Dynamical Mappings of Density Operators in Quantum Mechanics*

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The most general dynamical law for a quantum mechanical system is studied with particular reference to the necessary and sufficient conditions for such a law to represent Hamiltonian dynamics. The main results are stated in the form of three theorems.

I. INTRODUCTION

THE most general description of the state of a quantum mechanical system is afforded by the von Neumann density operator, and may be defined as a "real" linear functional which maps non-negative Hermitian operators on a Hilbert space to non-negative numbers and maps the unit operator to unity. It is well known that, in special cases, one can specify the states in terms of normalized vectors of the Hilbert space. The dynamical law is then usually given as a unitary transformation on these vectors. We refer to this law as Hamiltonian dynamics. But the most general dynamical law for a quantum mechanical system is to be formulated in terms of the density operator and may be described by a linear mapping of the set of density operators into itself. The question immediately arises as to the conditions under which a linear mapping of the space of all operators into itself maps the subset of density operators into itself, and then as to the conditions under which such a mapping represents Hamiltonian dynamics. This problem has been investigated[1] for the restricted case of a system described by a finite dimensional vector space. In this paper we will answer these questions for operators defined on any Hilbert space.

Since the density operators form a convex set, the possible dynamical mappings also form a convex set. It may then appear that all dynamical mappings can be formed as "probabalistic" combinations of some simple set of extremal mappings in the same fashion as all density operators can be formed as mixtures of pure state operators. That this set of extremal mappings cannot be limited to the Hamiltonian mappings is evident from the existence of a mapping of all pure state operators to a single pure state operator. Hence, such a limitation would be an additional physical postulate. If we limit ourselves to mappings of pure states to pure states as described by a linear mapping on the Hilbert space then, such a postulate is implicitly assumed.

In Sec. II we develop some properties of the convex set of density operators and the linear space to which they belong and in Sec. III we prove three theorems which contain the main results outlined above.

II. THE OPERATOR SPACE AND THE CONVEX SUBSET OF DENSITY OPERATORS

The quantum mechanical state of a physical system can be specified by a density operator ρ which satisfies the conditions[2]:

$$(\phi,\rho\psi)=(\rho\phi,\psi) \qquad \text{(Hermiticity)}, \qquad (1)$$

$$(\phi,\rho\phi)\geq 0 \qquad \text{(positive-definiteness)}, \qquad (2)$$

$$\mathrm{Tr}(\rho)=1 \qquad \text{(normalization)}, \qquad (3)$$

where ϕ and ψ are any vectors of the Hilbert space $\mathcal{H}$ on which the operator ρ is defined. It is well known[3] that an operator which satisfies these conditions has a purely discrete spectrum with real non-negative eigenvalues. From this it follows that $\mathrm{Tr}(\rho^2)\leq 1$, the equality holding if and only if ρ has just one nonzero eigenvalue, or equivalently if and only if $\rho^2=\rho$ which is the condition that ρ be a projection operator. In the latter case we say that ρ represents a pure state. Any density operator can then be expanded in its spectral representation as

$$\rho=\sum_i a_i\rho^{(i)},$$

where $\rho^{(i)}$ are orthogonal projection operators and a_i are real positive coefficients satisfying $\sum_i a_i=1$.

If we consider the linear operators on $\mathcal{H}$ as themselves forming a vector space, we can define an inner product in this space by

$$(\rho_1,\rho_2)=\mathrm{Tr}(\rho_1{}^+\rho_2).$$

Let $\mathcal{L}$ be the linear space of all operators ρ for which $\|\rho\|^2=\mathrm{Tr}(\rho^+\rho)<\infty$. Then all density operators belong to $\mathcal{L}$. If $\rho=a\rho^{(1)}+(1-a)\rho^{(2)}$, where $\rho^{(1)}$ and $\rho^{(2)}$ are density operators and $0\leq a\leq 1$, then by an application of Schwartz's inequality we get that

$$(\rho,\rho)\leq a^2(\rho^{(1)},\rho^{(1)})+(1-a)^2(\rho^{(2)},\rho^{(2)}) +2a(1-a)[(\rho^{(1)},\rho^{(1)})(\rho^{(2)},\rho^{(2)})]^{\frac{1}{2}} \leq a^2+(1-a)^2+2a(1-a)=1,$$

where the equality holds only if $\rho^{(1)}=\rho^{(2)}$ and $(\rho^{(1)},\rho^{(1)})=1$. Hence, if ρ represents a pure state, it cannot be

