I. THE STATISTICAL POSTULATE

Consider the collision of two particles with four-momenta $p_1$, $p_2$. The final state is in general a superposition of many states with different numbers of particles and with different momentum configurations. If $k_1, \ldots, k_n$ is a configuration of four-momenta for the final state, the transition amplitude is given by

$$S_{f1} = < f | S | i > = < p_1, p_2, \alpha | S | k_1, \ldots, k_n, \beta >$$

$$= u_1^+ u_2^+ v_1 \ldots v_n \tilde{T}_{\alpha \beta}(p_1, p_2; k_1, \ldots, k_n)$$

where $u$, $v$ are the "wave functions" of the respective particles and $T_{\alpha \beta}(p_1, p_2; k)$ is a function of the four-momenta, $\alpha, \beta$ being the additional labels (spin, isotopic spin, etc.) needed to specify the states completely. To simplify our notation let us consider the particles to be spin zero distinguishable particles with no internal symmetries. Then according to the usual conventions of relativistic quantum mechanics:

$$u_i = \frac{1}{\sqrt{2p_i^0}} \quad v_i = \frac{1}{\sqrt{2k_i^0}}.$$

Energy momentum conservation requires that $T(p_1, p_2; k)$ contain the factor $\delta(k_1 + \ldots + k_n - p_1 - p_2)$ and invariance under homogeneous Lorentz transformations requires it be an invariant function. Using a familiar result from tensor analysis it follows that $T(p_1, p_2; k)$ defined by $\tilde{T}(p_1, p_2; k) = \delta(k - p)T(p_2, k)$ is a function of only the scalar products $p_1 \cdot p_j$, $p_1 \cdot k$, $k \cdot k$ not all of which are independent. The transition rate to a set of states with momenta $k_i$ lying in a suitable interval $K(n)$ is proportional to the covariant integral:

$$\int_{K(n)} |T(p_1, p_2; k)|^2 d^4 k_i \Theta(k_i^0) \delta(k_i^2 - m_i^2) \ldots \delta(k_1^2 + \ldots + k_n^2 - p_1 - p_2)$$
where \( m_i \) is the mass of the \( i \)th particle. Note that we get an absolute "rate" only after adopting an appropriate convention for the incident "flux".

The statistical postulate of "equal opportunity" may now be invoked except for one minor difficulty: the various "rates" \( R(K(n)) \) must all have the same dimension and this requires that \( |T(p,k)|^2 \) must have dimensions dependent on the number of particles in the final state; more specifically, it must have the dimensions of the \( 2n-4 \)th power of a mass (we take \( h = c = 1 \) so that \( \text{length}^{-1} = \text{mass} = \text{momentum} = \text{energy} \)). To be able to formulate the statistical postulate we thus need a "scaling parameter". Two natural choices suggest themselves: the first one is obtained from a comparison with covariant perturbation theory, and the second by comparison with the usual formula for statistical weights.

Let us consider an arbitrary Feynman diagram contributing to the meson production process involving \( n \) particles in the final state; and we shall consider only interactions with dimensionless coupling constants (the so-called renormalizable interactions or interactions of the first kind).

Then there is a corresponding Feynman diagram for the production of \( n+1 \) mesons obtained by taking any nucleon propagator and attaching an external meson line to it. This modification introduces into the matrix element an extra factor of the form \( S_F \cdot g(2\pi)^{-2n} \cdot u \), where \( g \) is the coupling constant and \( S_F \) is a fermion propagator. The scaling parameter \( \kappa^2 \) in \( |T(p,k)|^2 \) in this case is simply

\[
\kappa^2 = 2\pi^2 \left( \frac{g^2}{4\pi} \right) \left| S_F \right|^2 \]

and has the dimensions of the square of a mass, but it is also proportional to the coupling strength \( (g^2/4\pi) \). Of course, this weak coupling approximation is hardly justified in the case of the multiple production process, but it does give the interpretation of the parameter \( \kappa^2 \) as the probability for an additional propagator (which is proportional to the square of the propagator and to the coupling strength). If \( g \) is taken to be the renormalized pion-nucleon coupling constant \( \frac{1}{2\pi} (g^2/4\pi) \) is of the order of unity and the parameter \( \kappa \) then corresponds to the average degree of deviation of the virtual nucleon from its mass shell for the dominant matrix elements. It is then to be expected that \( \kappa \) must be of the order of several pion masses. This is, in fact, the case. The statistical postulate now reads\(^{1,2}\)

