Chapter 11

The nature of space-time

11.1 The problem of coordinates

The basic problem of physics is to track in space and time the development of elements of a system. This requires that we have some method to communicate where and when something took place. In a three dimensional space the place is a set of three numbers; for instance, in a room you could use how far along the floor in a direction along one wall, how far along another wall, and how far up towards the ceiling. The time comes from a clock. This seems so obvious that we generally do not even think about it but, like all the things that we do, this is a subtle operation and we should understand what it is that we are doing when we make a coordinate system. In fact, the realization, that the establishment of the coordinate system is arbitrary is the key to understanding General Relativity. That will come later, Chapter 14.

First, let’s talk about places. The idea is to label the places. Think of a large parking lot, say at Disney Land. What you need is a unique label for every place. This could be done simply by going around and labeling spots on the lot with the name of a Disney character. This though is not an efficient way to label places. It is a unique label for each place which is how we started but there are many better ways to proceed. For one thing, this labeling scheme does not provide a guide for movement. If you are at Donald Duck, you do not know how far or in what direction to go to get to Goofy, the labels are not an ordered set. You could fix this by say ordering the characters alphabetically. This system is nice in that it provides a guide to how to move, it does not indicate how far. It is also not extendable.

Another reason that it won’t work because you are really in a two dimensional space and you are really only using one sequence of labels. Of course, you could wrap the ordered set of labels so that they would still cover the parking lot but this does not help you to know how far to go when you want to move between labeled points and compounds the extendability problem. Thus there
are two problems. First, you need a distance. You can use the length that we discussed in Section 2.2.1. In the present case, this means that we define length from how far light travels in a given time. Secondly, what happens with the idea of extension. What happens when you add to the lot? You have to relabel everything. You can still cover the lot with labels but it is not convenient. By the way, this fact that you can cover a two dimensional space with a wrapped one dimensional label is also a simple proof of the size of the spaces are the same and thus that, although it might appear that a two dimensional infinite space seems bigger than an infinite one dimensional, there are as many points on the plane as there are on a line. Thus since you want to extend in a direction that is not along the direction of the chosen sequence, you can improve things quite a bit by having two designators at each place and ordering each of the sets of designators so that a place is a doublet, i.e. (Goofy, Donald Duck).

The next improvement would be to replace the cartoon characters with the ultimate ordered set – the real line. The real line has two obvious advantages: it is dense so that each point has a label, no matter where you are there is a label, and it has a distance measure on it that we all know and understand. If you are at the place labeled (3,1) and want to go to the place labeled (7,2) you only need to go four places in the first direction and one place in the other, if you are at (9,0) and want to go to (7,0), you go 2 places in the backwards direction. In other words, the sign of the number of places is not only valid arithmetically, it is a code for directions. If it is positive, you go one way and, if negative, you go the other way. In two dimensions or more dimensions there are other interesting ways to designate direction. But in one dimension the role of the sign is as a direction indicator.

In our parking lot, we need two measures of distance, one in each of the independent directions. If both directions are the same, we could generate a combined measure of distance, i.e. not require that all movement be along one of the coordinate directions. More than that if we assume that the space is the same in all directions at any point, isotropic, we can make a measure of distance that is independent of how we chose the directions of the coordinate system. In the case of the parking lot, if we assume that it is isotropic we can adopt for our distance measure \( \Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2} \) where \( \Delta x \) is the displacement in the one direction and \( \Delta y \) is the displacement in the other direction. This distance has the advantage of being independent of the orientation of the axis system. I have to warn you that if we were really worried about a parking lot, we would most likely not have an isotropic pattern of labels. Automobiles are longer than they are wide.

In addition, if at each place the distance algorithm is the same regardless of where you are the space is homogeneous. In general, it could be that, at different places, the distance between places is different. For example, in the above case, this designation of 4 places to go is assumed to be universal. It does not matter if you are at 7 and have 4 places to go or at 27 and have 4 places. In both cases, the distance 4 places is the same. This idea of the distance being the same at all places is also an important simplification. When you think about it
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though it may not be possible. The space may not be homogeneous. Each place may be special. Length may depend on where you are. In our considerations of General Relativity, Chapter 14, these issues will become important.