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[1] E. C. G. Sudarshan, P. M. Mathews, and J. Rau, Phys. Rev. 121, 920 (1961). We refer the reader to this paper for a discussion of the physical motivation for the problem considered in the present paper and also for physical examples which illustrate the basic ideas and possible applications.

[2] J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1955), Chap. IV. For a general discussion of density operators see, e.g., U. Fano, Revs. Modern Phys. 29, 74 (1957).

[3] von Neumann, reference 2, p. 189.

formed as a linear combination with positive coefficients of any two distinct density operators. For any $\rho^{(1)}$ and $\rho^{(2)}$ as above, ρ is also a density operator, so the density operators form a convex set. We will call the set of all ρ formed as above for all values of a, $0<a<1$, the line segment between $\rho^{(1)}$ and $\rho^{(2)}$. We then define the set of extremal elements of the convex set as the set of all elements which do not belong to a line segment between any two distinct elements. From the above remarks, it is clear that the extremal elements of the set of all density operators is the set of operators which represent pure states, and that all other elements can be formed by positive combinations of these.

We will denote by $\psi\phi^+$ a linear operator defined on $\mathcal{H}$ by

$$(\phi^{(r)},\psi\phi^+\phi^{(s)})=(\phi^{(r)},\psi)(\phi,\phi^{(s)}),$$

where ψ and ϕ are vectors and the $\phi^{(r)}$ form an orthonormal basis in $\mathcal{H}$. Then the operators $\phi^{(r)}\phi^{(s)+}$ form an orthonormal basis in $\mathcal{L}$. For

$$\begin{aligned}(\phi^{(r)}\phi^{(s)+},\phi^{(r')}\phi^{(s')+})&=\mathrm{Tr}\{(\phi^{(r)}\phi^{(s)+})^+\phi^{(r')}\phi^{(s')+}\}\\&=\sum_{n,m}(\phi^{(m)},\phi^{(s)})(\phi^{(r)},\phi^{(n)})(\phi^{(n)},\phi^{(r')})(\phi^{(s')},\phi^{(m)})\\&=\delta_{rr'}\delta_{ss'}\end{aligned}$$

and any operator ρ in $\mathcal{L}$ can be expanded as

$$\rho=\sum_{rs}\rho_{rs}\phi^{(r)}\phi^{(s)+}$$

where

$$\rho_{rs}=(\phi^{(r)}\phi^{(s)+},\rho)=(\phi^{(r)},\rho\phi^{(s)})$$

and

$$\sum_{rs}|\rho_{rs}|^2=\mathrm{Tr}(\rho^+\rho)<\infty.$$

Since we are interested mainly in density operators, it is of interest that the pure state operators $\phi^{(r)}\phi^{(r)+}$ for all r and

$$\begin{aligned}&(1/\sqrt{2})(\phi^{(r)}+\phi^{(s)})(1/\sqrt{2})(\phi^{(r)}+\phi^{(s)})^+\\&\quad=\tfrac{1}{2}\phi^{(r)}\phi^{(r)+}+\tfrac{1}{2}\phi^{(s)}\phi^{(s)+}+\tfrac{1}{2}(\phi^{(r)}\phi^{(s)+}+\phi^{(s)}\phi^{(r)+})\\&(1/\sqrt{2})(\phi^{(r)}+i\phi^{(s)})(1/\sqrt{2})(\phi^{(r)}+i\phi^{(s)})^+\\&\quad=\tfrac{1}{2}\phi^{(r)}\phi^{(r)+}+\tfrac{1}{2}\phi^{(s)}\phi^{(s)+}+\tfrac{1}{2}i(\phi^{(s)}\phi^{(r)+}-\phi^{(r)}\phi^{(s)+})\end{aligned}\tag{4}$$

for all r, s, $r<s$, form a linearly independent set which spans $\mathcal{L}$.[4]

