The formulation of field theories by means of Wightman functions is studied. It is shown that, given two field theories that satisfy all the axioms, one can construct a family of Wightman fields with the same properties by a process of superposition of Wightman functions. The condition of unitarity is formulated without reference to asymptotic conditions, and it is proved that the Wightman fields constructed by the above superposition process (starting with "unitary" fields) fail to preserve unitarity, and a fortiori, the standard asymptotic condition.

1. INTRODUCTION

In the search for a dynamical scheme for describing elementary particle phenomena consistent with relativistic invariance and quantum mechanical principles, the theory of quantized fields has been favored with more study and has provided more insight than any other scheme. The use of manifestly covariant local Lagrangians as a starting point and the use of perturbation expansions lead to questionable mathematical operations with infinite quantities. In view of this, during the last few years the study of general field theories without starting with any specific Lagrangian has received much attention. The more fundamental part of such a program concerns the study of an abstract axiom system more or less suggested by earlier Lagrangian theories. In such a study it is worthwhile to know if the axioms are independent and whether they are compatible; while the axioms are "related" to general physical requirements their truth is neither "self-evident" nor can one trust intuitive "physical" justifications for the compatibility of these axioms.

Among the set of axioms usually taken as characterizing quantized fields, these comments apply particularly to the so-called "asymptotic condition" which enables one to relate the field operators to particle scattering amplitudes. The somewhat provisional nature of this axiom has been noted before; and perhaps not unconnected with this is the fact that the other "field axioms" have been the subject of a structure analysis by Wightman. Making use of the tools developed in this brilliant study we show in this paper that the "asymptotic condition" is an independent axiom and that one can construct systems satisfying all other axioms but not this axiom provided that at least one quantum field theory yielding a nontrivial scattering matrix exists. In the course of this study we have been able to construct several examples of fields with a trivial scattering matrix.

In Sec. 2 we review Wightman's theory and construct certain elementary families of Wightman fields using the technique of vacuum expectation values. Section 3 discusses the weak axiom of asymptotic particle interpretation and the normalization of the field. The main result of the present paper is to show that almost all members of the families of fields constructed in Sec. 2 do not satisfy the (weak) axiom of asymptotic particle interpretation; this result is stated and proved in Sec. 4. Certain related comments are made in the concluding section.

2. FAMILIES OF WIGHTMAN FIELDS

According to Wightman, a quantum field theory is defined in terms of a Hilbert space $\mathcal{H}$ and a set of hermitian linear operators (more specifically, operator-valued distributions) $\phi(x)$ labeled by a four-vector $x$ provided the following conditions are satisfied:

(I) Manifest Lorentz invariance. There must exist unitary operators $U(a,\Delta)$ such that

$$\phi(\Delta x + a) = U(a,\Delta)\phi(x)U^{-1}(a,\Delta)$$

for every proper orthochronous inhomogeneous Lorentz transformation.

(II) Absence of negative energy states. The spectrum of the Hamiltonian operator must be nonnegative, the Hamiltonian being defined as the hermitian generator of time translations.

(III) Local commutativity. The commutator of two field operators at space-like points must vanish,

$$[\phi(x),\phi(y)] = 0 \quad \text{for} \quad (x-y)^2 < 0.$$  

(IV) The existence of the "vacuum" state. There exists a unique state $|0\rangle$ invariant under all $U(a,\Delta)$.

We now form the vacuum expectation values of products of $n$ field operators labeled by the points $x_1, x_2, \ldots, x_n$:

$$W^{(n)}(x_1, x_2, \ldots, x_n) = \langle x_1 \rangle \cdots \langle x_n \rangle = \langle 0 | \phi(x_1) \cdots \phi(x_n) | 0 \rangle. \quad (1)$$

It can then be shown that, as a consequence of the conditions imposed on the Hilbert space $\mathcal{H}$ and the
function $W^{(n)}(\{x\})$ analytic for $\text{Im}\{z\}$ in the backward light cone (absence of negative energies).

(iii) $W^{(n)}(\{x\}) = W^{(n)}(\{x'\})$, where $\{x'\}$ is any permutation of the $n$ variables $\{x\}$, provided the permuted variables have space-like separations (local commutativity).