What is it that we want to get out of this rather extended discussion of the process for labeling a place. The most important thing is the realization that in contrast to what was our original ideas about labeling places, there is a great deal of choice. The choice, as is often the case, is arbitrary and cannot influence important issues. Later, when we discuss the General Theory of Relativity, Chapter 14, we will use this ambiguity as a part of the basis for understanding the theory. Suffice it to say, that we must develop a method for labeling places that must be consistent for all observers. It is the consistency requirement that allows us to derive the relationship between the different observers labeling of places and times.

Let us now go into the standard construction of the coordinate system. There are two general methods: the use of confederates at each place and the single observer method. We will start with the confederate method and then show its equivalence to the single observer method which is the one that we will subsequently use.

We begin by defining the spatial coordinates. We assume that we can fill space with a confederate at every place and that the distance between the origin observer and each of the confederates remains fixed. I must warn you about the intrinsic anthropomorphism of this action. Please be assured that the use of words like “confederate” and “observer” which is common to this business imply a humanity that is not really intended. In actually, by confederate or observer, we mean a measuring system – a clock and recording devices – not necessarily a person. It may appear that this assumption about our ability to fill space with fixed and uniform confederates must be true. In fact, one of the insights from general relativity is that this is the case only in the absence of gravity. Since they are fixed in space, we will label the confederate by how far away he/she is in each of the three coordinate directions. Obviously, if the space is homogeneous and isotropic, the location of the origin and the directions of the coordinate axis are arbitrary. For the definition of the distance, we will use the length defined earlier, Section ??, a defined speed of light and a time to label all distances. This speed will be universal for any observer establishing a coordinate system. This means that we need a standard clock and we choose the frequency of given emission line of a Cesium atom. In other words, our second is 9,192,631,770 oscillations of the light. To find the distance to any confederate, we send a light ray to that confederate who reflects it back and, with the standard clock, the observer at the origin can determine how far away that confederate is, \( d = \frac{c \Delta t}{2} \), where \( \Delta t \) is the time interval for the round trip of the light.

We have not discussed the problem of labeling the time. The situation is similar to the problem of labeling places. We need some ordered system at each place. What order do a series of events occur in? By endowing each confederate with a clock, we will have at each place a reference set of events to compare with the events being labeled. We use our standard clock. We tell each confederate
to make a standard clock. Since the space is assumed to be homogeneous, all the clocks must run at the same rate for each confederate. This is the first step in getting the time of an event that we want to label, to coordinatize. Since we have now endowed each confederate with a clock, we can use as the space and time label for any event as the time recorded on the nearest confederate’s clock and the location of the nearest confederate. You should realize that it is not enough to use the same clock at each place but we have to deal with the problem of synchronizing the several clocks; the confederates must synchronize their clocks – at some time agreeing on the time. It must also be consistent with our understanding that the speed of light is the same in all directions regardless of the velocity of the observer. Of course, this leads to the problem of the relativity of simultaneity and makes it important that we understand the process by which any observer synchronizes clocks. For now since we are dealing with only one frame, we do not need to worry about the relativity of simultaneity but it will cause some concern when we compare the coordinate systems constructed by two relatively moving observers. This is discussed in the next section, Section 11.2.1. For now, we can accomplish the synchronization by having a burst of light at some very early time released from the origin and, since we know the speed of light and that it is isotropic and we know the location of each confederate, we will know when it passes each confederate and they can set their clocks appropriately.

Let me summarize the confederate scheme for coordinatizing any event, see Figure 11.1. An observer establishes a lattice of confederates with identical synchronized clocks and the label of any event in space-time, for that observer, is the reading of the clock and the location of the nearest confederate to that event.

The a scheme that is equivalent to the confederate scheme can be accomplished with less elaboration by the simple mechanism of having a single clock at the spatial origin and requiring that observer to continuously send out light rays keeping track of the time of emission and in all directions that sample all space. At any event, the incoming light ray is reflected back to the observer. Therefore, the observer has thus two times and a direction that are associated with any event: the time the reflected ray left and the time of return of the reflected ray and the direction of the reflected light. To yield a spatial coordinatizing that is consistent with the confederate scheme, the spatial distance to the event is the difference in the two times times c divided by 2 or

\[ |\vec{x}| = \frac{c(t_2 - t_1)}{2} \] (11.1)

where \( t_2 \) is the later time and \( t_1 \) is the earlier time. The distance is resolved along the coordinate directions according to the direction of the incoming light ray. To be consistent with the time labeling of the confederate scheme, the time coordinate is

\[ t = \frac{t_2 + t_1}{2}. \] (11.2)
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The protocol for coordinatizing in one spatial dimension is shown in Figure 11.2.

