The formulation of field theories by means of Wightman functions is studied. It is shown that, given two field theories that satisfy all the axioms, one can construct a family of Wightman fields with the same properties by a process of superposition of Wightman functions. The condition of unitarity is formulated without reference to asymptotic conditions, and it is proved that the Wightman fields constructed by the above superposition process (starting with "unitary" fields) fail to preserve unitarity, and *a fortiori*, the standard asymptotic condition.

1. INTRODUCTION

In the search for a dynamical scheme for describing elementary particle phenomena consistent with relativistic invariance and quantum mechanical principles, the theory of quantized fields has been favored with more study and has provided more insight than any other scheme. The use of manifestly covariant local Lagrangians as a starting point and the use of perturbation expansions lead to questionable mathematical operations with infinite quantities. In view of this, during the last few years the study of general field theories without starting with any specific Lagrangian has received much attention.[1] The more fundamental part of such a program concerns the study of an abstract axiom system more or less suggested by earlier Lagrangian theories. In such a study it is worthwhile to know if the axioms are independent and whether they are compatible; while the axioms are "related" to general physical requirements their truth is neither "self-evident" nor can one trust intuitive "physical" justifications for the compatibility of these axioms.

Among the set of axioms usually taken as characterizing quantized fields, these comments apply particularly to the so-called "asymptotic condition"[2] which enables one to relate the field operators to particle scattering amplitudes. The somewhat provisional nature of this axiom has been noted before; and perhaps not unconnected with this is the fact that the other "field axioms" have been the subject of a structure analysis by Wightman.[3] Making use of the tools developed in this brilliant study we show in this paper that the "asymptotic condition" is an independent axiom and that one can construct systems satisfying all other axioms but not this axiom provided that at least one quantum field theory yielding a nontrivial scattering matrix exists. In the course of this study we have been able to construct several examples of fields with a trivial scattering matrix.

In Sec. 2 we review Wightman's theory and construct certain elementary families of Wightman fields using the technique of vacuum expectation values. Section 3 discusses the weak axiom of asymptotic particle interpretation and the normalization of the field. The main result of the present paper is to show that almost all members of the families of fields constructed in Sec. 2 do not satisfy the (weak) axiom of asymptotic particle interpretation; this result is stated and proved in Sec. 4. Certain related comments are made in the concluding section.

2. FAMILIES OF WIGHTMAN FIELDS

According to Wightman,[3] a quantum field theory is defined in terms of a Hilbert space $\mathcal{H}$ and a set of hermitian linear operators (more specifically, operator-valued distributions) $\phi(x)$ labeled by a four-vector x provided the following conditions are satisfied:

(I) Manifest Lorentz invariance. There must exist unitary operators $U(a,\Lambda)$ such that

$$\phi(\Lambda x+a)=U(a,\Lambda)\phi(x)U^{-1}(a,\Lambda)$$

for every proper orthochronous inhomogeneous Lorentz transformation.

(II) Absence of negative energy states. The spectrum of the Hamiltonian operator must be nonnegative, the Hamiltonian being defined as the hermitian generator of time translations.

(III) Local commutativity. The commutator of two field operators at space-like points must vanish,

$$[\phi(x),\phi(y)]=0 \quad \text{for} \quad (x-y)^2<0.$$

(IV) The existence of the "vacuum" state. There exists a unique state $|0\rangle$ invariant under all $U(a,\Lambda)$.

We now form the vacuum expectation values of products of n field operators labeled by the points $x_1, x_2, \cdots, x_n$:

$$W^{(n)}(x_1,x_2,\cdots,x_n)\equiv W^{(n)}(\{x\}) =\langle 0|\phi(x_1)\cdots\phi(x_n)|0\rangle. \quad (1)$$

It can then be shown that, as a consequence of the conditions imposed on the Hilbert space $\mathcal{H}$ and the

* Supported by the U. S. Atomic Energy Commission.

[1] See, for example, the Proceedings of the "Colloque sur les Problèmes Mathématiques de la Théorie Quantique des Champs" (Lille, 1957); see also, "Problemi Matematici della Teoria Quantistica delle Particelle e dei Campi" Suppl. Nuovo cimento 14 (1959) and references given there.

[2] R. Haag: Dan. Mat. Fys. Medd. 29, No. 1 (1955); H. Lehmann, K. Symanzik, and W. Zimmerman, Nuovo cimento 1, 205 (1955); O. W. Greenberg, Ph.D. thesis, Princeton University 1956 (unpublished).

