STATISTICAL DYNAMICS OF MULTIPLE MESON PRODUCTION

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I. THE STATISTICAL POSTULATE

Consider the collision of two particles with four-momenta p_1, p_2. The final state is in general a superposition of many states with different numbers of particles and with different momentum configurations. If $k_1 \ldots k_n$ is a configuration of four-momenta for the final state, the transition amplitude is given by

$$S_{fi} = \langle f|S|i \rangle = \langle p_1, p_2, \alpha|S|k_1, \ldots, k_n, \beta \rangle$$

$$= u_1^+ u_2^+ v_1 \ldots v_n \, \tilde{T}_{\alpha\beta}(p_1, p_2; k_1, \ldots, k_n)$$

where u, v are the "wave functions" of the respective particles and $T_{\alpha\beta}(p;k)$ is a function of the four-momenta, α, β being the additional labels (spin, isotopic spin, etc.) needed to specify the states completely. To simplify our notation let us consider the particles to be spin zero distinguishable particles with no internal symmetries. Then according to the usual conventions of relativistic quantum mechanics:

$$u_i = \frac{1}{\sqrt{2p_i^o}} \qquad v_i = \frac{1}{\sqrt{2k_i^o}} \, .$$

Energy momentum conservation requires that $T(p;k)$ contain the factor $\delta(k_1 + \ldots + k_n - p_1 - p_2)$ and invariance under homogeneous Lorentz transformations requires it be an invariant function. Using a familiar result from tensor analysis it follows that $T(p;k)$ defined by $\tilde{T}(p,k) = \delta(k-p)T(p,k)$ is a function of only the scalar products $p_i p_j$, $p_i k_j$, $k_i k_j$ not all of which are independent. The transition rate to a set of states with momenta k_i lying in a suitable interval $K_{(n)}$ is proportional to the covariant integral:

$$\int_{K(n)} |T(p,k)|^2 \, d^4k_1 \Theta(k_1^o) \delta(k_1^2 - m_1^2) \ldots \delta(k_1 + \ldots + k_n - p_1 - p_2)$$

where m_i is the mass of the ith particle. Note that we get an absolute "rate" only after adopting an appropriate convention for the incident "flu The statistical postulate of "equal opportunity" may now be invoked except for one minor difficulty: the various "rates" $R(K_{(n)})$ must all have the same dimension and this requires that $|T(p,k)|^2$ must have dimensions depen dent on the number of particles in the final state; more specifically, it must have the dimensions of the $(2n-4)^{th}$ power of a mass (we take $\hbar = c = 1$ so that $(\text{length})^{-1}$ = mass = momentum = energy). To be able to formulate the statistical postulate we thus need a "scaling parameter". Two natura choices suggest themselves: the first one is obtained from a comparison with covariant perturbation theory, and the second by comparison with the usual formula for statistical weights.

Let us consider an arbitrary Feynman diagram contributing to the meson production process involving n particles in the final state; and we shall consider only interactions with dimensionless coupling constants (th so-called renormalizable interactions or interactions of the first kind). Then there is a corresponding Feynman diagram for the production of $n+1$ mesons obtained by taking any nucleon propagator and attaching an external meson line to it. This modification introduces into the matrix element ar extra factor of the form $S_F \cdot g(2\pi)^{-3/2} \cdot u_{n+1}$ where g is the coupling consta and S_F is a fermion propagator. The scaling parameter κ^2 in $|T(p,k)|^2$ in this case is simply

$$\kappa^2 = 2\pi^2\{(g^2/4\pi)|S_F|^2\}^{-1}$$

and has the dimensions of the square of a mass, but it is also proportional to the coupling strength $(g^2/4\pi)$. Of course, this weak coupling approximation is hardly justified in the case of the multiple production process, but it does give the interpretation of the parameter κ^2 as the probability for an additional propagator (which is proportional to the square of the propagator and to the coupling strength). If g is taken to be the <u>renormalized</u> pion-nucleon coupling constant $\frac{1}{2\pi^2}(g^2/4\pi)$ is of the order of unity and the parameter κ then corresponds to the average degree of deviation of the virtual nucleon from its mass shell for the dominant matrix elements. It is then to be expected that κ must be of the order of several pion masses This is, in fact, the case. The statistical postulate now reads[1,2]

$$|T(p,k)|^2 \, \kappa^{4-2n} = \text{constant}.$$

An alternative choice of the scaling parameter is obtained by considering the transition rate in the form

$$R(K_{(n)}) = \text{const.} \int_{K_{(n)}} |\langle p|S|k\rangle|^2 \, d^3k$$

and considering the final states to form a discrete set corresponding to the particles in the final state normalized in a volume Ω so that

$$\frac{1}{(2\pi)^3} \int \ldots \, d^3k \to \frac{1}{\Omega} \Sigma \ldots$$

and the scaling parameter is now the volume $\Omega/(2\pi)^3$. The statistical postulate now may be stated as

$$|\langle p|S|k\rangle|^2 \, [\Omega/(2\pi)^3]^n = \text{constant} .$$

Comparing the two choices of the scaling parameters we find the correspondence

$$\kappa^2(2k^0)| \longleftrightarrow (2\pi)^3/\Omega$$

where k^0 is the typical energy of a particle in the final state; the correspondence is thus energy dependent. The choice of one or the other version seems a matter of taste at this point; Fermi originally chose the second version with Ω to be the volume of a "Lorentz contracted" sphere of radius equal to the pion Compton wavelength. We may find the first choice desirable since it leads to simpler computational algorithms[1,3].

We are now in a position to consider the question of spin and internal symmetries. If the particles in the final state have spin, one must sum over the spin labels; and similarly for internal symmetry labels like isotopic spin. The spin and isotopic spin of the initial state is either specified or it is to be averaged over, as the case may be. But the question of the summation of the spin and isotopic spin labels is complicated by the conservation laws. For the isotopic spin label this is simpler: we must conserve the isotopic spin in the initial and final state. For two particles only in the final state this completely fixes the isotopic spin wave function of the final state; but for more than two particles this is not, in general, true. There will be several isotopic spin eigenfunctions

with the same total isotopic spin (and the "third component"). In the spirit of the statistical theory every independent normalized isotopic wave function has the same weight. If one is interested only in the mu plicity without regard to the charge (i.e., isotopic spin) labels, the n ber of such independent isotopic spin functions may be counted[4,5] by a elementary method[4]. If more detailed information on the charge compl is needed it can be worked out using standard but tedious methods; a m elegant method which substantially reduces the labour of computations h been presented by Cerulus[6]. This method, involving invariant group i grations as projections, is ideally suited for a statistical theory in w all isotopic spin wave functions are given equal weight; it must be mod fied in those cases where an isotopic spin dependence is introduced into transition matrix element.

In addition to these complications one must further take into count the indistinguishability of the particles in the final states; an this may be done by treating them as identical, but for every set of n_i particles of type i include a factor $1/n_i!$ in the transition rate formul Combining the multiplicity $\mathcal{N}(\{n\},I)$ of functions with the same isotopic spin and the indistinguishability factors $1/n_i!$ we get a "symmetry factor" [4,5]

$$S(\{n\},I) = \frac{\mathcal{N}(\{n\},I)}{\prod_i n_i!} .$$

We may consider this factor as representing the integration over the discrete "phase space" for the internal symmetry labels.

The question of the spin labels is more complicated since "spin and "orbital" angular momenta are coupled; if one made the unrealistic a sumption that this coupling can be disregarded so that "spin" can be trea by itself, the treatment adopted for the isotopic spin can be extended to this case also: one merely has to multiply the symmetry factor by the mu plicity of spin wave functions for a total spin S. Since only nucleons have spin (pions being spinless), if we disregard nucleon pair production there are at most two spinning particles and the corresponding multiplici is zero or one; and correct results are obtained by completely disregard spin. This is what is usually done in standard statistical theories, though it is hardly realistic.