\[
|T(p,k)|^2 \kappa^{4-2n} = \text{constant.}
\]
An alternative choice of the scaling parameter is obtained by considering the transition rate in the form

$$R(K_{(n)}) = \text{const.} \int \frac{|<p|S|k>|^2 \, d^3k}{K_{(n)}}$$

and considering the final states to form a discrete set corresponding to the particles in the final state normalized in a volume $\Omega$ so that

$$\frac{1}{(2\pi)^3} \int \ldots \, d^3k \rightarrow \frac{1}{\Omega} \sum \ldots$$

and the scaling parameter is now the volume $\Omega/(2\pi)^3$. The statistical postulate now may be stated as

$$|<p|S|k>|^2 \frac{[\Omega/(2\pi)^3]^n}{\Omega} = \text{constant}.$$ 

Comparing the two choices of the scaling parameters we find the correspondence

$$k^2(2k^0)^n \rightarrow (2\pi)^3/\Omega$$

where $k^0$ is the typical energy of a particle in the final state; the correspondence is thus energy dependent. The choice of one or the other version seems a matter of taste at this point; Fermi originally chose the second version with $\Omega$ to be the volume of a "Lorentz contracted" sphere of radius equal to the pion Compton wavelength. We may find the first choice desirable since it leads to simpler computational algorithms$^1,^3$.

We are now in a position to consider the question of spin and internal symmetries. If the particles in the final state have spin, one must sum over the spin labels; and similarly for internal symmetry labels like isotopic spin. The spin and isotopic spin of the initial state is either specified or it is to be averaged over, as the case may be. But the question of the summation of the spin and isotopic spin labels is complicated by the conservation laws. For the isotopic spin label this is simpler: we must conserve the isotopic spin in the initial and final state. For two particles only in the final state this completely fixes the isotopic spin wave function of the final state; but for more than two particles this is not, in general, true. There will be several isotopic spin eigenfunctions
with the same total isotopic spin (and the "third component"). In the spirit of the statistical theory every independent normalized isotopic wave function has the same weight. If one is interested only in the multiplicity without regard to the charge (i.e., isotopic spin) labels, the number of such independent isotopic spin functions may be counted \(^4\) by an elementary method \(^5\). If more detailed information on the charge complexities needed it can be worked out using standard but tedious methods; a more elegant method which substantially reduces the labour of computations has been presented by Cerulus \(^6\). This method, involving invariant group integrations as projections, is ideally suited for a statistical theory in which all isotopic spin wave functions are given equal weight; it must be modified in those cases where an isotopic spin dependence is introduced into transition matrix elements.

In addition to these complications one must further take into account the indistinguishability of the particles in the final states; and this may be done by treating them as identical, but for every set of \(n_i\) particles of type \(i\) include a factor \(1/n_i!\) in the transition rate formula. Combining the multiplicity \(M(n_i, I)\) of functions with the same isotopic spin and the indistinguishability factors \(1/n_i!\) we get a "symmetry factor" \(^4\)

\[
S([n], I) = \frac{M([n], I)}{\prod_i n_i!}
\]

We may consider this factor as representing the integration over the discrete "phase space" for the internal symmetry labels.

The question of the spin labels is more complicated since "spin" and "orbital" angular momenta are coupled; if one made the unrealistic assumption that this coupling can be disregarded so that "spin" can be treated by itself, the treatment adopted for the isotopic spin can be extended to this case also: one merely has to multiply the symmetry factor by the multiplicity of spin wave functions for a total spin \(S\). Since only nucleons have spin (pions being spinless), if we disregard nucleon pair production there are at most two spinning particles and the corresponding multiplicity is zero or one; and correct results are obtained by completely disregarding spin. This is what is usually done in standard statistical theories, though it is hardly realistic.
We are now in a position to make detailed predictions. One can calculate the relative probability for any final state (or rather a group of final states with non-zero measure) and this is equal to the phase space measure for this final state configuration, with due account of the symmetry factor and "scaled" appropriately. In particular, the total transition rate to a state of \( n \) particles is obtained by integrating the phase space factor over the entire domain which conserves energy and momentum. We may also calculate angular correlations, energy spectra, two-particle and three-particle effective mass distributions, etc. These may be less "justified" than the calculation of total transition rates but are nevertheless direct consequences of the statistical postulate. Thus, for example, the unnormalized two-particle effective mass distribution is given by