[3] A. S. Wightman, Phys. Rev. 101, 860 (1956).

function $W^{(n)}(\{z\})$ analytic for $\mathrm{Im}\{z\}$ in the backward light cone (absence of negative energies).

(iii) $W^{(n)}(\{x\})=W^{(n)}(\{x'\})$, where $\{x'\}$ is any permutation of the n variables $\{x\}$, provided the permuted variables have space-like separations (local commutativity).

(iv)

$$\sum_{r,s}\int \int f_r^*(x_1,\cdots,x_r)W^{(r+s)}(x_1,\cdots,x_r,y_1,\cdots y_s)$$
$$\times f_s(y_1,\cdots,y_s)d^4x_1\cdots d^4x_rd^4y_1\cdots d^4y_s\geqslant 0, \quad (2)$$

where f_r are suitable arbitrary functions (positive definite metric). Wightman has also shown[3] that these conditions are sufficient, that is, given a set of functions $W^{(n)}(\{x\})$ satisfying these conditions, one can construct a theory of a (neutral scalar) field satisfying the four conditions stated at the beginning of this section which has these functions for its vacuum expectation values.

Before the field theory so defined can be used to describe a model of relativistic quantum theory of particles, one must introduce some particle concepts. The structure satisfying only the conditions introduced in this section is a more general system; we shall refer to this structure as a "Wightman field."

We now state two obvious properties of a Wightman field in terms of its Wightman functions in the form of two theorems.

Theorem I (*scale change*). If $W^{(n)}(\{x\})$ are a set of Wightman functions, the set of functions $k^nW^{(n)}(\{x\})$ defines a Wightman field for every real number k.

This statement is immediately verified by noting that if $\phi(x)$ is the Wightman field which corresponds to $W^{(n)}(\{x\})$, then $k\phi(x)$ corresponds to $k^nW^{(n)}(\{x\})$.

Theorem II (*convexity*). If $W_1^{(n)}(\{x\})$ and $W_2^{(n)}(\{x\})$ are two sets of Wightman functions, the convex set

$$W^{(n)}(\{x\})=\lambda W_1^{(n)}(\{x\})+(1-\lambda)W_2^{(n)}(\{x\}) \quad (3)$$

defines a Wightman field provided the real number λ lies between 0 and 1.

The theorem is proved by noting that the functions $W^{(n)}(\{x\})$ satisfy all the conditions imposed on Wightman functions: Lorentz invariance, analyticity in the future tube, permutation symmetry for space-like separated arguments, and finally the condition specified by Eq. (2). Hence they define a Wightman field. Note that, in this case, it is not easy to construct the field operator in a simple manner but these functions satisfy all the conditions imposed on Wightman functions; hence they define a Wightman field. If λ is real but not

Theorem I; however, out of this infinite set, a spec choice can be made by stating a normalization conditi We shall state such a condition in the next secti Theorem II allows us to construct an infinite set Wightman fields (normalized, if so required) from t (or more) distinct Wightman fields. Let us call the of all Wightman fields $W^{(n)}(\{x\})$ generated $W_1^{(n)}(\{x\})$ and $W_2^{(n)}(\{x\})$ the "family"; every po in this family is labeled by a parameter λ. We ha remarked above that while $0\leqslant\lambda\leqslant 1$ is allowed in cases, values of λ outside this interval are not necessar forbidden. It is then interesting to state the followi theorem regarding the boundedness of the allow values of λ:

Theorem III (*semibounded families*). There exi either a lower limit λ_1 or an upper limit λ_2 (or bo such that for either $\lambda<\lambda_1$ or $\lambda_2<\lambda$ (or both) combinations

$$V^{(n)}(\{x\})=\lambda W_1^{(n)}(\{x\})+(1-\lambda)W_2^{(n)}(\{x\})$$

cannot be a set of Wightman functions.

To prove the existence of such limits, we use positive definiteness condition showing that these violated for sufficiently large negative or positive valu of λ. Consider in particular $W_1^{(2)}(\{x\})$, which is no negative according to (2). It cannot be everywhere ze without making the field operator $\phi_1(x)$ trivial. Choo any suitable testing function $f(y)$ such that

$$\int f^*(y)W_1^{(2)}(x+\tfrac{1}{2}y,\, x-\tfrac{1}{2}y)f(y)d^4y=1,$$

and let

$$\int f^*(y)W_2^{(2)}(x+\tfrac{1}{2}y,\, x-\tfrac{1}{2}y)f(y)d^4y=a\geqslant 0.$$

Then,

$$\int f^*(y)V^{(2)}(x+\tfrac{1}{2}y,\, x-\tfrac{1}{2}y)f(y)d^4y=a+\lambda(1-a),$$

which becomes negative $\lambda<-a/|1-a|$ or for $a/|a-$ $>\lambda$ according as a is less than or greater than unit Hence, the statement made in the theorem is proved

This demonstration however does not guarant that provided $\lambda_1<\lambda<\lambda_2$ the set $V^{(n)}(\{x\})$ are Wightm functions since the positive definiteness condition its complete form may still be violated; it may even violated for other testing functions using $W^{(2)}(\{x$ only. However, from Theorem II we know that the exists the nontrivial family $0\leqslant\lambda\leqslant 1$ at least. In gener the family is, of course, larger.