We are now in a position to make detailed predictions. One can calculate the relative probability for any final state (or rather a group of final states with non-zero measure) and this is equal to the phase space measure for this final state configuration, with due account of the symmetry factor and "scaled" appropriately. In particular, the total transition rate to a state of n particles is obtained by integrating the phase space factor over the entire domain which conserves energy and momentum. We may also calculate angular correlations, energy spectra, two-particle and three-particle effective mass distributions, etc. These may be less "justified" than the calculation of total transition rates but are nevertheless direct consequences of the statistical postulate. Thus, for example, the unnormalized two-particle effective mass distribution is given by

$$P_2(\mu^2)d(\mu^2) = \int \delta(k-p)\delta\left([k_{n-1}+k_n]^2-\mu^2\right)S(\{n\},I) \times$$

$$\times \prod_i d^4k_i \;\delta(k_i^2-m_i^2)\Theta(k_i^0)\; d(\mu^2)$$

in the covariant model. Similar expressions can be written down for the non-covariant Fermi model

$$P_2'(\mu^2)d(\mu^2) = \int \delta(k-p)\delta\left([k_{n-1}+k_n]^2-\mu^2\right)S(\{n\},I) \times$$

$$\times \prod_i d^3k_i \;d(\mu^2)\;.$$

This statistical postulate (in either version) of "all or none", i.e., of equal probability for all allowed configurations, thus yields explicit formulae for the (unnormalized) spectra involving only the "phase space integrals"; and the major task of computation is obtaining an accurate value for these integrals. This is hardly an elementary task but can be accomplished by numerical methods; all the approximate analytic expressions obtained to date have proved too inaccurate. The best numerical results for the Fermi model have been obtained by Hagedorn[7]. The covariant model is somewhat simpler for computations in view of the recurrence relations[1,3,8]. It may be remarked here that in view of the interpretation of the scaling parameter (in the covariant model) as a weighted propagator squared, it is

not unnatural to ascribe different scaling parameters to different kinds of particles, particularly the strange particles.

II. SURVEY OF EXPERIMENTAL DATA ON MULTIPLE PROCESSES

In attempting a comparison with experiment we are hampered by the lack of precise and detailed experimental results on the one hand, and the labour of computations on the other hand. Most of the comparisons have been made with the approximate phase space integrals and these comparisons do not necessarily provide an evaluation of the statistical theory itself, as has been emphasized by Hagedorn. Before reviewing these comparisons, let us discuss some immediate qualitative features of the theory.

1) Since the probability is independent of the orientation of the momentum configurations of the final state in the c.m. frame, it follows that the angular distribution in c.m. frame must be isotropic; and more generally there must be no distinguished direction (i.e., no correlations) in the c.m. frame. Of course, the conservation laws impose mutual correlations between particles.

2) Since the probability is independent of the charge labels of the particles the various kinds of mesons must have the same energy spectra, etc. We may refer to this as "strong charge independence". Again, branching ratios will be determined by isotopic spin considerations, but, provided the multiplicities are held fixed, the spectra must be the same. Of course, the spectra will depend on the over-all multiplicity, and will not be the same if one sums over all multiplicities.

3) Since the matrix elements have no momentum dependence, all spectra are purely kinematical and as such expected to have only smooth dependences; they will in general have only a single maximum dictated by the energy-momentum conservation law.

4) Since all particles in the final state are treated on an equal footing, the energy distribution in the final state is dependent only on the masses; the energy carried by the created particles is a significant

fraction of the total available energy. Consequently, the inelasticity is expected to be large.

5) Since the (squared) matrix element is independent of the various particle labels, it is automatically symmetric in all the particles; hence taking account of the symmetry of the particles in the final state is not expected to introduce any modifications except to multiply the transition rate by the symmetry factor $S(\{n\},I)$. This leaves the spectra unchanged and only serves to change the branching ratios for different multiplicity groups.

Almost all these qualitative results are at variance with experiment. For threshold production one would have expected the statistical postulate to hold; but the strong energy, spin and isotopic spin dependence of the low-energy pion-nucleon interactions make this otherwise. Thus, for example, in the range 0.9 GeV to 1.5 GeV laboratory energies for pion-nucleon collisions[9] one finds the momentum spectra of the secondary pions in marked disagreement with experiment, but the recoil nucleon spectra are consistent; in general the statistical theory predicts energies of the pions too high. The ratio of single to double and higher multiplicity productions is also at variance with experiment, the theory preferring too low a multiplicity. At higher energies (6.2 GeV and 9 GeV) there is distinct disagreement with experiment[10] as regards the expected multiplicity (and branching ratios), the theory again predicting too small a multiplicity. Typical multiplicities are given in Table I. The angular distributions are definitely peaked along the axis of collision and the inelasticity is of the order of 0.5, but the experimental multiplicities are perhaps not accurate enough to supply any real comparison. Finally, in the process of antinucleon annihilation the multiplicity is greatly underestimated by the theory (prediction of about 3.3 against the experimental number[11] of about 5.5); and the angular correlations between pions as well as the effective two-particle mass distribution plots show a distinct dependence on the charge labels of the particles selected[12]. This last example shows the shortcomings of the simple statistical theory most graphically. And in the extremely high-energy region there is some evidence that there seems to be a limiting value for the pion momenta in the c.m. frame. This may be related to the existence of a fundamental length.

Table I

Typical pion multiplicities in N-N and π-N collisions

Energy (GeV)	Collision	Average multiplicity	Reference
5	π-p	2.3	Maenchen et al.[a]
7	π-N	4.8	Petrzilka [b]
6.2	p-N	2.27 ± 0.27	Daniel et al.[c]
6.2	p-N	2.7 ± 0.5	Kalbach et al.[d]
9	p-p	3.0	Van Shu Feng et al.[e]

a) Maenchen, Fowler, Powell and Wright, Phys.Rev. 108,850 (1957)
b) Petrzilka, Proc. of the 1960 Ann.Int.Conf. on High-Energy Physics, Session S1.
c) Daniel, Kameshwara Rao, Malhotra and Tsuzuki, Tata Institute preprint.
d) Kalbach, Lord and Tsao, Phys.Rev. 113, 330 (1959).
e) V.I. Veksler, Proc. of the 1960 Ann. Int. Conf. on High-Energy Physics, Session S1.

Modifications of the simple statistical theory to account for some or all of these features have been attempted by many authors. The phenomenological modification of replacing Ω by $n\Omega$ or of κ by $\kappa/\sqrt{n}$ introduced by Pomeranchuk goes in the right direction in increasing the relativ weights of higher multiplicity states and gives good agreement for the ave age multiplicities. This simultaneously improves the pion momentum spectr But this modified theory leaves unanswered the other two difficulties; no does it take into account the low-energy pion-nucleon resonance, and one hardly expects the detailed pion spectra and effective mass spectra to be correct without this.