III. DYNAMICAL MAPPINGS

The most general dynamical transformation on the system is represented by a linear mapping of the set of density operators into itself. But since there are sets of density operators spanning $\mathcal{L}$, this uniquely defines a linear mapping of $\mathcal{L}$ into itself,[5]

$$\rho\rightarrow\rho'=A\rho.\tag{5}$$

We shall call a linear mapping on $\mathcal{L}$ which maps the set of density operators into itself a dynamical mapping. The properties of such a mapping are described by the following.

Theorem 1. Necessary and sufficient conditions for a linear operator A on $\mathcal{L}$ to give a dynamical mapping are: For any set of basis vectors $\phi^{(r)}$ in $\mathcal{H}$

(a) $(\phi^{(r)}\phi^{(s)+},A\phi^{(r')}\phi^{(s')+})=(\phi^{(s)}\phi^{(r)+},A\phi^{(s')}\phi^{(r')+})^*$,

(b) $\sum_r(\phi^{(r)}\phi^{(r)+},A\phi^{(r')}\phi^{(s')+})=\delta_{r's'}$,

(c) The operator ω defined on $\mathcal{H}$ by $(\phi^{(s)},\omega\phi^{(r')})=(\phi^{(r)}\phi^{(r)+},A\phi^{(r')}\phi^{(s')+})$ is positive definite for each choice of r. In particular, this implies that

$$(\phi^{(r)}\phi^{(r)+},A\phi^{(r')}\phi^{(r')+})\geq0.$$

When these conditions are satisfied, Hermitian operators are mapped to Hermitian operators, the trace is preserved, and positive definite operators are mapped to positive definite operators.

Proof. We write Eq. (5) as

$$(\phi^{(r)},\rho'\phi^{(s)})=\sum_{r's'}(\phi^{(r)}\phi^{(s)+},A\phi^{(r')}\phi^{(s')})(\phi^{(r')},\rho\phi^{(s')}).\tag{5a}$$

If A satisfies (a), we can deduce that ρ' is Hermitian when ρ is Hermitian. If A satisfies (b) we can deduce that $\mathrm{Tr}(\rho')=\mathrm{Tr}(\rho)$. If A satisfies (c), then $(\phi^{(r)},\rho'\phi^{(r)})=\sum_{r's'}(\phi^{(s')},\omega\phi^{(r')})(\phi^{(r')},\rho\phi^{(s')})=\mathrm{Tr}(\omega\rho)\geq0$ if ρ is positive definite. Since this holds for each $\phi^{(r)}$ belonging to any set of basis vectors in $\mathcal{H}$, we deduce that ρ' is positive definite if ρ is positive definite. The sufficiency of these conditions as well as the final statement of the theorem have thus been proved. The necessity of (a) is obtained by noting that if ρ is taken to be $\tfrac{1}{2}(\phi^{(n)}\phi^{(m)+}+\phi^{(m)}\phi^{(n)+})$ or $\tfrac{1}{2}i(\phi^{(n)}\phi^{(m)+}-\phi^{(m)}\phi^{(n)+})$, each of which is, according to (4), a real combination of pure state operators, then ρ' must be Hermitian, or from (5a),

$$\begin{aligned}&\tfrac{1}{2}\{(\phi^{(r)}\phi^{(s)+},A\phi^{(n)}\phi^{(m)+})+(\phi^{(r)}\phi^{(s)+},A\phi^{(m)}\phi^{(n)+})\}^*\\&\quad=\tfrac{1}{2}\{(\phi^{(s)}\phi^{(r)+},A\phi^{(n)}\phi^{(m)+})+(\phi^{(s)}\phi^{(r)+},A\phi^{(m)}\phi^{(n)+})\},\\&\tfrac{1}{2}\{(\phi^{(r)}\phi^{(s)+},A\phi^{(n)}\phi^{(m)+})+(\phi^{(r)}\phi^{(s)+},A\phi^{(m)}\phi^{(n)+})\}^*\\&\quad=-\tfrac{1}{2}\{(\phi^{(s)}\phi^{(r)+},A\phi^{(n)}\phi^{(m)+})-(\phi^{(s)}\phi^{(r)+},A\phi^{(m)}\phi^{(n)+})\},\end{aligned}$$