\[
P_2(\mu^2) d(\mu^2) = \int \delta(k-p) \delta\left(\left[ k_{n-1} + k_n \right]^2 - \mu^2 \right) S([n],1) \times
\]

\[
\times \prod_i d^3k_i \delta(k_i^2 - m_i^2) \Theta(k_i^0) d(\mu^2)
\]

in the covariant model. Similar expressions can be written down for the non-covariant Fermi model

\[
P_2(\mu^2) d(\mu^2) = \int \delta(k-p) \delta\left(\left[ k_{n-1} + k_n \right]^2 - \mu^2 \right) S([n],1) \times
\]

\[
\times \prod_i d^3k_i d(\mu^2)
\]

This statistical postulate (in either version) of "all or none", i.e., of equal probability for all allowed configurations, thus yields explicit formulae for the (unnormalized) spectra involving only the "phase space integrals"; and the major task of computation is obtaining an accurate value for these integrals. This is hardly an elementary task but can be accomplished by numerical methods; all the approximate analytic expressions obtained to date have proved too inaccurate. The best numerical results for the Fermi model have been obtained by Hagedorn. The covariant model is somewhat simpler for computations in view of the recurrence relations. It may be remarked here that in view of the interpretation of the scaling parameter (in the covariant model) as a weighted propagator squared, it is
not unnatural to ascribe different scaling parameters to different kinds of particles, particularly the strange particles.

II. SURVEY OF EXPERIMENTAL DATA ON MULTIPLE PROCESSES

In attempting a comparison with experiment we are hampered by the lack of precise and detailed experimental results on the one hand, and the labour of computations on the other hand. Most of the comparisons have been made with the approximate phase space integrals and these comparisons do not necessarily provide an evaluation of the statistical theory itself, as has been emphasized by Hagedorn. Before reviewing these comparisons, let us discuss some immediate qualitative features of the theory.

1) Since the probability is independent of the orientation of the momentum configurations of the final state in the c.m. frame, it follows that the angular distribution in c.m. frame must be isotropic; and more generally there must be no distinguished direction (i.e., no correlations) in the c.m. frame. Of course, the conservation laws impose mutual correlations between particles.

2) Since the probability is independent of the charge labels of the particles the various kinds of mesons must have the same energy spectra, etc. We may refer to this as "strong charge independence". Again, branching ratios will be determined by isotopic spin considerations, but, provided the multiplicities are held fixed, the spectra must be the same. Of course, the spectra will depend on the over-all multiplicity, and will not be the same if one sums over all multiplicities.

3) Since the matrix elements have no momentum dependence, all spectra are purely kinematical and as such expected to have only smooth dependences; they will in general have only a single maximum dictated by the energy-momentum conservation law.

4) Since all particles in the final state are treated on an equal footing, the energy distribution in the final state is dependent only on the masses; the energy carried by the created particles is a significant
fraction of the total available energy. Consequently, the inelasticity is expected to be large.

Since the (squared) matrix element is independent of the various particle labels, it is automatically symmetric in all the particles; hence taking account of the symmetry of the particles in the final state is not expected to introduce any modifications except to multiply the transition rate by the symmetry factor $S([n], I)$. This leaves the spectra unchanged and only serves to change the branching ratios for different multiplicity groups.