11.2 The Lorentz Transformations

Now that we have developed a protocol for coordinatizing events, we need to find the transformation rules that one inertial observer must use to compare observations with another moving at relative velocity, \(v\). Actually, this is a special case of the more general problem of finding the transformation rules between any two coordinate systems. Since all inertial observers will see force-free motion as a straight line, you can convince yourself that the most general set of transformation rules is a set of linear transformations between the coordinates. Consider the particularly simple case of two coordinate systems that differ from each other only in the location of the origin. This is the case of two observers that have zero relative velocity, the same orientation of their coordinated axis and it is just the observer one says that observer two has her origin at the location \((x_2, y_2, z_2)\). An event measured at the coordinate, \((x, y, z, t)\) \(_1\), will have the label \((x - x_2, y - y_2, z - z_2, t)\) \(_2\) to observer two or

\[
\begin{align*}
x' &= x - x_2 \\
y' &= y - y_2 \\
z' &= z - z_2 \\
t' &= t
\end{align*}
\]
Figure 11.2: **General protocol for coordinatizing an event in one dimension.** The spatial coordinate of an event for an inertial observer with a clock is \( x = \frac{c(t_2 - t_1)}{2} \) and the time coordinate is \( t = \frac{t_2 + t_1}{2} \).

\[
\begin{align*}
    z' &= z - z_{20} \\
    t' &= t
\end{align*}
\]

This family of transformations has the general name of space translations labeled by the values \((x_{20}, y_{20}, z_{20})\) and they were discussed extensively, see Section ??.

Our process for finding the Lorentz transformations will be to develop a set of rules for the establishment of a coordinate system, see Section 11.1, and then to require that the same procedure be used in any inertial system. This process will lead to the fact that for two relatively moving systems, the same event will have two different coordinate designations. This should not come as a surprise since even prior to Einstein’s Theory of Special Relativity, the Galilean transformation, see Equation 9.1, gave different coordinate for an event when measured by two different inertial observers. In the case of the Special Theory, the rules connecting the different labels are called the Lorentz transformations. We will derive them in this section. The full family of transformations that include the rotations, translations, and velocity transformations is called the Poincare transformations.

To construct the Lorentz transformations, we will need to construct two independent inertial coordinate system. It should be clear that each inertial observer must have the same protocol for establishing their coordinate system, the same standard clock, and the same definition of the speed of light. For definiteness, we will assume that there is an event at which the two observers are at the same place and this event will be used as the origin of both coordinate systems. If this were not the case, a simple spatial coordinate translation, see
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Equation 11.3, will relocate the spatial origin. We set both observer’s clocks to \( t = 0 \) at this event. The two observers agree that their relative velocity, \( v \), is along the positive \( x \) axis of observer one. Since the orientation of the two sets of spatial coordinates is the same, the second observer will say that the first observer has relative velocity \( v \) along the negative \( x \) axis.

First, consider the nature of the agreements and disagreements about measurements that the two observers can have. Both observers are equivalent; neither is preferred. For instance, whatever of substance observer one says about observer two, two must also conclude about one. For instance, if one says that two’s standard clock runs the same as one’s, then two says that one’s clock runs the same. This is the case of Galilean transformations. The two observers would still be equivalent if one said that two’s standard clock ran slower if, at the same time, two also said that one’s standard clock ran slower. They both disagree in the same way. It would not work that one said that two’s clock ran slower and two agreed that his clock ran slower than one’s because then they would not be equivalent; one would have the faster clock. An analogy that I like to use is that in the class, all the students are equivalent even in John says that he is sane and the rest of the class is crazy if then Emily is also allowed to conclude that the rest of the class, including John, is crazy and she is sane.

Some coordinates are the same between the two relatively moving observers. Coordinates transverse to the direction of motion are the same. This can be argued this way. Consider two observers as shown in Figure 11.3. As stated earlier, the coordinate transformation between these must be linear so that \( z' = Bz \), where \( B \) is some function of the relative velocity. Now consider the configuration if the two observers had chosen instead a coordinate orientation that is obtained by a rotation about the \( z \) axis of \( \pi \) radians and invoking the principal that if one sees two moving along at \( v \) along the positive \( x \) axis then two sees one as moving along the negative \( x \) axis at speed \( v \). This reverses the roles of one and two and thus if the transformation was \( z' = Bz \) it is now \( z = Bz' \) which implies that \( B^2 = 1 \). We can dismiss the \( B = -1 \) solution so that we have \( z = z' \). A similar argument can be made for the other transverse direction, the \( y \) direction.