(iv) $\sum_{r,s} \int \int f^*_r(x_1, \ldots, x_r) W^{(r+s)}(x_1, \ldots, x_r, y_1, \ldots, y_s) \times f_s(y_1, \ldots, y_s) d^4x_1 \cdots d^4x_r d^4y_1 \cdots d^4y_s \geq 0$, (2)

where $f_r$ are suitable arbitrary functions (positive definite metric). Wightman has also shown\textsuperscript{3} that these conditions are sufficient, that is, given a set of functions $W^{(n)}(\{x\})$ satisfying these conditions, one can construct a theory of a (neutral scalar) field satisfying the four conditions stated at the beginning of this section which has these functions for its vacuum expectation values.

Before the field theory so defined can be used to describe a model of relativistic quantum theory of particles, one must introduce some particle concepts. The structure satisfying only the conditions introduced in this section is a more general system; we shall refer to this structure as a "Wightman field."

We now state two obvious properties of a Wightman field in terms of its Wightman functions in the form of two theorems.

**Theorem I (scale change).** If $W^{(n)}(\{x\})$ are a set of Wightman functions, the set of functions $k^n W^{(n)}(\{x\})$ defines a Wightman field for every real number $k$.

This statement is immediately verified by noting that if $\phi(x)$ is the Wightman field which corresponds to $W^{(n)}(\{x\})$, then $k\phi(x)$ corresponds to $k^n W^{(n)}(\{x\})$.

**Theorem II (convexity).** If $W_1^{(n)}(\{x\})$ and $W_2^{(n)}(\{x\})$ are two sets of Wightman functions, the convex set

$$W^{(n)}(\{x\}) = \lambda W_1^{(n)}(\{x\}) + (1-\lambda) W_2^{(n)}(\{x\})$$

(3)

defines a Wightman field provided the real number $\lambda$ lies between 0 and 1.

The theorem is proved by noting that the functions $W^{(n)}(\{x\})$ satisfy all the conditions imposed on Wightman functions: Lorentz invariance, analyticity in the future tube, permutation symmetry for space-like separated arguments, and finally the condition specified by Eq. (2). Hence they define a Wightman field. Note that, in this case, it is not easy to construct the field operator in a simple manner but these functions satisfy all the conditions imposed on Wightman functions; hence they define a Wightman field. If $\lambda$ is real but not

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3. ASYMPTOTIC PARTICLE INTERPRETATION AND THE SCATTERING MATRIX

If this field theory is to become a theory of interacting particles, one must introduce particle variables into the theory and identify at least some subspace of the Hilbert space \( \mathcal{H} \) as being associated with the particle states. Such a program has so far not been carried out except for free fields. There is however another type of particle interpretation which is less ambitious in the sense that certain linear combinations of vacuum expectation values of the fields are identified with a scattering amplitude for "asymptotically free" particles. Since there are certain properties to be satisfied by the scattering amplitude this identification in turn imposes some restrictions on the Wightman fields. However the scattering amplitudes themselves provide only an incomplete characterization of the field; and it appears that without the use of sufficiently strong additional postulates, the scattering amplitudes do not determine the Wightman field. In support of this, it is known that one can construct several distinct Wightman fields with a trivial associated scattering amplitude.

It is conventional to state the requirement of an asymptotic particle interpretation in terms of an appropriately stated "asymptotic condition" and then "derive" the scattering amplitude in terms of certain linear combinations of vacuum expectation values. We shall follow the alternative method of stating the connection between the scattering amplitude and the vacuum expectation values as the additional axiom. This apparenty arbitrary procedure has certain advantages: first of all, unlike the other axioms of quantum field theory, the asymptotic condition has so far been stated only in unsatisfactory forms and their plausibility is not immediately obvious. The best defense seems to be that it leads to a covariant expression for the scattering amplitude; but the expression itself could be obtained by other means, say for example, by a formal summation of the perturbation series.

Secondly the question of completeness of the particle scattering states which is generally a prerequisite to the axiomatization of the asymptotic condition seems too strong; it is conceivable that the field Hilbert space does not have a complete set of states associated with it. However, the condition imposes some restrictions on the Wightman functions. Hence, the \( T \) product vacuum expectation value is defined in terms of the Wightman functions by the equations

\[
\langle 0 | T \{ \phi(x_1), \ldots, \phi(x_n) \} | 0 \rangle = W^{(n)}(x_1, \ldots, x_n) \quad (5a)
\]

for \( x_1 \to x_2 \to \cdots \to x_n \),

\[
\langle 0 | T \{ \phi(x_1), \ldots, \phi(x_n) \} | 0 \rangle = \langle 0 | T \{ \phi(x_1'), \ldots, \phi(x_n') \} | 0 \rangle, \quad (5b)
\]

where \( x_1', \ldots, x_n' \) are any permutations of \( x_1, \ldots, x_n \).

