the energy, E, is mc^2. To an observer moving at a velocity v relative to that observer the particle has momentum $p' = \frac{-\frac{E}{c^2}v}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{-mv}{\sqrt{1-\frac{v^2}{c^2}}}$ and energy $\frac{E'}{c^2} = \frac{\frac{E}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{m}{\sqrt{1-\frac{v^2}{c^2}}}$. Remember that this observer sees the particle moving with a velocity of $-v$. In fact, this could be the another way to derive the addition of velocities formula, Section 9.3.4.

In this sense, p and $\frac{E}{c^2}$ form a transforming set like x and t. We call anything that transforms like this a four vector. This nomenclature comes from the fact that in the real world there are three space coordinates and one time.

13.6 The Energy, Momentum, and Mass of Light

When you discuss light in the classical sense of Maxwell, it is not clear what is meant by the energy and momentum of light as discussed above. For now though, think of light in the modern context – a particulate transfer of energy and momentum, Section **??**. Note that for a particle the ratio

$$\frac{pc}{E} = \frac{v}{c} \tag{13.22}$$

is independent of the mass that we start with and thus holds for all masses including zero. For particles that travel at the speed of light, this implies that the ratio is 1 and that $p = \frac{E}{c}$. In other words for these particles, $E^2 - p^2c^2 = 0$ which implies that the mass is zero. Particles that travel at the speed of light are massless and the converse also holds that massless particles travel at the speed of light. Note also that massless particles, for example light, have energy and momentum.

Another way to see that is the for any particulate system that carries energy and momentum,

$$E = \sqrt{m^2c^4 + p^2c^2}. \tag{13.23}$$

In the limit as $m \to 0, E = pc$, which implies that $v = c$.

In addition, the momentum and energy of light transform under the Lorentz transformations in the same way as they do for massive particles, Equations 13.20 and 13.21. You can easily convince yourself that for massless particles, you cannot find a transformation the can yield $p' = 0$ in Equation 13.20. Since $m^2 = \frac{E^2}{c^4} - \frac{p^2}{c^2}$ is the same for all Lorentz related E and p , if you had $p = 0$, and $m = 0$, you will also have $E = 0$. A massless particle with $E = 0$ and $p = 0$ is not there. Conversely, If a light beam, a beam of energy momentum transferred by massless particles, has energy E, it has momentum $p = \frac{E}{c}$ and another observer moving relative to the observer that measures that for the beam will measure an E' and p' given by Equations 13.20 and 13.21. Notice how my language here has segued into beam energetics regardless of the particulate nature of the energy.

13.7 Interactions

13.8 Multi-particle Systems

13.9 Rest energy of composite and elementary systems

When you are at rest with respect to a particle the p is zero and

$$E_{v=0} = mc^2 \tag{13.24}$$

This non-zero rest energy is an interesting aspect of special relativity. As stated above, Section 13.2, it is this result that is the basis of the statement that there is an equivalence of mass and energy, $E = mc^2$.

Note that you can never bring light to rest and thus it is consistent to say that light has energy and still meaningless to talk about rest mass.

13.10 Applications of Energy Momentum

1. If we had redone our collision problem with relativistic kinematics we would still have found that the energy and momentum are conserved. Said more generally, since we want all fundamental processes to be time translation invariant we want energy to be conserved. Prior to relativity we also assumed that mass – the amount of stuff – was also conserved. When you think about it this is just a prejudice. We do not have a symmetry requiring mass conservation. In other words, you can now think of processes in which the mass is changed. A popular example is

$$D + T \rightarrow He + n + 17.6 \text{ MeV}. \tag{13.25}$$

This example is popular because it is the basis for potential commercial fusion energy. A deuterium nucleus which is a heavy hydrogen nucleus, one neutron and one proton, and a triton, an even heavier hydrogen nucleus, which happens to be radioactive would serve as fuel for the fusion reactor.

The D and T are basically at rest. The incoming energy is $m_D c^2 + m_T c^2$. The outgoing energy is $\frac{m_{He}c^2}{\sqrt{1-\frac{v_{He}^2}{c^2}}} + \frac{m_n c^2}{\sqrt{1-\frac{v_n^2}{c^2}}}$.

Since there is no momentum coming in the momentum of the He and n are equal and opposite, $p_{He} = p_n \equiv p$. Writing the energy in terms of the momentum,

$$m_D c^2 + m_T c^2 = \sqrt{m_{He}^2 c^4 + p^2 c^2} + \sqrt{m_n^2 c^4 + p^2 c^2} \tag{13.26}$$

Solving for $\frac{p}{c}$

$$\frac{p}{c} = \sqrt{\frac{\{(m_D + m_T)^2 + m_{He}^2 - m_n^2\}^2}{4(m_D + m_T)^2} - m_{He}^2} \tag{13.27}$$

Looking up the values,

$$m_D = 2.01474 \text{ AMU} \tag{13.28}$$

$$m_T = 3.017 \text{ AMU} \tag{13.29}$$

$$m_{He} = 4.00387 \text{ AMU} \tag{13.30}$$

$$m_n = 1.00898 \text{ AMU} \tag{13.31}$$

These masses are in atomic mass units or AMU and the conversion is

$$1 \text{ AMU} = 931 \, \frac{\text{MeV}}{c^2}. \tag{13.32}$$

$$\frac{p}{c} = 0.174961 \text{ AMU}$$

This is one case where you need a calculator. You have to compute small differences.

Thus the value of $pc = 163$ MeV.

Note that the kinetic energy of the He and n are:

$$\frac{KE_{He}}{c^2} = \sqrt{m_{He}^2 + \frac{p^2}{c^2}} - m_{He} = 0.0038209 \text{ AMU} \tag{13.33}$$

and

$$\frac{KE_n}{c^2} = \sqrt{m_n^2 + \frac{p^2}{c^2}} - m_n = 0.0150571 \text{ AMU} \tag{13.34}$$

Thus the KE of the He is 3.56 MeV and the KE of the n is 14.01 MeV. The total energy that goes into KE or motional energy is 17.57 MeV

2.