A class of modifications of the statistical theory which enjoys good fit with experiment idealizes this strong interaction in terms of a quasi-bound state of a nucleon and pion with spin and isotopic spin $3/2$ whi is treated as an independent particle. The isobar is then supposed to de cay independently. This treatment is justified by saying that the "life-time" of the isobar is large compared with the "collision time"; but withou

too much justification one may simply consider this as a proper form for the matrix element in the presence of the resonant pion-nucleon interaction. If the resonance is "narrow enough" the interference terms between the resonant and non-resonant parts of the transition amplitude may be neglected so that one can talk in terms of probabilities. Isobar calculations[13] have been carried out for pion production in nucleon-nucleon collisions and pion-nucleon collisions using the $(3/2, 3/2)$ isobar and provide good fit with experimental data in the region of single and double pion production range (0.9 GeV to 1.5 GeV) and reproduce the double humped pion momentum spectra[14]. For meson production at higher energies (6.2 GeV and 9 GeV) the multiplicities are fitted by the theoretical predictions involving the isobar. But just as for the Pomeranchuk modifications here also the anisotropy and the inelasticity are not explained at all.

One is also led to ask whether the higher resonances in the pion-nucleon system and the inferred pion-pion resonances are to be taken into account. Perhaps one has justification for disregarding the higher nucleon isobars since they are rather wide; and further, at these higher energies higher multiplicity configurations are important and one may hope that in such configurations the resonant configurations are not as important and are well approximated by the statistical matrix element. This is borne out to a certain extent by the experimental data for collisions at 9 GeV[15]. On the other hand, the pion-pion interactions may be more important; they seem to be particularly so for nucleon-antinucleon annihilation. That the nucleon isobar model improves the fit of the single and double production data is not surprising since low-energy data are dominated by this interaction. A comparison of the calculated and experimental results for π-N and N-N collisions is given in Figs. 1 and 2 taken from the work of Barashenkov[16].

A further test of the statistical model is furnished by comparing the predicted and experimental total cross-sections as a function of the energy. We discuss this briefly in connection with partial wave analysis of these reactions below, particularly with regard to unitarity restrictions on maximum single partial wave cross-sections.

III. CONSERVATION OF ANGULAR MOMENTUM; PARTIAL WAVES

So far we have discussed the conservation laws of energy and momentum and isotopic spin. Let us now take up the question of angular momentum conservation. We have already remarked that all the conservation laws implied by the invariance under the (proper, orthochronous) inhomogeneous Lorentz group are satisfied if the transition matrix element contains an energy-momentum delta function as a factor and is an invariant function of the four-momenta. At this point we may require a stronger condition on the transition matrix element: namely, that it is a manifestly covariant function (i.e, it does not depend explicitly on the sign of the time component of time-like four-vectors). This is true of all the current field-theoretical matrix elements, where it is called the "substitution law". Such manifestly covariant functions are functions only of the invariant scalar products $k_i \cdot k_j$ and the invariant pseudoscalar products $\epsilon_{\alpha\beta\gamma\delta}\, k_i^\alpha\, k_j^\beta\, k_k^\gamma\, k_\ell^\delta$. We are thus led to consider analytic functions of these invariants and pseudoinvariants for the transition matrix elements; any such function completely satisfies all requirements of relativistic invariance; and in particular of angular momentum conservation!

This was not the way that angular momentum conservation was first considered; Fermi introduced angular momentum into his model in terms of impact parameters, and angular momentum conservation was taken care of using the method of undetermined multipliers familiar from statistical mechanics. This view involving a literal configuration interpretation is inconsistent with quantum mechanics in the same way that a Gaussian or exponential potential gives rise to qualitatively different expressions for scattering in classical and quantum mechanics. On the other hand, the problem of treating the spins of the particles (and isobars) in the general case suggests the need for a careful treatment of angular momentum conservation.

The clue to the proper treatment is to note that a system of two colliding particles is characterized not only by the momentum and energy, but by another quantum number: the "spin" or the angular momentum in the c.m. system. The existence of the additional "mechanical" quantum number is easily seen by considering two identical sets of two particles with th

same c.m. energy but relative momenta oriented two different ways: these states are orthogonal.

The situation is complicated by the fact that the state of two particles with well-defined momenta is an eigenstate of the energy and momentum but not of "spin". But this is not new to relativistic systems; the same is true for non-relativistic particles. In the reaction leading to the multiple production the partial amplitudes for the various spins do not mix with each other. Of course, the final state is not completely specified by the energy momentum and spin in complete parallel with the non-relativistic case and the problem of isotopic spin functions.

The relevance of these considerations is that the primitive statistical postulate enunciated in Section II is highly restrictive and corresponds only to interactions in the spin 0 states. This is most easily seen by noting that the dependence of the transition matrix elements on the initial (or final) particle momenta is at most through their energies in the c.m. frame; in the covariant statistical model there is no dependence at all except in imposing the energy-momentum conservation law. Hence only the $J = 0$ states interact. This is not only a highly restrictive and unphysical restriction, but is inconsistent with antinucleon annihilation from the $J \neq 0$ nucleonium bound states!

Note that the angular momentum about which we are talking is the total angular momentum of the system, the so-called "channel spin". In addition, one also has the relative angular momenta of subgroups of particles; for example, the "spins" of two-particle sub-systems in the final state. Such dependences on intermediate spins is important particularly in the modified statistical theories where some details of the strong interactions are to be introduced: the low-energy pion-nucleon interaction is strongly dependent on the relative spin state and the resonance is present only in the $J = 3/2$ states. Thus it is insufficient to introduce an isobar by constraining their relative momenta; we must also constrain the "partial spin". The same is true of the pion-pion resonance. In all calculations done so far these problems have not been taken care of; and it is not clear whether this omission leaves the qualitative agreements unaltered.

The statistical postulate, in addition, makes definite statements about the dependence on partial spins. It is interesting to note that the

covariant model has no dependence on partial spins, while the Fermi model involves a residual energy-dependent weighting of the partial spins. This may "explain" the simplicity of the computational algorithms for the covariant model.

But neither model is realistic; at the high energies where meson production processes arise there is no reason to suppose that the $J = 0$ waves must dominate. Even in lower energy ranges the $J \neq 0$ are important and in many cases dominate. It is also to be noted that the argument for $J \neq 0$ is not implied, but only $L = 0$ (i.e., the interaction is spin independent and momentum independent), is no better since higher partial waves are certainly important; both the most important pion-nucleon and pion-pion resonances seem to be p-wave resonances. It is true that in antinucleon annihilation one may expect the reactions to proceed mainly from the S state but even in this case $J \neq 0$ are very important since the final state consists entirely of spinless particles and hence the complete matrix element cannot be independent of the final state momenta. The only possible case is for K-$\bar{K}$ annihilation in the S state, one of the least pressing instances.

We are thus led to consider the primitive statistical matrix element to be the first term in a partial wave analysis of the reaction rate formula. The complete matrix element in the modified statistical model must be complicated and must involve not only the channel spin but also the intermediate spins. And in addition the isotopic spin dependence of the resonances of subgroups of particles must be properly inserted. If one can approximate resonant interactions by quasi-bound states in these modifications we can neglect the interference of this resonant matrix element with the rest and treat them in terms of new "particles" and the spin and isotopic spins of the particle can be appropriately chosen. But one must choose the complete matrix element to involve appropriate "contractions" of the spin wave functions with other vectors to produce an invariant matrix element. Needless to say, the same must be done for the isotopic spin functions but there is no orbital contribution in this case.

For making progress one must adopt specific forms for the invariant matrix elements. In the spirit of the statistical theory we must take the "simplest" matrix elements for each partial wave consistent with the conservation laws. In particular, the higher partial waves have a

initial wave function whose orbital part is a tensor of sufficiently high rank in the c.m. relative momentum $\underline{p}$ of the two colliding particles; and this must be contracted with the final state wave function. We are thus led to consider matrix elements of the form

$$\underline{p}\cdot\underline{k}_1 \; \underline{p}\cdot\underline{k}_2 \; \underline{p}\cdot\underline{k}_3 \; ; \qquad \underline{p}\cdot\underline{k}_1 \times \underline{k}_2 \; \underline{p}\cdot\underline{k}_3$$

and it is easily seen that these can lead to the peaked angular distributions. The symmetry requirements on the final state may necessarily couple the isotopic spins; for example, if $\underline{k}_1$ and $\underline{k}_2$ refer to two pions, the first and second matrix elements must respectively correspond to symmetric (I = 2,0) and to antisymmetric (I = 1) isotopic wave functions respectively. While we have written the above matrix elements in three-dimensional form, they are invariant since the c.m. frame is invariant; and they can be trivially transcribed into manifestly covariant forms.