Almost all these qualitative results are at variance with experiment. For threshold production one would have expected the statistical postulate to hold; but the strong energy, spin and isotopic spin dependence of the low-energy pion-nucleon interactions make this otherwise. Thus, for example, in the range 0.9 GeV to 1.5 GeV laboratory energies for pion-nucleon collisions, one finds the momentum spectra of the secondary pions in marked disagreement with experiment, but the recoil nucleon spectra are consistent; in general the statistical theory predicts energies of the pions too high. The ratio of single to double and higher multiplicity productions is also at variance with experiment, the theory preferring too low a multiplicity. At higher energies (6.2 GeV and 9 GeV) there is distinct disagreement with experiment as regards the expected multiplicity (and branching ratios), the theory again predicting too small a multiplicity. Typical multiplicities are given in Table I. The angular distributions are definitely peaked along the axis of collision and the inelasticity is of the order of 0.5, but the experimental multiplicities are perhaps not accurate enough to supply any real comparison. Finally, in the process of antinucleon annihilation the multiplicity is greatly underestimated by the theory (prediction of about 3.3 against the experimental number of about 5.5); and the angular correlations between pions as well as the effective two-particle mass distribution plots show a distinct dependence on the charge labels of the particles selected. This last example shows the shortcomings of the simple statistical theory most graphically. And in the extremely high-energy region there is some evidence that there seems to be a limiting value for the pion momenta in the c.m. frame. This may be related to the existence of a fundamental length.
### Table I

Typical pion multiplicities in N-N and π-N collisions

<table>
<thead>
<tr>
<th>Energy (GeV)</th>
<th>Collision</th>
<th>Average multiplicity</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>π-p</td>
<td>2.3</td>
<td>Maenchen et al. a)</td>
</tr>
<tr>
<td>7</td>
<td>π-N</td>
<td>4.8</td>
<td>Petrzilka b)</td>
</tr>
<tr>
<td>6.2</td>
<td>p-N</td>
<td>2.27 ± 0.27</td>
<td>Daniel et al. c)</td>
</tr>
<tr>
<td>6.2</td>
<td>p-N</td>
<td>2.7 ± 0.5</td>
<td>Kalbach et al. d)</td>
</tr>
<tr>
<td>9</td>
<td>p-p</td>
<td>3.0</td>
<td>Van Shu Feng et al. e)</td>
</tr>
</tbody>
</table>


Modifications of the simple statistical theory to account for some or all of these features have been attempted by many authors. The phenomenological modification of replacing Ω by nΩ or of κ by κ/√n introduced by Pomeranchuk goes in the right direction in increasing the relative weights of higher multiplicity states and gives good agreement for the average multiplicities. This simultaneously improves the pion momentum spectra. But this modified theory leaves unanswered the other two difficulties; namely, does it take into account the low-energy pion-nucleon resonance, and one hardly expects the detailed pion spectra and effective mass spectra to be correct without this.

A class of modifications of the statistical theory which enjoys good fit with experiment idealizes this strong interaction in terms of a quasi-bound state of a nucleon and pion with spin and isotopic spin ½ which is treated as an independent particle. The isobar is then supposed to decay independently. This treatment is justified by saying that the "lifetime" of the isobar is large compared with the "collision time"; but with
too much justification one may simply consider this as a proper form for
the matrix element in the presence of the resonant pion-nucleon interaction.
If the resonance is "narrow enough" the interference terms between the re-
sonant and non-resonant parts of the transition amplitude may be neglected
so that one can talk in terms of probabilities. Isobar calculations$^{13}$
have been carried out for pion production in nucleon-nucleon collisions and
pion-nucleon collisions using the $(3/2, 3/2)$ isobar and provide good fit with
experimental data in the region of single and double pion production range
(0.9 GeV to 1.5 GeV) and reproduce the double humped pion momentum spectra$^{14}$.
For meson production at higher energies (6.2 GeV and 9 GeV) the multipli-
cities are fitted by the theoretical predictions involving the isobar. But
just as for the Pomeranchuk modifications here also the anisotropy and the
inelasticity are not explained at all.

One is also led to ask whether the higher resonances in the pion-
nucleon system and the inferred pion-pion resonances are to be taken into
account. Perhaps one has justification for disregarding the higher nucleon
isobars since they are rather wide; and further, at these higher energies
higher multiplicity configurations are important and one may hope that in
such configurations the resonant configurations are not as important and are
well approximated by the statistical matrix element. This is borne out to
a certain extent by the experimental data for collisions at 9 GeV$^{15}$. On
the other hand, the pion-pion interactions may be more important; they seem
to be particularly so for nucleon-antinucleon annihilation. That the
nucleon isobar model improves the fit of the single and double production
data is not surprising since low-energy data are dominated by this interac-
tion. A comparison of the calculated and experimental results for $\pi$-N and
N-N collisions is given in Figs. 1 and 2 taken from the work of Barashenkov$^{16}$.