3. ASYMPTOTIC PARTICLE INTERPRETATION AND THE SCATTERING MATRIX

If this *field* theory is to become a theory of interacting *articles*, one must introduce particle variables into the heory and identify at least some subspace of the Iilbert space $\mathcal{H}$ as being associated with the particle tates. Such a program[4] has so far not been carried out xcept for free fields. There is however another type of article interpretation which is less ambitious in the ense that certain linear combinations of vacuum xpectation values of the fields are identified with a cattering amplitude for "asymptotically free" par-cles.[5] Since there are certain properties to be satisfied y the scattering amplitude *this identification* in turn *nposes some restrictions on the Wightman* fields. How-ver the scattering amplitudes themselves provide only n incomplete characterization of the field; and it ppears that without the use of sufficiently strong dditional postulates, the scattering amplitudes do *not* etermine the Wightman field. In support of this, it known that one can construct several distinct Vightman fields with a trivial associated scattering mplitude.[6]

It is conventional[2] to state the requirement of an symptotic particle interpretation in terms of an ppropriately stated "asymptotic condition" and then "derive" the scattering amplitude in terms of certain near combinations of vacuum expectation values. Ve shall follow the alternative method of stating the onnection between the scattering amplitude and the acuum expectation values as the additional axiom. his apparently arbitrary procedure has certain dvantages: first of all, unlike the other axioms of uantum field theory, the asymptotic condition has so r been stated only in unsatisfactory forms and their ausibility is not immediately obvious. The best efense seems to be that it leads to a covariant expres-on for the scattering amplitude; but the expression self could be obtained by other means, say for example, a formal summation of the perturbation series.[7] econdly the question of completeness of the particle attering states which is generally a prerequisite to e axiomatization of the asymptotic condition seems o strong; it is conceivable that the field Hilbert

[4] A. S. Wightman and S. S. Schweber, Phys. Rev. **98**, 812 (1955).

[5] This point of view is somewhat more general than the classi-ation of particle interpretations discussed by Wightman and hweber (reference 4).

[6] H. J. Borchers, Nuovo cimento **15**, 784 (1960).

[7] See, for example, Y. Nambu, Phys. Rev. **98**, 803 (1955).

$p_1, \cdots, p_r$ from a state containing s articles wi four-momenta $q_1, \cdots, q_s$ (with $p_1^2= \cdots =q_s^2=\mu^2$) given by the expression

$$S(p_1,\cdots,p_r;q_1,\cdots,q_s)$$

$$=\int d^4x_1\cdots d^4x_r d^4y_1 \cdots d^4y_s$$

$$\times\Lambda(p_1,x_1)\cdots\Lambda(p_r,x_r)\Lambda(-q_1,y_1)\cdots\Lambda(-q_s,y_s)$$

$$\times\langle 0|T[\phi(x_1),\cdots,\phi(y_s)]|0\rangle,$$

where

$$\Lambda(p,x)=\frac{-i}{(2\pi)^4}e^{ipx}(\Box_x{}^2-\mu^2)$$

and μ is a "mass" parameter. Hence, the T product vacuum expectation value is defined in terms of the Wightman functions by the equations

$$\langle 0|T\{\phi(x_1),\cdots,\phi(x_n)\}|0\rangle=W^{(n)}(x_1,\cdots x_n) \quad \text{(5a)}$$

for $x_1{}^0>x_2{}^0>\cdots>x_n{}^0$,

$$\langle 0|T\{\phi(x_1),\cdots,\phi(x_n)\}|0\rangle$$
$$=\langle 0|T\{\phi(x_1'),\cdots,\phi(x_n')\}|0\rangle, \quad \text{(5b)}$$

where $x_1', \cdots, x_n'$ are any permutations of $x_1, \cdots, x_n$. (Asymptotic particle interpretation.)