With the coordinates in the transverse directions the same, we can now show that the relatively moving observers will disagree about the rate at which the standard clock runs.

The Relatively Moving Clock

As discussed in Section 11.1, there is an atomic basis for the standard clock. Regardless, if we can make a system that repeats periodically this system will also be a clock. We will now use the agreement about the transverse lengths to construct a clock that proves that a moving clock must run slower than its identical cousin at rest. Since all observers will agree on the speed of light, we will use the speed of light and an agreed upon distance to make a clock.
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Figure 11.3: **Proof that the two observers agree on the transverse direction coordinates.** At the top of the figure are the coordinate frames for two observers moving relatively along the x axis. Below that are the same observers using frames rotated π radians about the z axis. In the lowest configuration, is the equivalent realization with the first observer moving to the left. This final configuration is the same as the original configuration with the roles of observers one and two reversed.

Figure 11.4: **A clock using light to time pulses.** Using the fact that the speed of light is the same to all inertial observers, we can use light as the basis for a clock. Setting two mirrors a distance, D apart, light bounces back and forth and the interval between passes is the unit of time. Since the light travels a longer distance, this same clock when observed by a relatively moving observer is seen to run slower.

We construct our clock by placing two mirrors a distance D apart and let a burst of light bounce between the two mirrors. The time that passes as the light travels from one mirror to the other and returns is the unit of time. Each observer constructs an identical clock; two mirrors set a distance D apart and
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held transverse to their relative motion so that they can agree that the mirrors are, in fact, the same distance apart. Consider Harry and Sally again. On her clock, Sally says that the interval between returns of the light is \( \Delta t_0 = \frac{2D}{c} \) but when she observes the operation of Harry’s clock, she says that the interval between ticks is longer since the light has to travel a greater distance. Said in another way, only the component of the velocity of the light perpendicular to the mirrors, \( v_\perp \) matters. Remember that the speed of light is the same in all directions and that both Sally and Harry have the same speed for light. Thus she says that his clock takes

\[
\Delta t = 2 \frac{D}{\sqrt{c^2 - v^2}} = 2 \frac{D}{c \sqrt{1 - \frac{v^2}{c^2}}} = \frac{\Delta t_0}{\sqrt{1 - \frac{v^2}{c^2}}}. 
\]

To Harry though, it is his clock that has a time interval of \( \Delta t_0 = \frac{2D}{c} \) and her clock that is running slow and has the interval \( \Delta t = \frac{2D}{c \sqrt{1 - \frac{v^2}{c^2}}} \). Remember that \( \sqrt{1 - \frac{v^2}{c^2}} \) is the same for \( v \) or \( -|v| \).

Look at this situation on space-time diagrams. First we draw the situation as represented by Sally. Here Sally’s time axis, her \( x = 0 \) line is vertical and Harry’s time axis is a line with slope \( \frac{1}{v} \). If she has a clock that reads at time \( t_0 \), she will record a time of \( \frac{tv_0}{\sqrt{1 - \frac{v^2}{c^2}}} \) for an identical clock carried by him when it reads \( t_0 \) to him. She will also record that the moving clock was located at \( v \) times that time, \( \frac{tv_0}{\sqrt{1 - \frac{v^2}{c^2}}} \), since the clock is traveling along Harry’s time axis.

![Figure 11.5: Operation of mirror clock in space time diagram.](image)

Sally’s time axis is vertical. Harry’s time axis has slope \( \frac{1}{v} \). If each observer carries an identical clock that to them ticks after a time \( t_0 \), the event of the tick on Sally’s clock has the coordinates \( (0, t_0) \) and, since the clocks are identical, the tick of Harry’s clock is labeled by Harry as \( (0, t_0)_h \). This same event though is labeled by Sally as \( (vt_v, t_v)_s \), where \( t_v = \frac{t_0}{\sqrt{1 - \frac{v^2}{c^2}}} \).