A further test of the statistical model is furnished by comparing
the predicted and experimental total cross-sections as a function of the
energy. We discuss this briefly in connection with partial wave analysis
of these reactions below, particularly with regard to unitarity restrictions
on maximum single partial wave cross-sections.
So far we have discussed the conservation laws of energy and momentum and isotopic spin. Let us now take up the question of angular momentum conservation. We have already remarked that all the conservation laws implied by the invariance under the (proper, orthochronous) inhomogeneous Lorentz group are satisfied if the transition matrix element contains an energy-momentum delta function as a factor and is an invariant function of the four-momenta. At this point we may require a stronger condition on the transition matrix element: namely, that it is a manifestly covariant function (i.e., it does not depend explicitly on the sign of the time component of time-like four-vectors). This is true of all the current field-theoretical matrix elements, where it is called the "substitution law". Such manifestly covariant functions are functions only of the invariant scalar products $k_i^* k_j$ and the invariant pseudoscalar products $\epsilon_{\alpha \beta \gamma \delta} k_i^\alpha k_j^\beta k_k^\gamma k_c^\delta$. We are thus led to consider analytic functions of these invariants and pseudoinvariants for the transition matrix elements; any such function completely satisfies all requirements of relativistic invariance; and in particular of angular momentum conservation.

This was not the way that angular momentum conservation was first considered; Fermi introduced angular momentum into his model in terms of impact parameters, and angular momentum conservation was taken care of using the method of undetermined multipliers familiar from statistical mechanics. This view involving a literal configuration interpretation is inconsistent with quantum mechanics in the same way that a Gaussian or exponential potential gives rise to qualitatively different expressions for scattering in classical and quantum mechanics. On the other hand, the problem of treating the spins of the particles (and isobars) in the general case suggests the need for a careful treatment of angular momentum conservation.

The clue to the proper treatment is to note that a system of two colliding particles is characterized not only by the momentum and energy, but by another quantum number: the "spin" or the angular momentum in the c.m. system. The existence of the additional "mechanical" quantum number is easily seen by considering two identical sets of two particles with th
same c.m. energy but relative momenta oriented two different ways: these states are orthogonal.

The situation is complicated by the fact that the state of two particles with well-defined momenta is an eigenstate of the energy and momentum but not of "spin". But this is not new to relativistic systems; the same is true for non-relativistic particles. In the reaction leading to the multiple production the partial amplitudes for the various spins do not mix with each other. Of course, the final state is not completely specified by the energy momentum and spin in complete parallel with the non-relativistic case and the problem of isotopic spin functions.

The relevance of these considerations is that the primitive statistical postulate enunciated in Section II is highly restrictive and corresponds only to interactions in the spin 0 states. This is most easily seen by noting that the dependence of the transition matrix elements on the initial (or final) particle momenta is at most through their energies in the c.m. frame; in the covariant statistical model there is no dependence at all except in imposing the energy-momentum conservation law. Hence only the \( J = 0 \) states interact. This is not only a highly restrictive and unphysical restriction, but is inconsistent with antinucleon annihilation from the \( J \neq 0 \) nucleonium bound states!

Note that the angular momentum about which we are talking is the total angular momentum of the system, the so-called "channel spin". In addition, one also has the relative angular momenta of subgroups of particles; for example, the "spins" of two-particle sub-systems in the final state. Such dependences on intermediate spins is important particularly in the modified statistical theories where some details of the strong interactions are to be introduced: the low-energy pion-nucleon interaction is strongly dependent on the relative spin state and the resonance is present only in the \( J = \frac{3}{2} \) states. Thus it is insufficient to introduce an isobar by constraining their relative momenta; we must also constrain the "partial spin". The same is true of the pion-pion resonance. In all calculations done so far these problems have not been taken care of; and it is not clear whether this omission leaves the qualitative agreements unaltered.

The statistical postulate, in addition, makes definite statements about the dependence on partial spins. It is interesting to note that the
covariant model has no dependence on partial spins, while the Fermi
model involves a residual energy-dependent weighting of the partial spins. It
may "explain" the simplicity of the computational algorithms for the
invariant model.