At this point, we must restrict our further discussion to Wightman fields for which the T-product vacuum expectation values exist. Given any Wightman field we can now calculate the particle scattering matrix in terms of this identification; but there is no guarantee that the scattering matrix so defined satisfies the conditions imposed on a scattering matrix, in particular unitarity. It is considered further necessary that the one-particle states are "steady" so that the S-matrix elements connecting one-particle states to any other state vanish identically (and that the two-particle scattering is elastic below the three-particle threshold). This condition can be used to normalize the field operator:

$$\int d^4x\Lambda(p,x)\int d^4y\Lambda(-q,y)\langle 0|T\{\phi(x),\phi(y)\}|0\rangle$$
$$=(2\pi)^4\delta(p-q)\delta(p^2-\mu^2) \quad (6)$$

with $p^2=q^2\rightarrow\mu^2$. It then follows that if $W^{(n)}(\{x\})$ denotes the Wightman functions for this normalized field of mass μ then $k^nW^{(n)}(\{x\})$ defines a field which is not normalized except for the special case $k=\pm 1$. The

[8] This choice is very closely related to the work of K. Nishijima, Phys. Rev. **119**, 485 (1960).

introduced here is weaker than the usual asymptotic condition in the sense that we do not assume either the completeness of the many particle states nor the existence of asymptotic fields. But if the asymptotic condition is postulated as an axiom of the theory in addition to the axioms for a Wightman field, we can *derive* the expression for the particle scattering matrix yielding the so-called reduction formulas.[9] Thus the axiom of asymptotic particle interpretation for a Wightman field yields a more general system than the Wightman field with the stronger axiom of asymptotic condition. Needless to say everything we have proved in the following sections apply *a fortiori* to fields satisfying the usual system of axioms including the asymptotic condition. We now proceed to show that Wightman fields in general do *not* have an asymptotic particle interpretation.

4. WIGHTMAN FIELDS WITHOUT ASYMPTOTIC PARTICLE INTERPRETATION

In terms of the scattering matrix S one may define the scattering amplitude f in the standard manner; and then note that the scattering amplitude so defined is *linearly* related to the Wightman functions. The unitarity relation imposed on $f(p_1,\cdots,p_r;q_1,\cdots,q_s)$ is

$$f(p_1,\cdots,p_r;q_1,\cdots,q_s)-f^*(q_1,\cdots,q_s;p_1,\cdots,p_r)$$
$$=i\sum_{n=0}^{\infty}\int d^4k_1\delta(k_1^2-\mu^2)\theta(k_1^0)\cdots\int d^4k_n\delta(k_n^2-\mu^2)$$
$$\times\theta(k_n^0)f(p_1,\cdots,p_r;k_1,\cdots,k_n)$$
$$\times f^*(q_1,\cdots,q_s;k_1,\cdots,k_n)\quad(7)$$

or symbolically,

$$(f-f^+)=iff^+.\quad(7')$$

In the summation, most of the terms contribute nothing since energy and momentum must be conserved if the scattering amplitude is not to vanish. Let f_1 and f_2 be the scattering amplitudes for two Wightman fields with asymptotic particle interpretation defined by their Wightman functions $W_1^{(n)}$ and $W_2^{(n)}$. We shall further specialize than to correspond to the same "mass." If we now define a field in terms of the Wightman functions

$$W^{(n)}=\lambda W_1^{(n)}+(1-\lambda)W_2^{(n)}$$

in view of the linear relation between the Wightman function and the scattering amplitude, it follows that the scattering amplitude f for this Wightman field

[9] H. Lehmann, K. Symanzik, and W. Zimmerman, Nuovo cimento 1, 205 (1955).

$$\{\lambda f_1+(1-\lambda)f_2\}\{\lambda f_1^++(1-\lambda)f_2^+\}$$
$$=\lambda f_1f_1^++(1-\lambda)f_2f_2^+,$$

which may be written

$$\lambda(1-\lambda)\sum_{n=0}^{\infty}\int d^4k_1\delta(k_1^2-\mu^2)\theta(k_1^0)\cdots\int d^4k_n\delta(k_n^2-$$
$$\times\theta(k_n^0)g(p_1,\cdots,p_r;k_1,\cdots,k_n)$$
$$\times g^*(q_1,\cdots q_s;k_1,\cdots,k_n)=0,$$

with

$$g=f_1-f_2.\quad(8$$

If we now specialize to the case of elastic scattering, t integrand is nonnegative and the vanishing of integral implies that either $g=0$ identically or $\lambda(1-$ $=0$. In the first case the two Wightman fields m have the same scattering matrix and all the Wightm fields in the allowed family $\lambda_1\leqslant\lambda\leqslant\lambda_2$ yield the *sa* scattering matrix; the second case is trivial. We m now prove the following theorem.