But a similar discussion is appropriate for Harry. He labels the event of that reading on his clock at \( (0, t_0)_h \). His coordinates for the event of the reading of
t_0 \text{ on her clock is at } (\frac{v t_0}{\sqrt{1-v^2/c^2}} , \frac{t_0}{\sqrt{1-v^2/c^2}}). \text{ Remember that, to him, Sally’s speed is } -|v|, \text{ a negative number, see Figure 11.6. The slope of her time axis in his space-time diagram is a negative number, } \frac{1}{v} = \frac{1}{-|v|}.

Figure 11.6: \textit{Operation of mirror clock in Harry’s frame.} The same pair of related events as in Figure 11.5 except as recorded by Harry

### 11.2.1 Derivation of the Lorentz Transformation

#### Coordinates of events

As stated in Section 11.1, Each observer is to send out a light ray that hits the event and one that returns. Record the times that the first ray is sent out and the time that the second ray comes back and the space coordinate and time coordinate are given by

\[
x = \frac{c(\tau_2 - \tau_1)}{2}, \\
t = \frac{\tau_1 + \tau_2}{2}.
\]  \tag{11.5}

This rule must be the same for all inertial observers.

When two relatively moving observers label an event, it is important to note though that all observers will use the same two light rays for any particular event, see Figure 11.7. In other words, any event is characterized uniquely by the two light rays that pass through it; all observers that are finding the labels of a particular event use the same transmitted and received rays. This apparent coincidence is actually a reflection of the fact that all observers agree on the speed of light and that the intersection of two light rays is an event and thus a unique label of an event.
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11.2.2 Details of the Derivation of the Lorentz Transformations

Now consider two observers, Sally and Harry, that share the same origin and want to coordinatize the same event. We have shown that the transverse coordinates must be the same for Harry and Sally, Figure 11.3, and, in fact, used this information to construct our clocks. Let us now show that this requirement is also obtained in the signaling method of coordinatizing.

In Figure 11.7, event 1 is coordinatized by Sally as \((x_s, t_s)\). By definition, Harry would label it \((x_h, t_h)\). The Lorentz transformations are the relationship between \((x_s, t_s)\) and \((x_h, t_h)\).

This is a rather tedious derivation, but a worthwhile exercise. Start by finding the coordinates of the events labeled \(\tau'_1\) and \(\tau'_2\) in terms of the coordinates of event 1 in Sally’s coordinates.

Event \(\tau'_1\) has the form \((vt_1, t_1)\) in Sally’s coordinates since it is on Harry’s worldline and he is moving at a speed \(v\) with respect to her. This event is also on a light ray with event 1. The equation of that light ray is

\[
x - x_s = c(t - t_s).
\]

Putting in the coordinates of the event \(\tau'_1\) which is on this line,

\[
vt_1 - x_s = c(t_1 - t_s).
\]

Solving for \(t_1\),

\[
t_1 = \frac{ct_s - x_s}{c - v}.
\]
Because of time dilation, see Section 11.2 and Figure 11.5,

\[ \tau_1' = t_1 \sqrt{1 - \frac{v^2}{c^2}}. \]  

(11.9)

Combining these:

\[ \tau_1' = \sqrt{1 - \frac{v^2}{c^2}} \frac{c t_s - x_s}{c - v} \]  

(11.10)

Similarly for event \( \tau_2' \)

\[ \tau_2' = t_2 \sqrt{1 - \frac{v^2}{c^2}} \]  

(11.11)

and

\[ \tau_2' = \sqrt{1 - \frac{v^2}{c^2}} \frac{c t_s + x_s}{c + v} \]  

(11.12)

Inserting this into the definitions, Figure 11.7 and Equation 11.5, and doing some straightforward algebra, we have

\[ x_h = \frac{x_s - vt_s}{\sqrt{1 - \frac{v^2}{c^2}}} \]

\[ t_h = \frac{t_s - \frac{v}{c^2} x_s}{\sqrt{1 - \frac{v^2}{c^2}}} \]  

(11.13)

which are the appropriate Lorentz transformations for this case.

### 11.3 Messing around with space-time diagrams

#### 11.3.1 Length contraction

In the transverse direction there is no ambiguity about length. In the direction of the motion we have to be careful. Sally holds a rod of length \( L_0 \). How long does Harry measure the rod to be?

In Harry’s coordinate system, the event at \( (L_0, \frac{L_0 v}{c^2})_s \) is written as \( (L', 0)_h \).

