Problem #1  You land on a strange planet and proceed to do physics. You begin with a simple chalk (0.5 kg) toss, and you find that the natural trajectory can be described by the following space-time diagram:

Use for your Lagrangian \( L = \frac{1}{2} m v^2 - kx \). (a) What is the action? (5 points). (b) What are the dimensions of \( k \)? (5 points). (c) Using once kinked trajectories at half the total time for the toss, find a value for \( k \)? (10 points). (d) Assuming that this strange planet is similar to a planet you already know a lot about, what is the value for the acceleration due gravity on this strange planet? (5 points). (e) Under space translation is the action invariant, symmetric, or neither? Is there a conserved quantity, if so what is it? (5 points). (f) Under time translation is the action invariant, symmetric, or neither? Is there a conserved quantity, if so what is it? (5 points). (g) Under space rescaling is the action invariant, symmetric, or neither? Is there a conserved quantity, if so what is it? (5 points). (h) Under time rescaling is the action invariant, symmetric, or neither? Is there a conserved quantity, if so what is it? (5 points). (i) Under both space and time rescaling is the action invariant, symmetric, or neither? Is there a conserved quantity, if so what is it? (5 points). (j) Analyze this action to see if it is Galilean invariant? (10 points). (k) Discuss why Noether's theorem is and isn't important. (10 points). (l) On our strange planet, there is a large deep body of water. Realizing that you cannot pile water and that this is the source of how water waves behave, how fast do we expect water waves to move on the planet relative to those on earth? (5 points) (m) What if the water is shallow? (5 points) (n) What is the meaning of shallow and deep? (5 points) TOTAL: 85 Points.

\[ \sum L \Delta t = \sum \left( \frac{1}{2} m v^2 - kx \right) \Delta t, \]

where the sum is over segments and trajectories.

(b) The dimensions of the Lagrangian are energy, so the dimensions of \( kx \) have to be energy as well. Therefore, the dimensions of \( k \) are energy divided by a length, which is a force, \( M L / T^2 \).

(c) Use the height up as the label for the trajectories.
The action then is,

\[ S = \sum \left( \frac{m}{2} v^2 - k x \right) \Delta t = \left[ \frac{m}{2} \frac{a^2}{4} - k \frac{a}{2} \right] 2 + \left[ \frac{m}{2} \frac{a^2}{4} - k \frac{a}{2} \right] 2 \\
= \frac{m}{2} a^2 - 2 k a \]

The \( a \) for which the action is least, is \( a = 2 k/m \). The least action trajectory is the natural trajectory. Since our trajectory label is the height and we know the natural trajectory went 40 meters high, this means that 40 meters = \( 2k/m \), or that \( k = \frac{0.5}{2} 40 = 10 \) Newtons (\( M L/T^2 \)).

(d) The other planet we know a lot about is the Earth. We know that chalk tosses on the earth have the same trajectory as shown in the figure, and we know what the action is for a flat earth, \( S = \sum (\frac{m v^2}{2} - m g x) \Delta t \). Therefore, \( m g = k \), or that the acceleration of gravity is \( k/m= 10 \) Newtons/0.5 kg = 20 m/s^2.

(e) Under space translation the action is invariant, so no conserved quantity.

(f) Under time translation the action is symmetric, so the conserved quantity is energy.

(g)/(h) Under either space or time rescaling the action is neither invariant or symmetric. No conserved quantity.

(i) With both space (\( x' = \lambda x \)) and time (\( t' = \beta t \)) rescaling the action is invariant if \( \lambda = \beta^2 \). There is no conserved quantity.

(j) The action is Galilean invariant. In an earlier homework problem we showed that the kinetic energy part of the action is Galilean invariant so all that we need to show now is that the potential energy part is also. Replacing \( x \) by \( x-ct \) in the potential energy part adds terms to the action of the form

\[ S = \sum_{\text{seg}} (k \text{ct}) \Delta t \]

but in each segment and for all \( a \) the \( t' \)s are the same. Thus this extra piece is the same for all \( a \) or in other words for all trajectories and thus a constant in trajectory space.

(k) Noether's theorem is not important since ultimately physics is about finding the trajectory of objects. We already have language to find the trajectory---the action. The trajectory that minimizes the action is the natural trajectory. However, using Noether's theorem we are able to find conserved quantities (from symmetries of the action) like energy or momentum. Now these conserved quantities allow us to talk about the trajectory of objects in a different language other than action. For example, in this problem, we found that energy was conserved (time translation symmetry). Therefore, as the chalk traces out its trajectory, energy is conserved over the entire trajectory. Knowing this I can write down the energy equation,
\[ E = \frac{1}{2} m v^2 + mgx \]. Which is constant throughout the motion, so I can solve the energy equation to get the trajectory of the object. I end up getting the same trajectory as we found with the least action language, but this time I didn't need to calculate the action over all possible trajectories---which we know is a complete mess.

(1) We know that the velocity of the water waves must depend on the gravity of the planet. Another important variable is the density of the water. To make a velocity though we need a length over a time and with \( g_{\text{planet}} \) and a density you cannot make a speed. If you had a length you could make a speed from that length and \( g_{\text{planet}} \) and the form would be \( \sqrt{g_{\text{planet}}L} \) and for waves we have a length, the wavelength or the width of the pile of water you start with. Thus the ratio of the speeds for the same size pile is \( \sqrt{2} \). Note also that the speed depends on the shape of the disturbance and thus these are not really waves by our definition. Note also that if the water is shallow, there is another unit of length the depth. In this case, a speed can be constructed that is independent of the size of the disturbance and thus have a speed independent of the size of the disturbance and thus the waves all travel at the same speed and this is a wave in our definition. What is the meaning of shallow and deep? If the lumps of water are large compared to the depth the water is shallow.