from which (a) follows. Similarly, using these same operators for ρ we see from (4) that we must have $\mathrm{Tr}(\rho')=0$ while if $\rho=\phi^{(n)}\phi^{(n)+}$ we must have $\mathrm{Tr}(\rho')=1$. Using (5a), these imply (b). To prove the necessity of (c) we note that for every pure state operator $\rho=\psi\psi^+$

[4] This space of operators with the inner product defined by the trace has been considered by J. Schwinger, Proc. Natl. Acad. Sci. U. S. 46, 257 (1960).

[5] To avoid confusion between the two types of operators we will use capital letters A for operators on $\mathcal{L}$ and Greek letters ρ, ω, σ for operators on $\mathcal{H}$ (elements of $\mathcal{L}$). Greek letters ϕ, ψ, χ, ξ will denote vectors in $\mathcal{H}$, while small letters a will denote scalars.

we must have

$$(\phi^{(r)},\rho'\phi^{(r)})=\sum_{r's'}(\phi^{(s')},\omega\phi^{(r')})(\phi^{(r')},\psi\psi^+\phi^{(s')})$$
$$=\sum_{r's'}(\phi^{(s')},\psi)^*(\phi^{(s')},\omega\phi^{(r')})(\phi^{(r')},\psi)$$
$$=(\psi,\omega\psi)\geq 0$$

for every $\phi^{(r)}$ belonging to any set of basic vectors in $\mathcal{H}$. This completes the proof of Theorem 1.

In order to characterize the Hamiltonian dynamical transformations we will need one more definition. If the dynamical mapping maps all pure state operators to pure state operators then for each normalized vector ϕ in $\mathcal{H}$ we have $\phi\phi^+=\rho\rightarrow\rho'=\phi'\phi'^+$. The mapping $a\phi\rightarrow a\phi'$ of $\mathcal{H}$ into $\mathcal{H}$ will be called the mapping induced on $\mathcal{H}$ by the dynamical mapping. Note that this induced mapping is not necessarily linear and is defined only to within a (unimodular) phase factor. When we can choose these phase factors so as to make the mapping linear, we will say that the dynamical mapping induces a linear mapping on $\mathcal{H}$.

Theorem 2. Equivalent necessary and sufficient conditions for a dynamical transformation to represent Hamiltonian dynamics are[6]:

(i) There exists a linear unitary operator ω on $\mathcal{H}$ such that $\rho\rightarrow\rho'=\omega\rho\omega^+$. That is the operator A has the form

$$(\phi^{(r)}\phi^{(s)+},A\phi^{(r')}\phi^{(s')+})=(\phi^{(r)},\omega\phi^{(r')})(\phi^{(s)},\omega\phi^{(s')})^*.$$

(ii) The dynamical mapping gives a mapping of the set of pure state operators into itself and induces a linear mapping on $\mathcal{H}$.

(iii) For each member $\chi^{(i)}$ of any set on basis vectors in $\mathcal{H}$, there exists a normalized vector $\xi^{(i)}$ such that the dynamical transformation maps

$$\chi^{(i)}\chi^{(j)+}\quad\text{to}\quad\xi^{(i)}\xi^{(j)+}.$$

(iv) The operator A of the dynamical mapping can be factored in the form

$$(\phi^{(r)}\phi^{(s)+},A\phi^{(r')}\phi^{(s')+})=(\phi^{(r)},\omega\phi^{(r')})(\phi^{(s)},\sigma\phi^{(s')})^*,$$

where ω and σ are linear operators on $\mathcal{H}$.