Using the invariant form this must be \( (L_0 \sqrt{1 - \frac{v^2}{c^2}}, 0)_h \) so that he says that the length is

\[ L' = L_0 \sqrt{1 - \frac{v^2}{c^2}} \]  

(11.14)
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11.3.2 The Doppler effect

We are all familiar with the classical Doppler effect. An approaching fire truck is racing to the chemistry building and the siren is at a high frequency. When the fire truck passes, the frequency of the siren drops.

Sally is moving by Harry at a speed \( v \). Sally sends a ray of light to Harry at time \( \tau_e \). This implies that \( x_e = vt_e \). This means \( \tau_e = \sqrt{t_e^2 - \frac{x^2}{c^2}} = \sqrt{1 - \frac{v^2}{c^2}}(t_e) \).

We can find the time of arrival, \( t_a \), of the light ray emitted by Sally to Harry. We use the equation of the light ray going through the emission event. The equation of this line is \( (x - x_e) = -c(t - t_e) \). Thus \( ct_a = ct_e + x_e = (c + v)t_e \) or \( t_a = \frac{(1+\frac{v}{c})}{\sqrt{1-\frac{v^2}{c^2}}} \tau_e = \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} \tau_e \) which is the relativistic doppler effect.

Problem: Convince yourself that this is the same Doppler effect that you see in the book.
11.3.3 Addition of velocities

It is very easy to get the formula for the addition of velocities using the Lorentz transformations. Consider Harry, Sally and Tom. Harry moves by Sally to increasing $x$ at $v_h$, where $v_h$ is Harry’s velocity as measured by Sally. Tom moves by Sally at $v_t$, where $v_t$ is Tom’s velocity as measured by Sally. How fast does Harry say that Tom is moving? The situation is shown Graphically in Figure 11.10.

Although we do not need it, note that $t'' = \sqrt{t'^2 - \frac{v^2 t^2}{c^2}} = \sqrt{1 - \frac{v^2}{c^2}} t$.

Also note that the $v_t$ and $v_h$ in the figure should be $v_{t,s}$ and $v_{h,s}$. The graphics package does not allow for stacked subscripts.

Drawing this in terms of Harry’s coordinates is

Similarly to above the $v_t$ in the figure should be $v_{t,h}$. Using the Lorentz transform for this event in Harry’s coordinates.

$$v_{t,h} t' = \frac{v_{t,s} t - v_{h,s} t}{\sqrt{1 - \frac{v^2}{c^2}}}$$
11.3. MESSING AROUND WITH SPACE-TIME DIAGRAMS

\[ t' = \frac{t - \frac{v_{hs}}{c^2} v_{ts} t}{\sqrt{1 - \frac{v_{hs}^2}{c^2}}} \]

Dividing these equations

\[ v_{th} = \frac{v_{ts} - v_{hs}}{1 - \frac{v_{hs}^2}{c^2}} \] (11.15)

11.3.4 Time over different trajectories

Sally and Dorothy are inertial and are at rest with respect to each other. They are separated by a distance of one light year. Harry is traveling at 3/5 of c toward Sally and Dorothy. He passes Sally and continues to Dorothy, turns around instantly and at the same speed goes back to Sally. How long is the trip from Sally and back to Sally according to Sally? According to Harry? How far apart are Sally and Dorothy according to Harry? When he is at Sally, how far away is Dorothy?

Sally says that Harry reached Dorothy in 5/3 years. So she says that the round trip takes 10/3 years. The event of Harry meeting Dorothy is (1,5/3) to Sally. Using the Lorentz transformations, this same event is labeled by Harry as (0,4/3). She says that she and Dorothy are 1 light year apart and he says that they are 4/5 of a light year apart. (He says that Dorothy is coming at him at (3/5)c and it takes 4/3 of a year for her to get there.) On a space time diagram, the event of his being at Sally and the event of where Dorothy is are different for Harry and Sally.

![Figure 11.12: The rules for coordinatizing and event on a space time diagram](image-url)

We see that we need a vocabulary to discuss the situation.
11.4 Events and World lines

An event is at a place and a time. For instance, when Harry is at Dorothy. Note that even though there is one event it has different coordinate descriptions depending on the observer. Harry says that when he is at Dorothy they are zero distance apart but that this occurred 4/3 years he left Sally. Sally says that when Harry is at Dorothy they are 1 light year away and it was 5/3 years after she and he were together.