The partial wave matrix elements must in general depend upon the c.m. energy and have different "energy" dependences for different channel spins. One guiding principle is that they must be simple functions of the invariants. The present understanding of the analytic structure of many-particle amplitudes is too fragmentary to provide any guidance; and one is best advised to choose such forms as are most amenable to computations and represent knowledge about resonant interactions as best as one can.

Some qualitative points deserve mention: firstly, since the channels are now labelled by c.m. energy and spin (as well as isotopic spin), these high-energy stars admit a two-dimensional classification rather than simply by energy; such ideas have been advanced before[17]. Secondly, the statistical matrix elements also give an expression for the total interaction cross-sections; and unitarity sets an upper limit to the interaction cross-sections. For two particles colliding in a state of angular momentum J the maximum cross-section consistent with unitarity is

$$4\pi \lambda\!\!\!^{-2} (2J+1)$$

where $\lambda\!\!\!^{-}$ is the reduced wavelength of the nucleons. A cross-section of about 22.6 mb for 6.2 GeV nucleon-nucleon collisions is certainly inconsistent with J = 0 interactions (see Tables II, III and IV). Finally, the higher partial waves lead, in general, to the collimation of the produced

mesons into axial cones; the quantitative features must, of course, de on the specific matrix elements (compare Tables V and VIII).

We have thus arrived at the conclusion that the primitive sta tical model matrix element is the first term in an expansion in partial waves and one must include contribution from higher partial waves for ev qualitative agreement with experiment as regards total cross-sections an axial collimation. In this connection it is interesting to note the ob servations of Daniel et al.[10] who find isotropy for large multiplicity axial collimation for low multiplicity; these may be interpreted to tak place in low and high orbital angular momentum channels respectively. the detailed predictions and numerical results do depend on the full mat element; the situation is thus very similar to, say, nucleon-nucleon sc tering results analysed in terms of partial wave phase shifts. In the l ter case there is a limit to the number of partial waves one can reasona include and the present problem is similar.

Table II

Typical total π-N and N-N cross-sections

Energy (GeV)	Collision	Total cross-section (mb)	Reference
5	π^--p	22.5 ± 2.4	Maenchen et al.[a]
7	π-N	23 ± 3	Petrzilka[b]
2.75	p-p	15	Block et al.[c]
5	p-p	43.7 ± 0.7	von Dardel et al.[d]
7	p-p	43.3 ± 0.4	
10	p-p	42.1 ± 0.4	

a) Maenchen, Fowler, Powell and Wright, Phys.Rev. 108, 850 (1957).

b) Petrzilka, Proc. of the 1960 Ann.Int.Conf. on High-Energy Physics, Session S1.

c) Block, Harth, Cocconi, Hart, Fowler, Shutt, Thorndike and Whittemore, Phys.Rev. 103, 1484 (1956).

d) von Dardel, Frisch, Mermod, Milburn, Piroue, Vivargent, Weber, and Winter, Proc. of the 1960 Ann.Int.Conf. on High-Energy Physics, Session S1.

Table III

Proton-proton elastic scattering cross-section measurements from 0.81 to 6.2 GeV

Energy (GeV)	Method	Cross-section (mb)	Reference
0.81	Cloud chamber	24 ± 2	Morris et al.[a]
0.925	Emulsion	17 ± 3	Duke et al.[b]
1.00	Counter	19 ± 3	Smith et al.[c]
1.50	Cloud chamber	15 ± 2	Fowler et al.[d]
2.24	Counter	17	Cork et al.[e]
2.75	Cloud chamber	15 ± 2	Block et al.[f]
2.85	Cloud chamber	17.3 ± 1.5	Smith et al.[g]
3.00	Emulsion	8.9 ± 1.0	Cester et al.[h]
4.4	Counter	10	Cork et al.[e]
5.7	Emulsion	13 ± 6	Giles [i]
6.15	Counter	8	Cork et al.[e]
6.2	Emulsion	8.8 ± 2.8	Kalbach et al.[j]
8.2	Emulsion	8.7 ± 0.4	Azimov et al.[k]

a) Morris, Fowler and Garrison, Phys.Rev. 103, 1472 (1956).

b) Duke, Lock, March, Gibson, McEwen, Hughes and Muirhead, Phil.Mag. 2, 204 (1957).

c) Smith, McReynolds and Snow, Phys.Rev. 97, 1186 (1955).

d) Fowler, Shutt, Thorndike and Whittemore, Phys.Rev. 103, 1479 (1956).

e) Cork, Wenzel and Causey, Phys.Rev. 107, 859 (1957).

f) Block, Harth, Cocconi, Hart, Fowler, Shutt, Thorndike and Whittemore, Phys.Rev. 103, 1484 (1956).

g) Smith, Courant, Fowler, Kraybill, Sandweiss and Taft, Proc. of the 1960 Ann.Int.Conf. on High-Energy Physics (Rochester).

h) Cester, Hoang and Kernan, Phys.Rev. 103, 1443 (1956).

i) P.C. Giles, UCRL-3223.

j) Kalbach, Lord and Tsao, Phys.Rev. 113, 325 (1959).

k) V.I. Veksler, Proc. of the 1960 Ann.Int.Conf. on High-Energy Physics (Rochester), Session S1.

Table IV

Typical antiproton total cross-sections

Momentum (GeV/c)	Cross-section (mb)	Reference
2	49 ± 6	Armenteros et al.[a]
5	67.0 ± 2.1	
7	63.0 ± 2.3	von Dardel et al.[b]
10	46.0 ± 2.0	
10.7	53.0 ± 1.1	

a) W.A. Wenzel, Proc. of the 1960 Ann.Int.Conf. on High-Energy Physics (Rochester).

b) G. von Dardel, Proc. of the 1960 Ann.Int.Conf. on High-Energy Physics (Rochester).

In the nucleon-nucleon scattering problem there is a possibility of including the higher angular momentum contribution without expanding it into partial waves by explicitly taking account of the one-pion exchange contribution. In terms of analytic functions this corresponds to taking the nearest singularities in the (cos Θ) plane (where Θ is the scattering angle); but this may be restated in a potential language as taking account of the longest range ("peripheral") interactions. This allows one to take fewer phenomenological phase shifts to achieve the same fit with experiments.

A similar procedure is suggested here also. The higher partial waves may be taken phenomenologically in terms of one-particle poles, so that we consider two subgroups of the final particles with partial statistical matrix elements connected by a single pion propagator. The magnitude of the matrix element will depend on the magnitude of the propagator and the dominant contributions will come from almost real pions. With such a modification, retaining a few partial waves in the purely phenomenological term would be expected to be a fair approximation. Thus we are led to the single-pion exchange models as a simple refinement of the primitive statistical theory to bring it in closer agreement with the actual situation.

: verifies that the one-pion exchange contribution generates higher par-
al waves and axial collimation of the final state particles and provides
ough total cross-section.

To be complete one must take more complicated groupings, but it
to be expected that the primitive matrix element along with the one-pion
change contribution and any isobars is sufficiently detailed to provide a
tisfactory model.