(Asymptotic particle interpretation.)

At this point, we must restrict our further discussion to Wightman fields for which the \( T \)-product vacuum expectation values exist. Given any Wightman field we can now calculate the particle scattering matrix in terms of this identification; but there is no guarantee that the scattering matrix so defined satisfies the conditions imposed on a scattering matrix, in particular unitarity. It is considered further necessary that the one-particle states are "steady" so that the \( S \)-matrix elements connecting one-particle states to any other state vanish identically (and that the two-particle scattering is elastic below the three-particle threshold). This condition can be used to normalize the field operator:

\[
\int d^4x \Lambda(p,x) \int d^4y \Lambda(-q, y) \langle 0 | T \{ \phi(x), \phi(y) \} | 0 \rangle = (2\pi)^4 \delta(p-q) \delta(p^2-\mu^2) \quad (6)
\]

with \( p^2 = q^2 - \mu^2 \). It then follows that if \( W^{(n)}(\{x\}) \) denotes the Wightman functions for this normalized field of mass \( \mu \) then \( k^n W^{(n)}(\{x\}) \) defines a field which is not normalized except for the special case \( k = \pm 1 \). The

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2. This point of view is somewhat more general than the classification of particle interpretations discussed by Wightman and Schweber (reference 4).
4. See, for example, Y. Nambu, Phys. Rev. 98, 803 (1955).
introduced here is weaker than the usual asymptotic condition in the sense that we do not assume either the completeness of the many particle states nor the existence of asymptotic fields. But if the asymptotic condition is postulated as an axiom of the theory in addition to the axioms for a Wightman field, we can derive the expression for the particle scattering matrix yielding the so-called reduction formulas. Thus the axiom of asymptotic particle interpretation for a Wightman field yields a more general system than the Wightman field with the stronger axiom of asymptotic condition. Needless to say everything we have proved in the following sections apply a fortiori to fields satisfying the usual system of axioms including the asymptotic condition. We now proceed to show that Wightman fields in general do not have an asymptotic particle interpretation.

4. WIGHTMAN FIELDS WITHOUT ASYMPTOTIC PARTICLE INTERPRETATION

In terms of the scattering matrix $S$ one may define the scattering amplitude $f$ in the standard manner; and then note that the scattering amplitude so defined is linearly related to the Wightman functions. The unitarity relation imposed on $f(p_1, \ldots, p_r; q_1, \ldots, q_s)$ is

$$f(p_1, \ldots, p_r; q_1, \ldots, q_s) - f^*(q_1, \ldots, q_s; p_1, \ldots, p_r)$$

$$= i \sum \int d^4k_1 \delta(k_1^2 - \mu^2) \theta(k_1^0) \int d^4k_n \delta(k_n^2 - \mu^2) \times \theta(k_n^0) f(p_1, \ldots, p_r; k_1, \ldots, k_n) \times f^*(q_1, \ldots, q_s; k_1, \ldots, k_n)$$

or symbolically,

$$(f - f^+) = i_ff^+.$$  \hspace{1cm} (7')

In the summation, most of the terms contribute nothing since energy and momentum must be conserved if the scattering amplitude is not to vanish. Let $f_1$ and $f_2$ be the scattering amplitudes for two Wightman fields with asymptotic particle interpretation defined by their Wightman functions $W_1(n)$ and $W_2(n)$. We shall further specialize than to correspond to the same "mass." If we now define a field in terms of the Wightman functions

$$W(n) = \lambda W_1(n) + (1 - \lambda) W_2(n)$$

in view of the linear relation between the Wightman function and the scattering amplitude, it follows that the scattering amplitude $f$ for this Wightman field

$$\lambda f_1 + (1 - \lambda) f_2$$

which may be written

$$\lambda(1 - \lambda) \sum \int d^4k_1 \delta(k_1^2 - \mu^2) \theta(k_1^0) \int d^4k_n \delta(k_n^2 - \mu^2) \times \theta(k_n^0) g(p_1, \ldots, p_r; q_1, \ldots, q_s; k_1, \ldots, k_n) = 0,$$

with

$$g = f_1 - f_2.$$  \hspace{1cm} (8)

If we now specialize to the case of elastic scattering, the integrand is nonnegative and the vanishing of integral implies that either $g = 0$ identically or $\lambda(1 - \lambda) = 0$. In the first case the two Wightman fields must have the same scattering matrix and all the Wightman fields in the allowed family $\lambda_1 \leq \lambda \leq \lambda_2$ yield the same scattering matrix; the second case is trivial. We now prove the following theorem.