A world line is a connected set of events that represent the places and times that an object moves through. Sally’s time axis is her world line. In the above example the bent line labeled Harry is his world line. Since Dorothy and Sally are inertial observers their world lines are straight. Until Harry met Dorothy, he was an inertial observer. In the next section, I will add one more criteria to the world line. It is not only continuous it is everywhere time like. Before we can discuss that we need the concept of the interval.

11.5 Intervals, Four vectors, and Invariants

An event is a place and a time. Given two events we can talk about the interval between them. This is like the “distance” in space time that two events are separated. It is the length of a vector in a 4 dimensional space.

\[ I^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - c^2(t_2 - t_1)^2 \]  

(11.16)

Note that, because of the minus sign in front of the time part, this “distance” can be positive or negative. For simplicity of notation and understanding, place the origin in space time on the first event. Then \( x_1 = 0, y_1 = 0, z_1 = 0, \) and \( t_1 = 0, \) and the interval is simply

\[ I^2 = x_2^2 + y_2^2 + z_2^2 - c^2t_2^2 \]  

(11.17)

One important feature of this definition is that it is a form invariant for Lorentz transformations. For two different observers you have the same interval but different values for the coordinates. This form, \((\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - (c\Delta t)^2\) is unchanged under the Lorentz transformations. Just like distance, \((\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2,\) for rotations in the usual space; distance is a form invariant for rotations. The "length squared" of the interval between two events is independent of the velocity of the Galilean observer that happens to be "coordinatizing" the events.

The other important feature is that the definition is for the interval squared. Yet because of the minus sign in front of the time term, the interval squared can be positive or negative.

In the Harry and Sally example above the interval of Harry meeting Dorothy is
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\[ \vec{I}_{bd} = \vec{x}_s + \frac{5}{3} \vec{t}_s = \frac{4}{3} \vec{t}_h \]  
(11.18)

Notice what at the moment appears to be a coincidence for both the point at which Harry meets Dorothy and where Dorothy is when Harry is at Sally.

\[ \left( \frac{5}{3} \right)^2 - 1^2 = \left( \frac{4}{3} \right)^2 - 0^2 \text{ and } 0^2 - \left( \frac{4}{5} \right)^2 = \left( \frac{3}{5} \right)^2 - 1^2 \]  
(11.19)

Take the Lorentz transformations and look at the form

\[(ct)^2 - x^2 - y^2 - z^2 = (ct')^2 - x'^2 - y'^2 - z'^2\]  
(11.20)

Problem: Plug in the Lorentz transformations and show that this is true.

The interval length squared (ILS) comes in two signs. Time-like separations in which the ILS is positive and space-like separations in which the ILS is negative. Note that the designation time-like or space-like is invariant— all inertial observers agree that two events are time-like related or space-like as the case may be. There are also light-like related events. Here the interval vanishes. The events can be connected with a light ray.

For any time-like separations there exists a Galilean observer for whom the separation is a pure time:

\[ \vec{I} = \tau \hat{t} \]  
(11.21)

where

\[ \tau \equiv \sqrt{(c(t_2 - t_1))^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2} \]  
(11.22)

This time is called the proper time and is the actual time.

11.5.1 Time Dilation

Using the invariants it is easy to derive the formula for time dilation. We had already treated the problem of time dilation in Section ?? but using invariants it is direct. If you are moving with a clock and it reads an interval of time \( \Delta t_0 \). If an identical clock that is moving relative to you at a speed \( v \), at that instant it will read \( \Delta t = \sqrt{1 - \frac{v^2}{c^2}} \Delta t_0 \). This is proven as follows:

Similarly for space-like separations, there exists a Galilean observer for whom the separation is purely space and

\[ \vec{I} = d\hat{x} \]  
(11.23)
Figure 11.13: Time dilation in a moving clock. Two observers with identical clocks are moving relative to each other at a speed v. At the time that the one observer notes the time $\Delta t_0$, that observer records the other clock as the event $(v\Delta t_0, \Delta t_0)$. The observer moving with that clock though records the event as $(0, \Delta t)_0$. Since the invariant form must take on the same value for all coordinatizations of the same event, $(\Delta t)^2 - (\frac{\Delta x}{c})^2 = (\Delta t_0)^2 - (\frac{\Delta x_0}{c})^2$ or

$$\Delta t = \sqrt{1 - \frac{v^2}{c^2}} \Delta t_0.$$ 

where

$$d \equiv \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 + (c(t_2 - t_1))^2}$$

(11.24)

This distance is the actual distance.