Proof. (i) represents the usual form of Hamiltonian dynamics; we need only prove that (i) implies (ii) implies (iii) implies (iv) implies (i).

That (i) implies (ii) is obvious. For $\phi\phi^+\rightarrow\phi'\phi'^+=\omega\phi\phi^+\omega^+=\omega\phi(\omega\phi)^+$ induces the linear mapping

$$a\phi\rightarrow a\phi'=a\omega\phi\quad\text{on}\quad\mathcal{H}.$$

To prove that (ii) implies (iii) we note that for any $\chi^{(i)}$ belonging to a set of basis vectors in $\mathcal{H}$, the pure state operator $\chi^{(i)}\chi^{(i)+}$ is mapped to a pure state operator, say $\xi^{(i)}\xi^{(i)+}$. Also, according to (ii), all the $\xi^{(i)}$ must be normalized vectors in $\mathcal{H}$. Since the induced mapping on $\mathcal{H}$ must be linear it can be determined by $\xi^{(i)}=\omega\chi^{(i)}$. Using Eq. (4) and the fact that $(1/\sqrt{2})\times(\chi^{(i)}+\chi^{(j)})$ and $(1/\sqrt{2})(\chi^{(i)}+i\chi^{(j)})$ are mapped to $(1/\sqrt{2})(\xi^{(i)}+\xi^{(j)})$ and $(1/\sqrt{2})(\xi^{(i)}+i\xi^{(j)})$, respectively, we see that $\frac{1}{2}(\chi^{(i)}\chi^{(j)+}+\chi^{(j)}\chi^{(i)+})$ and $\frac{1}{2}i(\chi^{(i)}\chi^{(j)+}-\chi^{(j)}\chi^{(i)+})$ are mapped to $\frac{1}{2}(\xi^{(i)}\xi^{(j)+}+\xi^{(j)}\xi^{(i)+})$ and $\frac{1}{2}i(\xi^{(i)}\xi^{(j)+}-\xi^{(j)}\xi^{(i)+})$ from which it follows that $\chi^{(i)}\chi^{(j)+}$ is mapped to $\xi^{(i)}\xi^{(j)+}$ which establishes (iii).

To obtain (iv) we use Eq. (5a) with (iii) to write

$$(\phi^{(r)},\xi^{(i)}\xi^{(j)+}\phi^{(s)})$$
$$=\sum_{r's'}(\phi^{(r)}\phi^{(s)+},A\phi^{(r')}\phi^{(s')+})(\phi^{(r')},\chi^{(i)}\chi^{(j)+}\phi^{(s')}).$$

Then,

$$\sum_{ij}(\phi^{(r)},\xi^{(i)})(\chi^{(i)},\phi^{(r'')})(\phi^{(s'')},\chi^{(j)})(\xi^{(j)},\phi^{(s)})$$
$$=\sum_{r's'}(\phi^{(r)}\phi^{(s)+},A\phi^{(r')}\phi^{(s')+})\sum_i(\phi^{(r')},\chi^{(i)})(\chi^{(i)},\phi^{(r'')})$$
$$\times\sum_j(\phi^{(s'')},\chi^{(j)})(\chi^{(j)},\phi^{(s')})$$
$$=\sum_{r's'}(\phi^{(r)}\phi^{(s)+},A\phi^{(r')}\phi^{(s')+})\delta_{r'r''}\delta_{s's''}$$

since both the $\phi^{(r)}$ and the $\chi^{(i)}$ were assumed to form sets of basis vectors in $\mathcal{H}$. Hence $(\phi^{(r)}\phi^{(s)+},A\phi^{(r')}\phi^{(s')+})=\sum_i(\phi^{(r)},\xi^{(i)}\chi^{(i)+}\phi^{(r')})\sum_j(\phi^{(s)},\xi^{(j)}\chi^{(j)+}\phi^{(s')})^*$ and setting $\omega=\sigma=\sum_i\xi^{(i)}\chi^{(i)+}$ gives (iv).

