ADDITIVE AND MULTIPLICATIVE QUANTUM NUMBERS

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ADDITIVE AND MULTIPLICATIVE QUANTUM NUMBERS

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1. Introduction. In the current formalism of elementary particle physics, a single particle state is characterised by a set of quantum numbers some of them taking on only a discrete set of values (like charge, spin, etc.) while some others take on any of a set of continuous eigenvalues (like momentum, energy, etc.). In the passage from one particle states to many particle states, the various quantum numbers of the many particle states may be formed out of those of the one particle states generally by simple addition. For example the total charge or momentum is the sum of the corresponding quantities for one particle states.

However, there exists a distinct class of quantum numbers whose law of composition is multiplicative; and these are the quantum numbers associated with the discontinuous operations of the various symmetry groups admitted by the fields under consideration. The more important examples are the quantum numbers of parity (space reflection) and charge parity (charge conjugation). It is the purpose of this note to point out the general correspondence between additive and multiplicative quantum numbers and to investigate the physical implications of stating an empirical conservation law (like " strangeness " conservation) in a multiplicative or additive form.

2. Additive and Multiplicative Operators. We shall start with the elementary remark that to every operator $A$ with eigenvalues $a_i$ for which the law of composition (for many particle states) is additive, there corresponds the operator $U$:

$$ U = \exp (\lambda A) $$

with eigenvalues $u_i$

$$ u_i = \exp (\lambda a_i) $$

for which the law of composition is multiplicative. Since $U$ is a function of $A$; $U$ and $A$ can be made simultaneously diagonal. The
parameter $\lambda$ is completely arbitrary. If we now choose $\lambda$ to be pure imaginary and a submultiple of $(2\pi i)$, $U$ will be idempotent (i.e. $U^n = I$ for some finite $n$) and hence $U$ can be interpreted to be a "parity".

If we now consider, simultaneously, more than one quantum number, the preceding correspondence can be extended. Thus to a set of operators $\{A^{(i)}\}$ with eigenvalues $\{\alpha^{(i)}\}$ combining additively there corresponds a set of operators $\{U^{(i)}\} = \{\exp[\lambda, A^{(i)}]\}$ with eigenvalues $\{\alpha^{(i)}\} = \{\exp \lambda, \alpha^{(i)}\}$ combining multiplicatively. Here again the set of parameters $\{\lambda\}$ is completely arbitrary. We may now choose any one of these to be a submultiple of $(2\pi i)$ and interpret the corresponding multiplicative quantum number to be a "parity".

3. The Gödel Operator. This purely algebraic correspondence can be made use of in writing a single operator $G$ and quantum number $g$ in place of the (ordered) set of quantum numbers using Gödel numbers. For this purpose, label the operators (and corresponding eigenvalues) in some arbitrary, but definite, order; now choose $\lambda_i$ to be the logarithm of the $i$th prime number, so that

$$e^{\lambda_1} = 3, e^{\lambda_2} = 5, e^{\lambda_3} = 7, e^{\lambda_4} = 11, \text{ etc.}$$

Then the set of quantum numbers $\{\alpha^{(1)}, \alpha^{(2)}, \alpha^{(3)}, \ldots\}$ is equivalent to the single quantum number

$$g = 3\alpha^{(1)} 5\alpha^{(2)} 7\alpha^{(3)} \ldots$$

It is easily verified that this correspondence is unique; and that conservation of each of the quantum numbers $\alpha^{(1)}, \alpha^{(2)}, \ldots$ is equivalent to the conservation of the single Gödel number $g$. An equivalent Gödel number $g'$ can be got by replacing the exponents in the defining equation for $g$ by any linearly independent combinations of the $\alpha^{(i)}$. This corresponds to the well-known result that if $\alpha, \beta$ refer to two operators whose eigenvalues are conserved, $\lambda\alpha + \mu\beta$ also corresponds to an operator with conserved eigenvalues for all values of the numbers $\lambda, \mu$. Conservation of $g'$ is completely equivalent to the conservation of $g$. 
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Naturally the Gödel number for a many particle states is the

product of Gödel numbers of the one particle states. Conservation

of \( g \) takes care of all conservation laws corresponding to additive

quantum numbers. A single multiplicative quantum number \( \pi \) can

be included in this scheme by multiplying the Gödel operator \( G \)

by the multiplicative operator \( \pi \) to give \( G' \):

\[ G' = G \pi \rightarrow g' = g \pi. \]

Then conservation of \( G' \) is equivalent to all conservation laws

considered.

