Problem #1

Harry is moving toward Sally at $\frac{4}{5}c$. Harry has a mass of 50 kg and Sally is 40 kg. At the instant that he passes her, she decides to pursue him and accelerates at $g$ to catch him. Where and when does she catch him? (10 points) During the trip, she sends out a light signal at intervals of $\Delta t$, where $\Delta t$ is once a month. When does Harry say he gets these signals? (10 points) How fast does he say she is traveling when they collide? (10 points) What does he say is her acceleration when they collide? (10 points)

**Solution Problem #1**

Harry and Sally's worldlines:

$$x_s = \sqrt{\left(\frac{c^2}{g}\right)^2 + c^2 t_e^2 - \frac{c^2}{g}}$$
$$x_h = vt_h$$

Setting the positions and times equal, the collision time is

$$\sqrt{\left(\frac{c^2}{g}\right)^2 + c^2 t_e^2 - \frac{c^2}{g}} = vt_e$$

Solve: $\left[\sqrt{\frac{c^4}{g^2} + c^2 t_e^2 - \frac{c^2}{g}} = vt_e, t_e\right]$

$$\{ t_e \rightarrow \frac{2 c^2 \nu}{g (c^2 - \nu^2)} \}$$

$$t_e = \frac{2 c^2 \nu}{g (c^2 - \nu^2)}$$

and

$$x_e = \frac{2 c^2 \nu^2}{g (c^2 - \nu^2)}$$
\[ v = \frac{4}{5} \]
\[ c = 1 \]
\[ g = 1 \]
\[ t_c = \frac{2 \, c^2 \, v}{g \, (c^2 - v^2)} \]
\[ x_c = \frac{2 \, c^2 \, v^2}{g \, (c^2 - v^2)} \]
\[ \frac{4}{5} \]
\[ 1 \]
\[ 1 \]
\[ \frac{40}{9} \]
\[ \frac{32}{9} \]

we get for our case \( t_c = \frac{40}{9} \) yrs and \( x_c = \frac{32}{9} \) ltyrs.

To find the rate at which Sally lets out light rays we need to know the proper time along her worldline. In parametric form it is

\[ x_s = \frac{c^2}{g} \cosh \left( \frac{gt}{c} \right) - \frac{c^2}{g} \]
\[ t_s = \frac{c}{g} \sinh \left( \frac{gt}{c} \right) \]

Light rays emitted at \( \tau \) have the following trajectory

\[ (x - x_s) = c \, (t - t_s) \]

Putting this on Harry's worldline \( x = vt \):

\[ (vt - x_s) = c \, (t - t_s) \]

\[ t = \frac{x_s - c \, t_s}{c - v} = \frac{c^2}{g} \left( 1 + \sinh \left( \frac{gt}{c} \right) - \cosh \left( \frac{gt}{c} \right) \right) \]
\[ \frac{c - v}{c - v} \]
\[ x = vt = \frac{c^2}{g} \left( 1 + \sinh \left( \frac{gt}{c} \right) - \cosh \left( \frac{gt}{c} \right) \right) \]
\[ \frac{1 - \frac{v}{c}}{1 - \frac{v}{c}} \]

The proper times that he sees are thus

\[ t_h = \frac{1}{c} \sqrt{c^2 \, t^2 - x^2} = \frac{\sqrt{c^2 \, t^2 - v^2 \, t^2}}{c} = \sqrt{1 - \frac{v^2}{c^2}} \, t = \sqrt{1 - \frac{v^2}{c^2}} \, \frac{c^2}{g} \left( 1 + \sinh \left( \frac{gt}{c} \right) - \cosh \left( \frac{gt}{c} \right) \right) \]
Setting $\tau = n\Delta \tau$ and looking at $n = 1$ through 20

\{0.239867, 0.460555, 0.663598, 0.850406, 1.02228, 1.18041, 1.32589, 1.45975, 1.5829, 1.69621, 1.80045, 1.89636, 1.9846, 2.06579, 2.14049, 2.20921, 2.27244, 2.33061, 2.38413, 2.43337\}

Look at the differences of the arrival times. This will show the doppler effect.

\{0.220688, 0.203043, 0.186808, 0.171872, 0.15813, 0.145487, 0.133854, 0.123152, 0.113305, 0.104246, 0.0959106, 0.088242, 0.0811866, 0.0746953, 0.068723, 0.0632282, 0.0581727, 0.0535215, 0.0492422\}
To get her velocity relative to him at the time of collision get the four velocity and transform to his frame.

\[
\begin{align*}
  u_x &= \frac{dx}{dt} = c \sinh \left( \frac{gt}{c} \right) = gt \\
  u_t &= c \frac{dt}{dt} = c \cosh \left( \frac{gt}{c} \right) = c \left( \frac{gx}{c^2} + 1 \right)
\end{align*}
\]

In this case, \( u_x = \frac{40}{9} \text{ yrs} \) and \( u_t = \frac{41}{9} \text{ yrs} \). Transforming to his frame:

\[
\begin{align*}
  u_{x_h} &= \frac{u_x - \frac{v}{c} u_t}{\sqrt{1 - \frac{v^2}{c^2}}} \\
  u_{t_h} &= \frac{u_t - \frac{v}{c} u_x}{\sqrt{1 - \frac{v^2}{c^2}}}
\end{align*}
\]

We get \( u_{x_h} = \frac{4}{3} \text{ yrs} \) and \( u_{t_h} = \frac{5}{3} \text{ yrs} \).