Finally to show that (iv) implies (i) we use condition (a) of Theorem 1 which in the factored form of (iii) is

$$(\phi^{(r)},\omega\phi^{(r')})(\phi^{(s)},\sigma\phi^{(s')})^*=(\phi^{(s)},\omega\phi^{(s')})^*(\phi^{(r)},\sigma\phi^{(r')}).$$

Then $\sigma=c\omega$ where c is a real number. For $(\phi^{(r)},\sigma\phi^{(r')})=0$ if and only if $(\phi^{(r)},\omega\phi^{(r')})=0$, and for all $(\phi^{(r)},\sigma\phi^{(r')})\neq 0$, $(\phi^{(s)},\sigma\phi^{(s')})\neq 0$ we have

$$\frac{(\phi^{(r)},\omega\phi^{(r')})}{(\phi^{(r)},\sigma\phi^{(r')})}=\frac{(\phi^{(s)},\omega\phi^{(s')})^*}{(\phi^{(s)},\sigma\phi^{(s')})^*}=\frac{1}{c}.$$

Then,

$$(\phi^{(r)}\phi^{(s)+},A\phi^{(r')}\phi^{(s')+})=(\phi^{(r)},c^{\frac{1}{2}}\omega\phi^{(r')})(\phi^{(s)},c^{\frac{1}{2}}\omega\phi^{(s)})^*.$$

Now if $\chi^{(i)}$ is any set of basis vectors in $\mathcal{H}$, let $\xi^{(i)}=c^{\frac{1}{2}}\omega\chi^{(i)}$. Then using condition (b) of Theorem 1 we have that

$$(\xi^{(i)},\xi^{(j)})=\sum_{rr's'}(\phi^{(r)},c^{\frac{1}{2}}\omega\phi^{(r')})(\phi^{(r)},c^{\frac{1}{2}}\omega\phi^{(s')})^*$$
$$\times(\phi^{(r')},\chi^{(j)})(\chi^{(i)},\phi^{(s')})$$
$$=\sum_{r's'}(\chi^{(i)},\phi^{(s')})\delta_{r's'}(\phi^{(r')},\chi^{(j)})=(\chi^{(i)},\chi^{(j)})=\delta_{ij}.$$

Then all the $\xi^{(i)}$ are non-null distinct vectors and form a basis in $\mathcal{H}$; $c^{\frac{1}{2}}\omega$ must be a unitary operator which gives (i) and completes the proof of Theorem 2.

[6] Another criteria for a Hamiltonian mapping, that the mapping preserve multiplication properties, has been given by J. Schwinger, Proc. Natl. Acad. Sci. U. S. 46, 570 (1960).

To show that the condition of the linear induced mapping in (ii) of the preceding theorem is actually needed we will include two examples (one a one-to-one mapping, the other not) of dynamical mappings which map the pure state operators to pure state operators, but are not Hamiltonian, in the proof of the following theorem which describes some of the possible mappings.

Theorem 3. The set of possible dynamical mappings, or the set of operators A giving these mappings, forms a convex set. The set of extremal elements of this set contains those which map all pure state operators to pure state operators. These include Hamiltonian mappings and (one-to-one and non-one-to-one) mappings which induce a nonlinear mapping on $\mathcal{H}$.

Proof. If two operators $A^{(1)}$ and $A^{(2)}$ each give a dynamical mapping of $\mathcal{L}$, then if $0\leq a\leq 1$, $A=aA^{(1)}+(1-a)A^{(2)}$ also gives a dynamical mapping. For if ρ is any density operator in $\mathcal{L}$, it must be mapped to a density operator $\rho^{(1)\prime}$ by $A^{(1)}$ and to a density operator $\rho^{(2)\prime}$ by $A^{(2)}$. But then A maps ρ to $\rho'=a\rho^{(1)\prime}+(1-a)\rho^{(2)\prime}$ which is also a density operator. Hence the set of possible dynamical mappings or the set of operators A giving these mappings forms a convex set.

If a mapping takes all pure state operators to pure state operators it cannot be on the line segment between two distinct mappings. For this would mean that at least one pure state operator would be on the line segment between two distinct density operators.