IV. ANTINUCLEON ANNIHILATION

The multiple production of mesons in antinucleon annihilation is
haps the most ideal case for the application of a statistical theory.
final state consists only of pions; and the isotopic spin and exten-
l charge conjugation selection rules can be easily accommodated. The
tial nucleon pair can annihilate at essentially zero kinetic energy.
phenomenon lends itself to a careful experimental study since the c.m.
me is practically the same as the laboratory frame. And a great deal
experimental results are available on this process.

The earliest calculations[4,5,18] using the Fermi model gave multi-
cities of the order of 3, too small compared with the observed[19] multi-
city of about 5. In these calculations the symmetry factors were pro-
ly included and the standard Fermi volume Ω_0 was used. Recalculation of
se results using the covariant model seems to make no great difference[8].
obtain agreement with observed multiplicities it was necessary to take
10 Ω_0 which seems difficult to justify directly. There are a great
ber of selection rules applicable to this reaction (see below) but their
lusion did not improve the simple statistical model calculations. On
other hand, we already know that the primitive statistical matrix ele-
t is inadequate; we must include appropriate channel spin and isotopic
as well as include resonant interactions. The introduction of
onances by introducing isobars has been tried by various authors. Thus
lus[6] introduced an $I = 1$, $J = 1$, $G = +1$ pion isobar with a mass of 3
pion masses and was able to fit the multiplicity. A simpler method
lves taking the Pomeranchuk version of the Fermi model, i.e., $\Omega = n\Omega_0$

and this gives the correct multiplicity[8]. But from our comm
previous section this treatment of the annihilation matrix ele
particularly justified.

This is most clearly indicated by the observation by
and co-workers[12] that the angular correlations of the annihil
are different for pions of like and unlike charges, the like pi
more strongly correlated. The forward-to-backward ratio for
was 1.23 ± 0.10 and for unlike pions 2.18 ± 0.12 which is to b
with the value 1.80 predicted by the covariant model. A relat
tion is the very sharp maximum in the c.m. momentum distributi
charged pions (which is equivalent to a maximum in the two-pio
mass distribution around 4.7 ± 0.2 pion masses), the maximum b
pronounced in the $\pi^+\pi^0$ and $\pi^-\pi^0$ distributions for the three-bo
state in the reaction

$$p + \bar{p} \rightarrow \pi^+ + \pi^- + \pi^0 .$$

These considerations suggest that the pion-pion interaction is
spin dependent. The calculations of Cerulus[6] and of Pinski e
took such an interaction. Maglić has also observed a marked c
of the pions with the direction of the incident antiprotons at
He finds that the negative pions tend to go forward, while the
pions show the opposite tendency. The degree of correlation i
the following table.

Table V

Antiproton-pion correlations in
$\bar{p}p \rightarrow 2\pi^+ + 2\pi^-$ and $pp \rightarrow 2\pi^+ + \pi^0 + 2\pi^-$
at 1.65 GeV/c

Type	Experimental correlation
$\bar{p}\ \pi^-$	$1 + (0.25 \pm 0.08) \cos\Theta + (0.415 \pm 0.12)$ co
$\bar{p}\ \pi^+$	$1 - (0.25 \pm 0.08) \cos\Theta + (0.415 \pm 0.12)$ co

As usual, the statistical theory (or any of its modifications d
crease the multiplicity) will have no explanation for this effe

In passing we note that there were two rather detailed theories for explaining the large multiplicity and the large antiproton total cross-sections, one by Ball and Chew[21] and the other by Koba and Takeda[22]. Both these theories provide detailed mechanisms for the reactions. The first one is aimed at explaining the interaction cross-sections rather than the multiplicity. The latter explains the multiplicity by a two-stage mechanism, a "core-annihilation" and a "cloud dispersion". In this case again while the multiplicity is explained the dependence of the angular correlations, etc., cannot be explained; but these authors note the necessity to introduce higher partial waves to explain the total cross-sections.

Let us now briefly recall the selection rules in nucleon-antinucleon annihilation. The invariance of the strong interactions under charge conjugation and isotopic spin rotations lead to the conservation of the extended charge conjugation parity (G-parity). For a state of the nucleon-antinucleon system with orbital angular momentum L, spin S, and isotopic spin I, the G-parity is $(-1)^{L+S+I}$ while for n pions it is $(-1)^n$. Hence we must have $L+S+I \equiv n$ (modulo 2). On the other hand, a two-spin system with angular momentum J has space parity $(-1)^J$ so that a state with J = L of the nucleon-antinucleon system cannot decay into two pions. (Recall that a Dirac particle-antiparticle system has odd intrinsic parity.) In addition the $^3P_0$ state cannot decay into three pions from parity conservation. Consequently the several states cannot decay into less than four pions (see Table VI).

Table VI

Minimum number of pions
in antinucleon annihilation

State	I = 0	I = 1	Minimum no. of pions I = 0	Minimum no. of pions I = 1
$^1S_0$	+	-	2	3
$^3S_1$	-	+	3	2
$^1P_1$	-	+	3	4
$^3P_0$	+	-	4	4
$^3P_1$	+	-	4	3
$^3P_2$	+	-	2	3

Note that we must use the complete channel specification to fully utili these selection rules. In general one does not know in antiproton ann lation the labels for the state from which capture takes place. But t important role played by the Stark effect shows that the capture will t place predominantly from the S states, according to the calculations of Desai[23]; both $^3S_1$ and $^1S_0$ contribute. And the $p\bar{p}$ state is an equal mixture of I = 0 and I = 1.

We have remarked before the need for taking a large "volume" explaining the observed multiplicities by a statistical theory. Since covariant model is more convenient for calculations it has sometimes bee considered significant to introduce a volume parameter into the covarian model by the correspondence

$$2\, m_\pi \cdot \kappa^2 \longleftrightarrow \frac{(2\pi)^3}{\Omega'} .$$

The calculated values (for various choices of this adjusted volume Ω') o the charged pion multiplicities in antinucleon annihilation are compared with the latest available experimental data by Lynch[24] (Fig. 3) and sho very good agreement for a choice

$$\Omega' \simeq 5\Omega_0$$

which is significantly smaller than previously quoted values. More deta specification of the annihilation at 1.61 GeV/c in terms of charged pron branching ratio is also given[24]. (See Table VII.)

Table VII

Pion modes in antiproton annihilations at 1.61 GeV/c

Type of event	Measured fraction of pion annihilations in per cent	Values of Ω' needed to fit the measurement
0-prong	$1.0 \begin{smallmatrix} +3 \\ -0.6 \end{smallmatrix}$	------
2-prong	36.0 ± 5.2	4.6 ± 1.3
4-prong	54.6 ± 1.3	5.1 ± 0.9
6-prong	8.4 ± 0.3	4.6 ± 0.3
8-prong	0.15 ± 0.4	5.8 ± 0.7
$\pi^+\pi^-$	0.20 ± 0.05	4.8 ± 0.4

We shall now report briefly on a calculation by G. Pinski in progress at Rochester. The model aims at using the multiplicity, angular correlations and effective mass distributions as a method of finding evidence for the pion-pion interaction and to determine its nature. The basic idea is similar to that of Cerulus but all the calculations are done covariantly. The usual license for using probabilities rather than matrix elements is invoked and the partial spins have been treated only approximately, but isotopic spin weights have been carefully included. Rather than calculate energy spectra of particles weighted over many multiplicities, the distribution in energy and two-particle masses has been treated for charge-identified prongs in events with small multiplicity in terms of appropriate phase space integrals.