To get her acceleration to him we start with her acceleration in the original frame. Differentiating the trajectory twice, we have

\[
\begin{align*}
  a_x &= gcosh \left( \frac{gt}{c} \right) = \left( \frac{gx}{c^2} + 1 \right) g \\
  a_t &= gsinh \left( \frac{gt}{c} \right) = \frac{g^2 t}{c}
\end{align*}
\]

At the collision \( t_c = \frac{40}{9} \text{ yrs} \) and \( x_c = \frac{32}{9} \text{ ltyrs} \). So that \( a_t = \frac{40}{9} \text{ lbyrs} / \text{yr}^2 \) and \( a_x = \frac{41}{9} \text{ lbyrs} / \text{yr}^2 \). But Harry is moving at 4/5 c and he sees

\[
\begin{align*}
  a_{x_h} &= \frac{a_x - \frac{v}{c} a_t}{\sqrt{1 - \frac{v^2}{c^2}}}
\end{align*}
\]
Another way to do this is to realize that to her comover at the collision she is moving with an acceleration of \((g, 0)\). Knowing her velocity relative to Harry at the collision and transforming will give you the answer.

**Problem #2**

In the previous problem, just before Harry and Sally collide, Sally turns off the acceleration and they then move together. What is their velocity from the point of view of Sally's original frame of reference? (10 points) What is their energy and momentum in this frame? (10 points) What is their mass after the collision? (5 points)

**Solution Problem #2**

Harry is trivial he is still moving at \(4/5 \, c\). To get the momentum and energy we need the four velocity. \(u_{th} = \frac{4}{5} \, c\) and \(u_{th} = \frac{5}{3} \, c\). Sally's velocity at the time of the collision from Problem 1 is

\[
\begin{align*}
u_{xs} &= \frac{dx_s}{dt} = c \sinh \left( \frac{gt}{c} \right) = gt \\
u_{ts} &= \frac{dt_s}{dt} = c \cosh \left( \frac{gt}{c} \right) = c \left( \frac{gx}{c^2} + 1 \right)
\end{align*}
\]

At the collision time we had \(t_c = \frac{40}{9} \, \text{yrs}\) and \(x_c = \frac{32}{9} \, \text{lyrs}\), so that \(u_{xs} = \frac{40}{9} \, \text{lyrs} \, \text{yr}^{-1}\) and \(u_{ts} = \frac{41}{9} \, \text{lyrs} \, \text{yr}^{-1}\). Their energy and momentum after the collision is the sum of each ones before the collision.

\[
\begin{align*}
p &= m_h \, u_{th} + m_s \, u_{ts} = 50 \, \text{kg} \, \frac{4}{3} \, c + 40 \, \text{kg} \, \frac{40}{9} \, c = \frac{2200}{9} \, 3 \times 10^8 \, \text{kg m sec}^{-1} \\
E &= m_h \, c \, u_{th} + m_s \, c \, u_{ts} = 50 \, \text{kg} \, \frac{5}{3} \, c^2 + 40 \, \text{kg} \, \frac{41}{9} \, c^2 = \frac{2390}{9} \, (3 \times 10^8)^2 \, \text{kg m}^2 \, \text{sec}^{-2}
\end{align*}
\]

The velocity is

\[
\frac{v}{c} = \frac{220}{239} \, c
\]

The mass is given by

\[
m = \frac{1}{c^2} \sqrt{E^2 - p^2 \, c^2} = 104 \, \text{kg}
\]
Problem #3

Once again, Harry is moving toward Sally at \( \frac{v}{c} \). Sally notes that simultaneous with the arrival of Harry, there are three explosions at positions 2 ltyrs, 3 ltyrs, and 4 ltyrs. When and where does Harry say that these explosions took place? (10 points) From when on were each of these explosions in Sally's past? (5 points) From when on were each of these explosions in Harry's past according to Sally? (10 points) According to Harry? (10 points)

**Solution Problem #3**

The coordinates of the three explosions are \((2,0)\), \((3,0)\), and \((4,0)\). Using the Lorentz transformations to get Harry's coordinates and using \( t_s = 0 \)

\[
x_h = \frac{x_s - vt_s}{\sqrt{1 - v^2/c^2}} = \frac{5}{3} x_s
\]

\[
t_h = \frac{t_s - \frac{v}{c} x_s}{\sqrt{1 - v^2/c^2}} = -\frac{4}{5} x_s = -\frac{4}{3} x_s
\]

or \( \left( \frac{10}{3}, -\frac{8}{3} \right)_h \), \( (5, -4)_h \), and \( \left( \frac{20}{3}, -\frac{16}{3} \right)_h \)

The first explosion is in Sally's past after 2 yrs, the second after 3 yrs and the third after 4 yrs. Light rays from the explosions are on the lines \((x-x_0) = -c t_s\). Harry's worldline is \(x=v t\). Putting these together, \( t = \frac{x_0}{v + c} \). Thus light from the explosions reaches Harry at coordinates \( \left( \frac{x_h}{\sqrt{1 + \frac{v^2}{c^2}}, \frac{x_h}{\sqrt{1 + \frac{v^2}{c^2}}} \right)_h \), so Sally says that they are in his past after \( \frac{5}{3} \) yrs, \( \frac{3 + 5}{3} \) yrs, and \( \frac{4 + 8}{3} \) yrs. Harry records these as his proper time

\[
c_t x_h = \sqrt{\left( \frac{c x_0}{v + c} \right)^2 - \left( \frac{v x_0}{v + c} \right)^2} = \sqrt{1 - \frac{v^2}{c^2}} \frac{c x_0}{v + c} = x_0 \frac{\frac{3}{5}}{\frac{3}{5} \frac{3}{3}} = x_0 \frac{3}{3}
\]

thus Harry's times are \( \frac{2}{3} \) yr, 1 yr, and \( \frac{4}{3} \) yr.