We have seen examples of Hamiltonian mappings. As an example of a dynamical mapping which gives a mapping of the set of all pure state operators one-to-one onto itself but induces a nonlinear mapping on $\mathcal{H}$,[7] we consider the following: Let $\phi^{(r)}\phi^{(s)+}$ be mapped to $\phi^{(s)}\phi^{(r)+}$ for all r, s. Consider any pure state operator $\psi\psi^{+}$, $\psi=\sum_r(\phi^{(r)},\psi)\phi^{(r)}$. Then

$$\psi\psi^{+}=\sum_{rs}(\phi^{(r)},\psi)(\psi,\phi^{(s)})\phi^{(r)}\phi^{(s)+}$$

is mapped to

$$\sum_{rs}(\phi^{(r)},\psi)(\psi,\phi^{(s)})\phi^{(s)}\phi^{(r)+}$$
$$=\sum_{rs}(\phi^{(s)},\psi)(\psi,\phi^{(r)})\phi^{(r)}\phi^{(s)+}$$
$$=\sum_{rs}(\phi^{(r)},\psi)^{*}(\psi,\phi^{(s)})^{*}\phi^{(r)}\phi^{(s)+}=\psi'\psi'^{+},$$

where $\psi'=\sum_r(\phi^{(r)},\psi)^{*}\phi^{(r)}$. All pure state operators are clearly mapped one-to-one to pure state operators so we do have a dynamical mapping, but the induced mapping $\psi\rightarrow\psi'$ is clearly not linear.

An example of a non-one-to-one mapping is the mapping of all pure state operators to a single pure state operator, $\phi^{(r)}\phi^{(r)+}\rightarrow\psi\psi^{+}$, $\phi^{(r)}\phi^{(s)+}\rightarrow 0$ for $r\neq s$. Note that this induces a nonlinear mapping on $\mathcal{H}$. For, according to Eq. (4), $(1/\sqrt{2})(\phi^{(r)}+\phi^{(s)})\rightarrow\psi\neq\sqrt{2}\psi$. These examples complete the proof of Theorem 3.

We have seen that the set of all dynamical mappings is larger than the convex subset having the Hamiltonian mappings as its boundary. To limit ourselves to this latter subset would require an additional physical postulate. The non-one-to-one mappings of pure state operators to pure state operators could be thought of as describing a kind of measurement process but this does not exhaust the non-Hamiltonian mappings of pure state operators to pure state operators. It is interesting to note that if one describes the mappings of pure states to pure states in terms of linear mappings of the vectors in $\mathcal{H}$ the postulate limiting these to Hamiltonian mappings is implicitly contained in the linearity. From the density operator point of view this cannot readily be interpreted as resulting from the kinematical structure of the theory.

As a final note we observe that the operator A for a dynamical mapping can have the form

$$(\phi^{(r)}\phi^{(s)+},A\phi^{(r')}\phi^{(s')+})=(\phi^{(r)},\omega\phi^{(s)})(\phi^{(r')},\sigma\phi^{(s')})$$

only if $(\phi^{(r')},\sigma\phi^{(s')})=\delta_{r's'}$ and ω is a density operator. For condition (b) of Theorem 1 requires that $\sum_r(\phi^{(r)},\omega\phi^{(r)})(\phi^{(r')},\sigma\phi^{(s')})=\delta_{r's'}$ which implies that $(\phi^{(r')},\sigma\phi^{(s')})=\delta_{r's'}$ and $\mathrm{Tr}(\omega)=1$. Conditions (a) and (c) then require that ω be Hermitian and positive definite.

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[7] Such nonlinear mappings are used for representing antilinear discrete operations in quantum mechanics. The most familiar example is time inversion; see E. P. Wigner, Gottinger Nachr., 546, (1932); J. Math. Phys. 1, 409 (1960); R. G. Sachs, *Nuclear Theory* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1953), Appendix; another example is charge conjugation in one-particle theories, see e.g., L. L. Foldy, Phys. Rev. 102, 568 (1956).