**Problem #2:** Consider a Young's double slit apparatus. It has a separation between the slits of \( w \). If we put light of wavelength \( \lambda \) through it, it has repeating pattern of equal brightness spots on the screen separated by a distance \( d \). We now shoot electrons through the apparatus and count the electron impacts on the screen and an pattern of maximum counts develops. It has the same separation between the bright spots as the light had. What speed did the electrons have? Give a formula using the information above. (10 points) We go back to using light in the apparatus and replace the screen with an array of Einsteinian photo detectors. Again we accumulate counts in the photo detectors. At any place \( x \) on the screen what is the relationship between the counts per unit time and the brightness we had before. Each photo detector has a given small area exposed to the beam. (10 points) We examine the places of the maximums. Do we expect that they all have the same number of counts? (5 points) If not why? If so why? (5 points) If we do the experiment again with our electrons do we have the same kind of behavior for the electron counts at the maximums that we had for the light? (5 points) Why do we have to wait for a fairly large number of counts of either the electrons or the light before we can see a pattern emerge? (5 points) Total points: 35

- **Solution**

If the pattern is the same, the wavelength of the electrons and light is the same. Therefore the velocity of the electron is related to the wavelength by \( p = mv = h\lambda \) or \( v = \frac{h\lambda}{m} \). If the photo detectors had an area of \( A \), the pattern on number of hits per unit time at a place labeled \( x \) follows the pattern of the brightness at the place as \( n(x) \frac{\Delta t}{A} = n(x) \frac{hc}{2\lambda A} = I(x) \), where \( I \) is the brightness. We do not expect the count pattern to follow the brightness pattern exactly. The counting process is stochastic and thus the preceding formula only gives the average number of counts per unit time at that place and there will be scatter about the average that is Poisson, the standard deviation is \( \sqrt{n(x)} \) at any place. We expect the electron counts to have the same scatter of values as the light. In order to get sharp contrast between the maximums and the minimums, we need to have \( n_{\text{max}} \) large enough that the stochasticity of the counts in the maximum is not apparent. This would mean that \( \sqrt{n_{\text{max}}} > \frac{1}{10} \) or the counts in the max locations should get up to the hundreds. This should be true for all the several illuminated maxes.

**Problem #3:** A light source (shown below) emits two photons with opposite polarization back-to-back at a rate of 1 photon/second in each direction. On the left and right there is a series of calcite crystals. Each side has a detector---that counts photons---connected to a match, after 40 photons the left match will ignite, while the right match will ignite after 45 photons.
A table of sines and cosines for you to reference:

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<th>45°</th>
<th>60°</th>
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<td>$\sin^2(\theta)$</td>
<td>1/10</td>
<td>1/4</td>
<td>1/2</td>
<td>3/4</td>
<td>9/10</td>
</tr>
</tbody>
</table>

(a) What is the photon count rate at the left detector? (5 points). (b) at the right detector? (5 points). (c) On average how long does the left match take to ignite? (5 points). (d) On average how long does the right match take to ignite? (5 points). (e) Ignoring the fact that the two photon polarizations are related, what is the least amount of time we need to wait to be close to 98% sure that both matches are on fire? (10 points). (f) Now recalling that the two photon polarizations are related, what is the probability that the right detector counts a photon if the left detector counts a photon? (10 points). (g) what is the amplitude? (5 points). (h) In this case, do we need to wait a shorter, longer, or the same amount of time to be 98% sure that both matches are on fire? Why? (5 points).

TOTAL: 50 Points.

- **Solution**

(a) The photon rate at the left detector is $\frac{1}{16}$ photons/second.

(b) The photon rate at the right detector is $\frac{1}{8}$ photons/second.

(c) It takes 40 photons for the left match to ignite, and photons are coming in at a rate of $\frac{1}{16}$ photons/second, so we would expect to have to wait 640 seconds on average.

(d) On the right match it takes 45 photons to ignite, and the photon rate is $\frac{1}{8}$ photons/second, so we would expect to have to wait 360 seconds on average.

(e) We know that photons follow a Poisson distribution, which we approximate with a normal distribution. The distributions for both the left and right detectors are shown below (recall that the standard deviation is the square root of the expected average):
Notice that if we wait 640 seconds the probability of getting 45 photons on the right detector is essentially 1. But after 640 seconds there is only a 50% chance of getting 40 photons on the left detector. That means after only 640 seconds there is only a 50% chance (have to multiply the probabilities together) that both matches are on fire. To be 98% sure that both are on fire we need to wait at least 690 seconds.

(f) If the left detector registers a click, that means the left photon has a polarization that is $105^\circ$. Since the two photons are oppositely polarized the right photon has a polarization of $15^\circ$. Running that through the series of crystals on the right, the probability of a click on the right detector is only $\frac{3}{16}$.

(g) ... and the amplitude is $\sqrt{\frac{3}{8}} \cos(45 - 90) = \sqrt{\frac{3}{8}} \sqrt{\frac{1}{2}} = \sqrt{\frac{3}{16}}$.

(h) Since the two photon polarizations are correlated every time we get a click on the left detector there is only a $3/16$ chance of getting a photon on the right detector as well. So after 40 photons on the left, we should only expect about 7.5 photons on the right on average. Since we need 45 photons on the right to ignite the match, we have to wait quite a bit longer.