4. The Gödel Matrix. This transcription fails if one has more than

a single "parity" that is conserved in any interaction. In such a

case the scheme can be extended by replacing the Gödel number \( g \) by

a Gödel matrix. The matrix \( g \) would correspond to an operator

matrix \( g \). The matrix \( g \) is obtained by multiplying by \( g \) a matrix which is

the direct (Kronecker) product of a group of idempotent matrices.

Each matrix factor would correspond to the eigenvalue of a uni-

modular multiplicative ("parity") quantum number. This is simply

an extension of the Gödel scheme in which we include in addition to

prime number factors also idempotent matrices. Thus

\[ g = g P_1 \# P_2 \# P_3 \# ... \]

where \( P_1, P_2, P_3, ... \) are related to the conserved multiplicative

quantum numbers \( \pi_i \) by the mapping of powers of idempotent

matrices \( P \) (with \( P^n = I \)) on the \( n \)th roots of unity.

One can now pass from "strong" interactions (which conserve

all the quantum numbers) to weak interactions (which conserve only

some of them) by choosing a reduced Gödel number \( g' \) from \( g \) by

substituting unity (or the unit matrix I) for those factors of \( g \) corre-

sponding to non conserved quantities, and requiring the conservation

of only this reduced Gödel number \( g' \).

5. Physical Interpretation. We may now consider the physical

interpretation of the mathematical formalism. As mentioned earlier

physically interpretable multiplicative quantum numbers are those

corresponding to the disjoint operations of a continuous group; and
conservation of such a quantum number is equivalent to the con-
servation of the symmetry under this disjoint operation. One would
hence associate a multiplicative operator (i.e. one whose eigenvalue
for a many particle states is got by multiplication of the corresponding
single particle eigenvalues) with symmetry under a disjoint operation.
This in fact is the case. To demonstrate this, it is sufficient to recall
that if $A$ is the generator of infinitesimal displacements, the operator

$$U = e^{iA}$$

is the generator of a finite displacement $\lambda$. For example, $P_x$ generates
infinitesimal translations along the $x$ direction; then $e^{iP_x}$ generates
a finite displacement $\lambda$ along the $x$ direction. Hence conservation
of a Gödel number $g$ is equivalent to a lattice type symmetry of the
system considered under a series of finite displacements. The prime
factors correspond to the "lattice constants."

A similar interpretation can be given for the matrix factors of the
Gödel matrix $g$. These correspond to finite 2 dimensional rotations
and their conservation is equivalent to a "wheel" type symmetry;
the index of idempotency $n$ is the number of "spokes" of the wheel.
The connection between multiplicative unimodular operators and
rotational symmetries is thus evident.

Since a finite displacement can be built up of infinitesimal dis-
placements but not vice versa, the Gödel scheme can be made
less restrictive than the usual scheme which demands invariance
under infinitesimal displacements and/or rotations (and thus the
transition to a particular Gödel scheme would permit a "structure"
to the base space.) The usual requirement of invariance under
infinitesimal displacements would be equivalent to the conservation
requirement for all possible assignment of the Gödel numbers.

We could, as well, have assigned abstract elements of a multi-
plicative group by replacing each factor by the generating element
of a (distinct) group and consider the group element formed by the
direct product of these elements. The idempotent matrix factors
would then correspond to cyclic groups. Since all the operators are
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Simultaneously diagonal they commute and hence the corresponding abstract group should be Abelian (commutative). We shall however omit this transcription in view of the physical interpretation given above.

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