The relative transition probability for annihilation into n pions in a configuration $K_{(n)}$ (including the momentum and isotopic spin labels) is given by

$$R(K_{(n)}) = \int \Lambda(K_{(n)}) \prod_{i=1}^{n} d^4k_i \; \Theta(k_i^0)\delta(k_i^2 - m_\pi^2) \times$$

$$\times \; \delta(k - p_1 - p_2) f(p_1, p_2, k, \zeta)$$

where Λ is a projection operator to the configuration $K_{(n)}$, $f(p_1, p_2, k, \zeta)$ is an arbitrary invariant function of the momenta and of the isotopic spin labels determined by the dynamics of the problem. The covariant statistical theory implies that $f(p_1, p_2, k, \zeta)$ is a constant; the "resonant" <u>model</u> obtained by choosing

$$f = 1 + \lambda^2 \sum_{i>j} \mathcal{F}\left[(p_i + p_j)^2 - M^2\right] \Lambda_{ij}$$

can be used to represent the suggested $T = J = 1$ resonance by requiring Λ_{ij} to vanish unless the partial "spin" and isotopic spin are both unity. Here both λ^2 and M^2 are parameters with the dimensions of the square of a mass; and $\mathcal{F}$ is a sharply peaked function and approximated by the smeared-out δ-function

$$\mathcal{F}(\mu^2 - M^2) = \frac{1}{2M} \frac{\epsilon/\pi}{(\mu - M)^2 + \epsilon^2}$$

where ϵ is the half-width. In our calculation we took ϵ to be half a mass and M to be four pion masses.

We have already remarked about the complications introduced b channel spin; to make progress we had to ignore the restriction J = 1 proceed to do the numerical calculations. Hence the results of our ca lation do not test the J = 1 restriction; but the qualitatively intere predictions made must survive in the more complicated exact treatment. reservation is nevertheless to be borne in mind.

Let us now compute the effective mass of a pair of particles; distribution can be separated into two parts, one coming through the no resonant part of $f(p_1,p_2,k,\zeta)$ and the other through the resonant terms p portional to λ^2. The non-resonant term is identical with what has bee calculated in Section I; so we need to compute only the coefficient o in the (unnormalized) effective mass distribution. There are three di tinct cases according to whether the particles are chosen both resonan only one resonant, or neither resonant. We denote these terms by P_{rr}, P_{rf} and P_{ff} respectively; they are proportional to

$$P_{rr} \sim \int \prod_{i=1}^{n} \left[d^4k_i \ \delta(k_i^2 - m_\pi^2)\Theta(k_i^0) \right] \delta(\Sigma\, k_i - p_1 - p_2) \times$$

$$\times \mathcal{F}(\mu^2 - M^2)\delta([k_{n-1} + k_2]^2 - \mu^2)$$

$$P_{rf} \sim \int \prod_{i=1}^{n} \left[d^4k_i \ \delta(k_i^2 - m_\pi^2)\Theta(k_i^0) \right] \delta(\Sigma\, k_i - p_1 - p_2) \times$$

$$\times \mathcal{F}([k_{n-1} + k_{n-2}]^2 - M^2)\delta([k_{n-1} + k_n]^2 - \mu^2)$$

$$P_{ff} \sim \int \prod_{i=1}^{n} \left[d^4k_i \ \delta(k_i^2 - m_\pi^2)\Theta(k_i^0) \right] \delta(\Sigma\, k_i - p_1 - p_2) \times$$

$$\times \mathcal{F}([k_{n-3} + k_{n-2}]^2 - M^2)\delta([k_{n-1} + k_n]^2 - \mu^2)$$

respectively. All these covariant integrals can be computed (using app priate recurrence relations) by numerical methods. To complete the cal tion it is necessary to determine the relative weights of these three co butions in the resonant part of the effective mass distributions; and t contributions correspond to the factor $\Lambda_{n-1,n}$ in the resonance function.

to compute the relative weights of P_{rr}, P_{rf} and P_{ff} for two particles (of known charges) we compute the expectation value of $\Lambda(n,I)A_{n,n-1}$ for the isotopic spin wave function of the n pions where the factor $\Lambda(n,I)$ is the projection operator to a state of total isotopic spin I. These projection operators can be written down as matrices in the charge labels ζ, but in actual practice it seems more efficient to construct actual isotopic spin wave functions with total isotopic spin I and two particles coupled to isotopic spin 1. This is particularly so in the case of small multiplicities $n = 3$ and $n = 4$. The results of the calculation of relative weights are summarized in Table VIII. The computed squared mass distributions for three-pion final states are illustrated in Fig. 4 and are to be compared with the experimental results of Maglić et al. in Fig. 5.

In the resonant case we see that in $\bar{n}p$ annihilation for the (+-) case the normalized distribution is concentrated around the resonance mass and the difference between (+-) and (++) or (--) curves is unmistakable. For the $p\bar{p}$ annihilation into $\pi^+\pi^-\pi^0$ the resonance must show up most prominently in the $\pi^{\pm}\pi^0$ distribution in view of the isotopic spin weight factors. For the four-pion modes the situation is more complicated since the number of independent isotopic spin functions is considerably larger. (However, compare Table VI.) In general, as the number of annihilation pions increases, the number of ways of picking the two particles also increases. Therefore the effect of a pion-pion resonance is more prominent for smaller multiplicities and it may be worth while to confine our attention to events with few pions only.

A corresponding analysis can be carried out for three-pion resonances also. The computations are more involved than for the two-particle resonance but no new principle is involved. In this case again one expects to find a peaked distribution for the three-particle effective mass; for the four-pion channel this would experimentally be seen as a peak in the momentum distribution of pions, and should be easy to detect if it exists.

More complete investigations including further refinements appropriate to a more realistic matrix element including pion-pion interactions are suggested, particularly as regards the treatment of partial spins, extended charge conjugation, and multiple resonances. Only a complete fit with all data is significant since it is clear that correct multiplicities can be obtained by a variety of models.

Table VIII

Relative weights of rr, rf and ff contributions in two-particle effective mass distributions

Reaction		Correlation	rr	rf	ff
$p + \bar{p} \to$	$\pi^+\pi^-$	(+-)	1	0	0
	$\pi^+\pi^-\pi^0$	(+0),(0-)	$5/12$	$7/12$	0
		(+-)	$1/6$	$5/6$	0
	$\pi^+\pi^-\pi^0\pi^0$	(+0),(0-)	$3/14$	$8/14$	$3/14$
		(+-)	$1/7$	$6/7$	0
		(00)	0	$6/7$	$1/7$
	$\pi^+\pi^-\pi^+\pi^-$	(+-)	$1/4$	$1/2$	$1/4$
		(+-),(--)	0	1	0
$n + \bar{p} \to$	$\pi^0\pi^-$	(0-)	1	0	0
	$\pi^-\pi^+\pi^-$	(+-)	$1/2$	$1/2$	0
		(--)	0	1	0
	$\pi^0\pi^-\pi^0$	(0-)	$1/2$	$1/2$	0
		(00)	0	1	0
	$\pi^+\pi^-\pi^0\pi^-$	(0-)	$3/13$	$8/13$	$2/13$
		(+-)	$2/13$	$8/13$	$3/13$
		(+0)	$3/13$	$10/13$	0
		(--)	0	$10/13$	$3/13$
	$\pi^0\pi^-\pi^0\pi^0$	(0-)	$1/3$	$2/3$	0
		(00)	0	$1/3$	$2/3$