Theorem IV (equivalent scattering matrices). Wightman field defined in terms of the Wightman functions

$$W(n) = \sum \lambda_\alpha W_\alpha(n), \sum \lambda_\alpha = 1, \lambda_\alpha \geq 0,$$

the functions $W_\alpha(n)$ admitting asymptotic particle interpretations with the same "mass," has an asymptotic particle interpretation if and only if all the Wightman fields have the same scattering matrix.

This more general statement is proved essentially in the same way as used above; one derives in place of the equation

$$\sum \lambda_\alpha \lambda_\beta (f_\alpha^+ - f_\beta^+) (f_\alpha - f_\beta) = 0,$$

from which it follows that $f_\alpha = f_\beta$ unless $\lambda_\alpha$ or $\lambda_\beta$ vanishes provided all the $\lambda_\alpha$ are nonnegative. Note that, unlike the case of two fields only, here the condition $\lambda_\alpha \geq 0$ cannot be simply relaxed; in general, grounds of continuity, one expects the domain of validity of $\lambda_\alpha$ (with sum unity) for which the theorem holds is somewhat larger in view of the demonstration above regarding only two fields.

5. DISCUSSION

The results of the preceding section imply that the axiom of asymptotic particle interpretation is independent of the other axioms of field theory and is derivable from them; a conclusion already indicated by the existence of several distinct fields with the same $S$ matrix. We have actually used only a weaker axi
\[ \langle 0 | T\{\phi(x_1), \ldots, \phi(x_n)\} | 0 \rangle \]

of the Wightman functions for momenta on the mass shell; without additional restrictions this is not sufficient to determine the field in any sense. Yet here we see that the unitarity requirement on the particle scattering matrix excludes most Wightman fields from having an asymptotic particle interpretation.

Perhaps the weakest point of the present investigation is that it has not provided any example of a field theory with asymptotic particle interpretation with a nontrivial scattering matrix; rather it asserts that if there exists at least one such theory there exists an infinity of Wightman fields not having an asymptotic particle interpretation belonging to the family generated by this one field together with the free field of the same mass.

We have worked here within the framework of the conventional axiomatization of quantum field theory. The purpose of the field theory is only to provide a quantum theory of interacting particles invariant under the complex Lorentz group, the conventional axiomatization is too rigid in that it imposes "physical requirements" on the field. This is most easily seen in the case of the axiom of positive definiteness: in a theory where the physical particle states do not form a complete set of states in the generalized Hilbert space in which the field operators are defined, it is sufficient if the particle states constitute a subspace with positive definite metric. That these considerations are not devoid of physical interest is seen from the example of the quantized Maxwell field. One of the present authors has discussed\(^{10}\) examples of quantum field theories formulated in terms of a generalized Hilbert space with an indefinite metric where again the physical particle states are not complete in the generalized space but constitute only a subspace with positive definite metric. In such theories the physical interpretation requires an


\[ W(n) = \sum_\alpha \lambda_\alpha k_\alpha^n W_\alpha^{(n)}, \quad \sum_\alpha \lambda_\alpha = 1, \quad \lambda_\alpha \geq 0, \quad (9) \]

which provide Wightman fields, the functions \( W_\alpha^{(n)} \) corresponding to known theories; say either free fields with arbitrary masses, or the Wick polynomials of free fields or terminating Haag expansions.\(^{8}\) By a limiting procedure in forming such linear combinations one can produce any two-point function

\[ \langle 0 | \phi(x)\phi(y) | 0 \rangle = \int d\rho(m^2) \Delta^{(1)}(m; x-y). \quad (10) \]

(where \( \Delta^{(1)}(m; x-y) \) is the two-point Wightman function for a free field of mass \( m \)) by taking for the Wightman functions

\[ W^{(n)}(\{x\}) = \int d\rho(m^2) W^{(n)}(m; \{x\}), \quad (11) \]

where \( W^{(n)}(m; \{x\}) \) are the Wightman functions for a free field of mass \( m \), and \( \rho(m^2) \) is a nonnegative measure. But all these fields have a trivial scattering matrix.

Finally the present study illustrates the validity of Wightman's statement\(^3\) that the consequences of positive definiteness are distinct from the consequences of unitarity. The Wightman fields constructed above satisfy positive definiteness but do not yield unitary scattering matrices, while certain indefinite metric theories (including quantum electrodynamics)\(^9\) provide examples of theories in which the field operators are defined in a generalized Hilbert space but the scattering matrices are unitary.

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