Theorem IV (*equivalent scattering matrices*). Wightman field defined in terms of the Wightm functions

$$W^{(n)}=\sum\lambda_\alpha W_\alpha^{(n)},\quad\sum\lambda_\alpha=1,\quad\lambda_\alpha\geqslant0,$$

the functions $W_\alpha^{(n)}$ admitting asymptotic part interpretations with the same "mass," has an asyr totic particle interpretation if and only if all the Wig man fields have the same scattering matrix.

This more general statement is proved essentially same way as used above; one derives in place of the equation

$$\sum_{\alpha>\beta}\lambda_\alpha\lambda_\beta(f_\alpha^+-f_\beta^+)(f_\alpha-f_\beta)=0,$$

from which it follows that $f_\alpha=f_\beta$ unless λ_α or vanishes provided all the λ_α are nonnegative. N that, unlike the case of two fields only, here the co tion $\lambda_\alpha\geqslant0$ cannot be simply relaxed; in general, grounds of continuity, one expects the domain values of λ_α (with sum unity) for which the theo holds is somewhat larger in view of the demonstra above regarding only two fields.

5. DISCUSSION

The results of the preceding section imply that axiom of asymptotic particle interpretation is indepe ent of the other axioms of field theory and is derivable from them; a conclusion already indica by the existence of several distinct fields with the s S matrix. We have actually used only a weaker ax

$$\langle 0|T\{\phi(x_1),\cdots,\phi(x_n)\}|0\rangle$$

f the Wightman functions for momenta on the mass hell; without additional restrictions this is not sufficient o determine the field in any sense. Yet here we see hat the unitarity requirement on the particle scattering natrix excludes most Wightman fields from having an symptotic particle interpretation.

Perhaps the weakest point of the present investigation s that it has not provided any example of a field theory vith asymptotic particle interpretation with a nontrivial cattering matrix; rather it asserts that if there exists at east one such theory there exists an infinity of Wightnan fields not having an asymptotic particle interpretaion belonging to the family generated by this one field ogether with the free field of the same mass.

We have worked here within the framework of the onventional axiomatization of quantum field theory. f the purpose of the field theory is only to provide a uantum theory of interacting particles invariant under e complex Lorentz group, the conventional axiomatation is too rigid in that it imposes "physical requireents" on the field. This is most easily seen in the case f the axiom of positive definiteness: in a theory where e physical particle states do not form a complete set f states in the generalized Hilbert space in which the eld operators are defined, it is sufficient if the particle ates constitute a subspace with positive definite etric. That these considerations are not devoid of ysical interest is seen from the example of the uantized Maxwell field. One of the present authors has scussed[10] examples of quantum field theories forulated in terms of a generalized Hilbert space with an definite metric where again the physical particle ates are not complete in the generalized space but nstitute only a subspace with positive definite metric. n such theories the physical interpretation requires an

[10] E. C. G. Sudarshan, Phys. Rev. 123, 2183 (1961).

also have more generally

$$W^{(n)}=\sum_\alpha \lambda_\alpha k_\alpha{}^n W_\alpha{}^{(n)}, \quad \sum_\alpha \lambda_\alpha=1, \quad \lambda_\alpha\geqslant 0, \tag{9}$$

which provide Wightman fields, the functions $W_\alpha{}^{(n)}$ corresponding to known theories; say either free fields with arbitrary masses, or the Wick polynomials of free fields or terminating Haag expansions.[6] By a limiting procedure in forming such linear combinations one can produce any two-point function

$$\langle 0|\phi(x)\phi(y)|0\rangle=\int d\rho(m^2)\Delta^{(1)}(m;x-y) \tag{10}$$

(where $\Delta^{(1)}(m;x-y)$ is the two-point Wightman function for a free field of mass m) by taking for the Wightman functions

$$W^{(n)}(\{x\})=\int d\rho(m^2)W^{(n)}(m;\{x\}), \tag{11}$$

where $W^{(n)}(m;\{x\})$ are the Wightman functions for a free field of mass m, and $\rho(m^2)$ is a nonnegative measure. But all these fields have a trivial scattering matrix.

Finally the present study illustrates the validity of Wightman's statement[3] that the consequences of positive definiteness are distinct from the consequences of unitarity. The Wightman fields constructed above satisfy positive definiteness but do not yield unitary scattering matrices, while certain indefinite metric theories (including quantum electrodynamics)[10] provide examples of theories in which the field operators are defined in a generalized Hilbert space but the scattering matrices are unitary.

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