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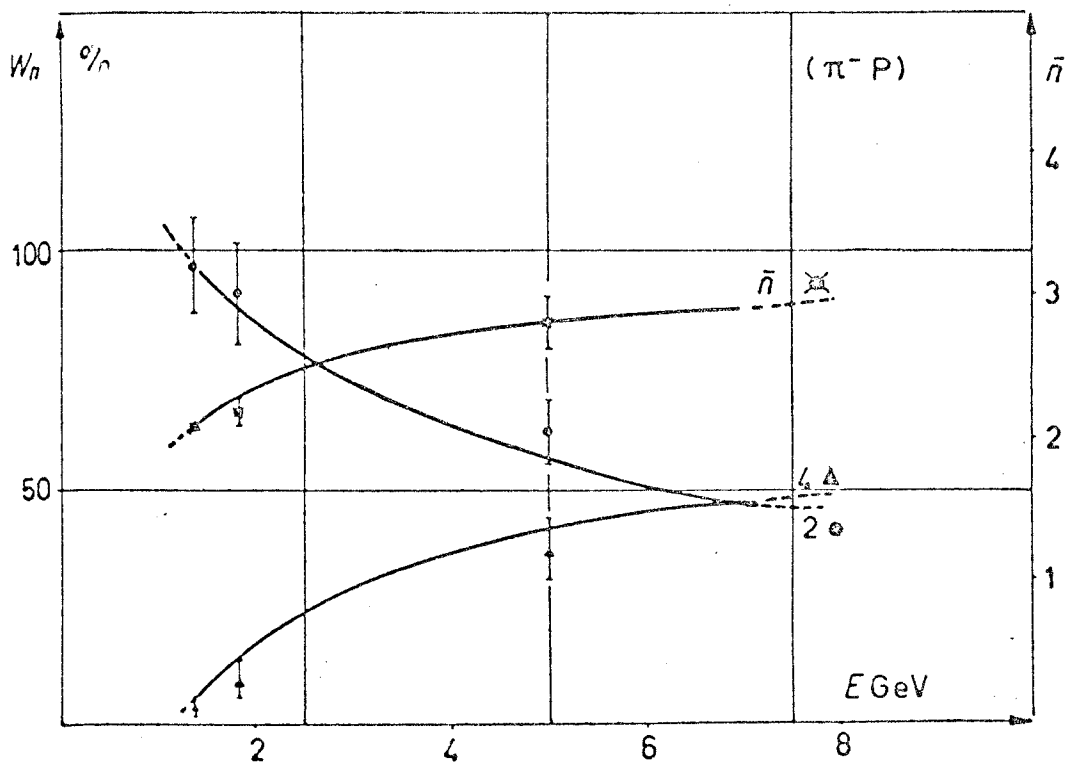

Fig. 1. Comparison between isobar calculations and experiments for the probabilities W_n of 2-, 4- and 6-prong star production and the average number $\bar{n}$ of produced charged particles in $(\pi^- p)$ collisions.

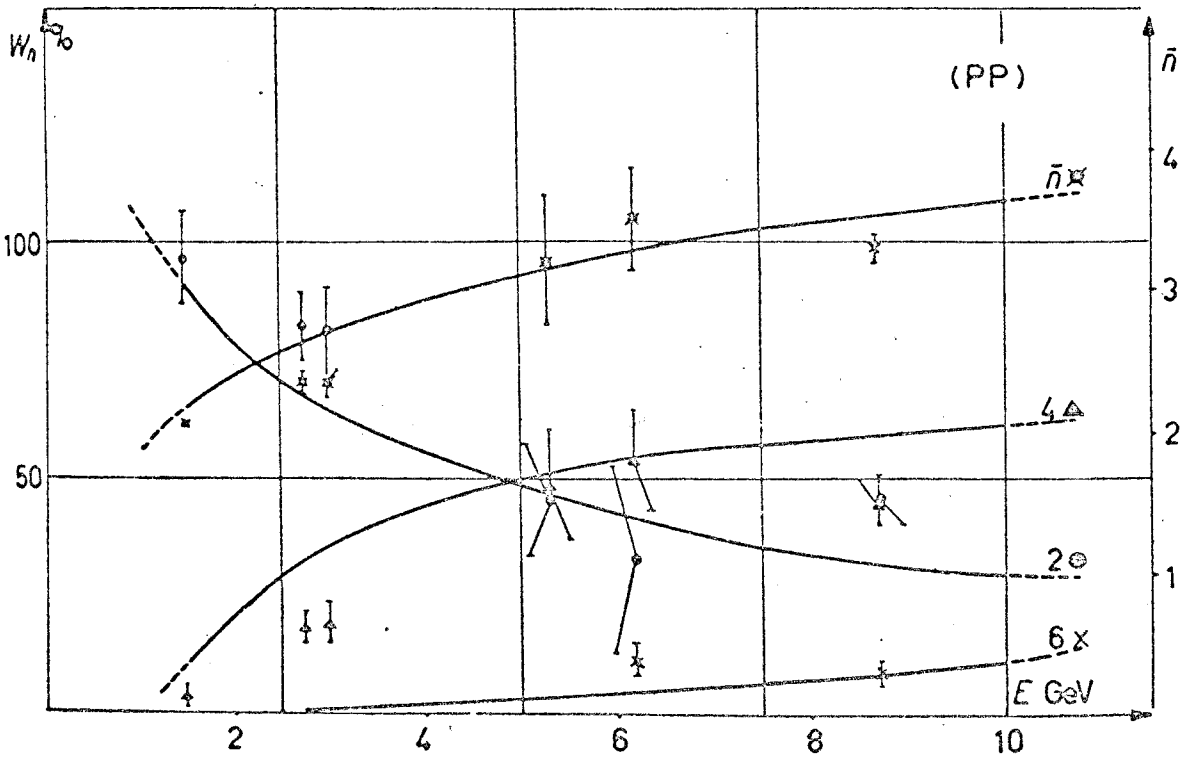

Fig. 2. Comparison between isobar calculations and experiments for the probabilities W_n of 2-, 4- and 6-prong star production and the average number $\bar{n}$ of produced charged particles in (pp) collisions.

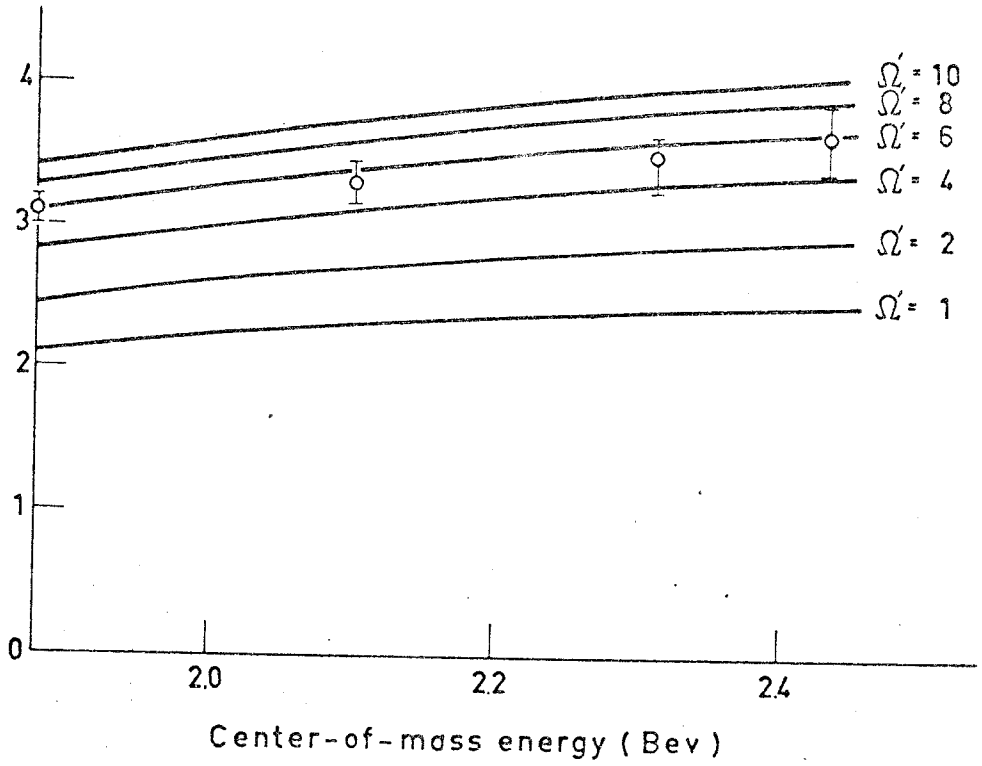

Fig. 3. Calculated values for charged pion multiplicities for various values of the adjusted volume Ω′ compared to experiments.