11.6 Future, Past and Elsewhere

See the book, "Travelers Guide". We are all experienced with the idea that at any given instant there is a future and a past to that time. We are also used to the idea that every one observing the situation would agree about which events were in the future and which were in the past. In order to accommodate these ideas, because of the relativity of simultaneity, Section 10.3, we will have to modify these notions slightly. In our new language, for every event, E, we can use the invariant to talk about relationship to other events. Remember, also, that an event is a place and a time.

Consider a certain event, E. For convenience, place the origin of the coordinate system for some observer at that event, i.e. for that event this observer designates it as $(0, 0)$. Pick any other event, $(x, t)$. This new event will have either a timelike or spacelike relationship to the original event depending on whether $(ct)^2 - x^2$ is greater or less than zero. The events that have the invariant greater than zero are timelike and there are two cases: $t > 0$ and $t < 0$, i.e. the second event is either before or after the origin event. All events that are
11.7. CURVED TRAJECTORIES IN SPACE TIME

Timelike and have $t > 0$ are said to be in the future of our origin event and all events that have $t < 0$ are said to be in the past of the origin event. Because we used the invariant to define the future and past this designation is the same for all inertial observers. In Figure 11.14, the events that are in the future of the origin event are in the upper or forward light cone and the events that are in the past of the origin event are in the lower or past light cone. Events that are in the past can have had a causal influence on the origin event. Events that are in the future can be influenced by the origin event.

Events that have $(ct)^2 - x^2 < 0$, are spacelike with respect to the origin event and are said to be in the elsewhere. For any event in this part of space time, different inertial observers will say that these events occurred before or after the origin event. Thus there is no agreement among inertial observers about on the temporal order of any of these events relative to the origin event. Events that are spacelike related cannot have any causal relationship to each other.

Figure 11.14: Future Past and Elsewhere. For any event, in this case the event at the vertex of the two cones, all the other events in space-time can be categorized into a future, a past, and an elsewhere. This separation of events is the same for all inertial observers.

11.7 Curved trajectories in space time

A connected set of events is a path in space time. These are often called trajectories. If the points are all time-like with respect to each other, it is a worldline; a possible trajectory for a material particle. Harry’s, Sally’s and Dorothy’s paths through space time are all worldlines. The set of events that are the
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Events simultaneous with \((0,0)\) for Harry is not a world line because the events on it are spacelike related. For an arbitrary time-like path in space time it is possible to rectify it and then add the times for each straight line segment. The sum of these times is the actual time the object experiencing that path goes through. Since the intervals are invariant, the elapsed time over any path is the same to all inertial observers. An important and interesting result is that the time between events depends on the path and in space-time the straight line is the longest path.

\[ \tau \equiv \sum_{i=1}^{n} \Delta \tau_i = \sum_{i=1}^{n} \sqrt{(\Delta t_i)^2 - \left(\frac{\Delta x_i}{c^2}\right)^2} = \sum_{i=1}^{n} \sqrt{1 - \left(\frac{v_i}{c^2}\right)^2} \Delta t_i \]  

(11.25)

This time is the actual time taken as the object moves over the path.

11.8 Paradoxes of Relativity

11.8.1 The Twin Paradox

Alphonse and Gaston are twins and they are authors. Alphonse writes advertising copy and has to travel to town every day and Gaston writes novels and stays home. Each day when Alphonse is on the train going to town he is observed by Gaston. Due to their relative motion, Gaston sees Alphonse’s clock running slower and thus Alphonse is aging slower than he does. At the end of the day, when Alphonse has returned home he has not aged as much as Gaston and is therefore younger. The problem is that, during the trip, Alphonse observes Gaston. He notes that Gaston’s clock is the one that runs slow. He expects that, when they get back together, Gaston will be younger. When they get back
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together are they the same age? If there is a difference in their ages, who is younger. The clue to the problem is that Alphonse spills a drink on his shirt every day.

11.8.2 The boy in the barn

A boy is a pole vault freak. He runs around a track all day to practice. He has to pass through a barn. In fact, the pole that he practices with is taken from the roof beam of the barn and is the same length as the barn when they are at rest together. He practices all day and his parents worry about him. They want to stop him and make him come in for dinner. He agrees that, if he and his pole are ever entirely in the barn, they can close the front and back doors. Since his pole is much longer than the barn, there is no problem. They will never get him. They agree to do as he says. What happens?

11.8.3 The Bandits and the Bullet Train