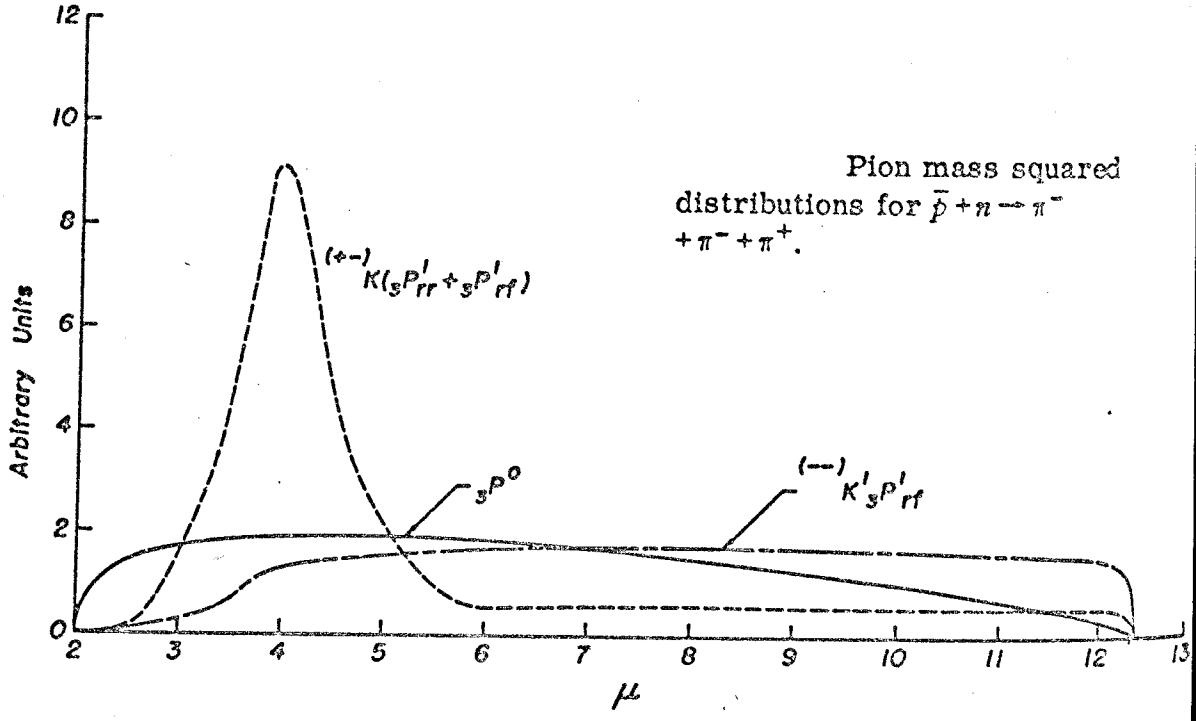

Fig. 4. Computed squared mass distribution for three-pion final states.

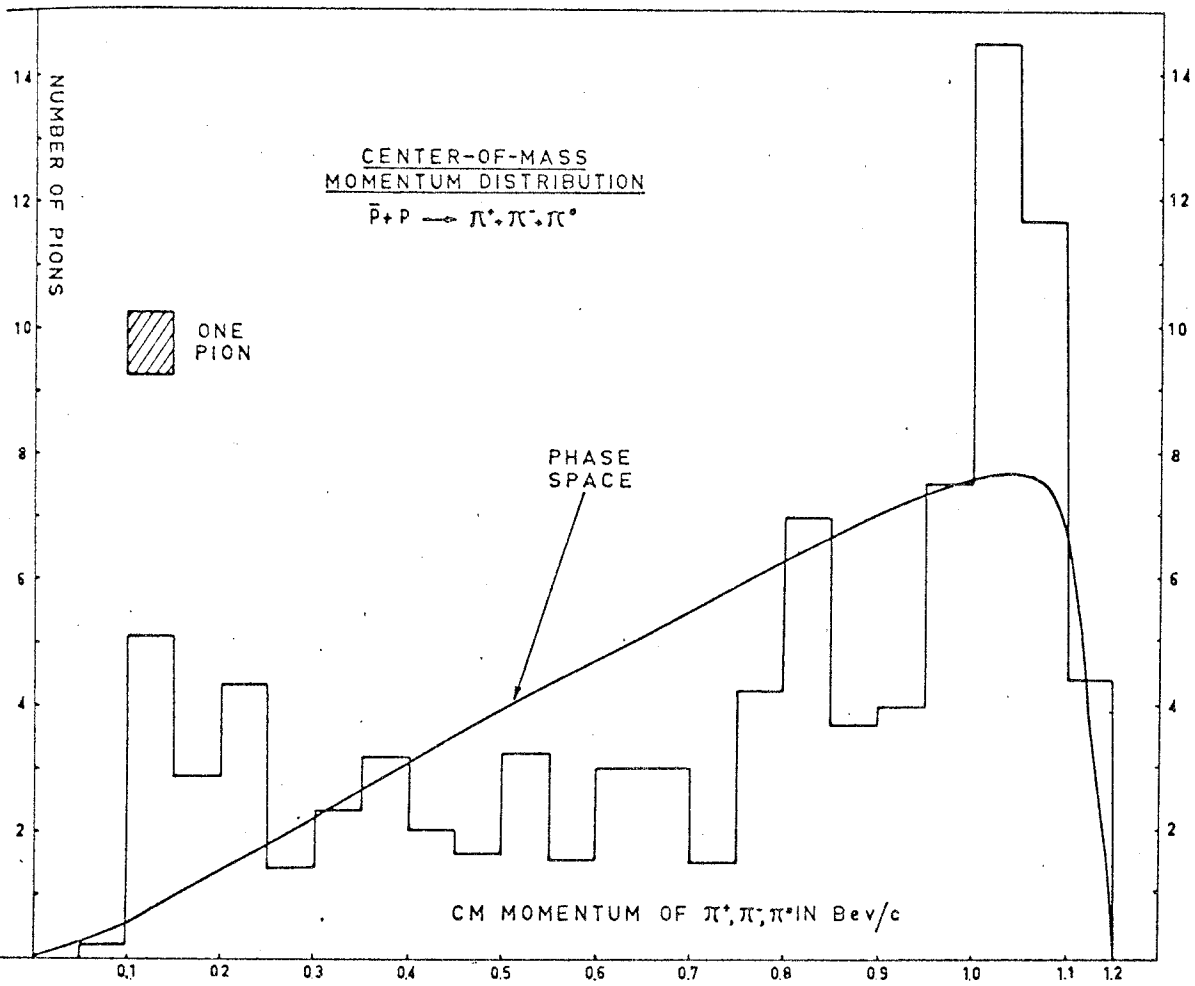

Fig. 5. CM momentum distribution of pions from annihilation.