But neither model is realistic; at the high energies where production processes arise there is no reason to suppose that the J = 0
waves must dominate. Even in lower energy ranges the J ≠ 0 are important and in many cases dominate. It is also to be noted that the argument J ≠ 0 is not implied, but only L = 0 (i.e., the interaction is spin independent and momentum independent), is no better since higher partial waves are certainly important; both the most important pion-nucleon and pion
resonances seem to be p-wave resonances. It is true that in antinucleon annihilation one may expect the reactions to proceed mainly from the S
state but even in this case J ≠ 0 are very important since the final state consists entirely of spinless particles and hence the complete matrix cannot be independent of the final state momenta. The only possible is for K-K annihilation in the S state, one of the least pressing inters...

We are thus led to consider the primitive statistical matrix element to be the first term in a partial wave analysis of the reaction process formula. The complete matrix element in the modified statistical model must be complicated and must involve not only the channel spin but also intermediate spins. And in addition the isotopic spin dependence of the resonances of subgroups of particles must be properly inserted. If we approximate resonant interactions by quasi-bound states in these modifications we can neglect the interference of this resonant matrix element plus the rest and treat them in terms of new "particles" and the spin and isotopic spins of the particle can be appropriately chosen. But one must choose the complete matrix element to involve appropriate "contractions" of the spin wave functions with other vectors to produce an invariant matrix element. Needless to say, the same must be done for the isotopic functions but there is no orbital contribution in this case.

For making progress one must adopt specific forms for the invariant matrix elements. In the spirit of the statistical theory we might take the "simplest" matrix elements for each partial wave consistent with the conservation laws. In particular, the higher partial waves have
initial wave function whose orbital part is a tensor of sufficiently high
rank in the c.m. relative momentum $p$ of the two colliding particles; and
this must be contracted with the final state wave function. We are thus
led to consider matrix elements of the form

$$p^*k_1 \ p^*k_2 \ p^*k_3 ; \ \ p^*k_1 \times k_2 \ p^*k_3$$

and it is easily seen that these can lead to the peaked angular distribu-
tions. The symmetry requirements on the final state may necessarily
couple the isotopic spins; for example, if $k_1$ and $k_2$ refer to two pions,
the first and second matrix elements must respectively correspond to sym-
metric ($I = 2, 0$) and to antisymmetric ($I = 1$) isotopic wave functions re-
spectively. While we have written the above matrix elements in three-
dimensional form, they are invariant since the c.m. frame is invariant;
and they can be trivially transcribed into manifestly covariant forms.

The partial wave matrix elements must in general depend upon the
c.m. energy and have different "energy" dependences for different channel
spins. One guiding principle is that they must be simple functions of the
invariants. The present understanding of the analytic structure of many-
particle amplitudes is too fragmentary to provide any guidance; and one
is best advised to choose such forms as are most amenable to computations
and represent knowledge about resonant interactions as best as one can.

Some qualitative points deserve mention: firstly, since the
channels are now labelled by c.m. energy and spin (as well as isotopic
spin), these high-energy stars admit a two-dimensional classification
rather than simply by energy; such ideas have been advanced before\textsuperscript{17}).
Secondly, the statistical matrix elements also give an expression for the
total interaction cross-sections; and unitarity sets an upper limit to the
interaction cross-sections. For two particles colliding in a state
of angular momentum $J$ the maximum cross-section consistent with unitarity is

$$\frac{4\pi}{3} \xi^2 (2J+1)$$

where $\xi$ is the isospin wavefunction dependent parameter $\xi$. A cross-section of
about 22.6 mb for 6.2 GeV nucleon-nucleon collisions is certainly inco-
herent with $J = 0$ interactions (see Tables II, III and IV). Finally, the
higher partial waves lead, in general, to the collimation of the produced
mesons into axial cones; the quantitative features must, of course, depend on the specific matrix elements (compare Tables V and VIII).

We have thus arrived at the conclusion that the primitive statistical model matrix element is the first term in an expansion in partial waves and one must include contribution from higher partial waves for even qualitative agreement with experiment as regards total cross-sections and axial collimation. In this connection it is interesting to note the observations of Daniel et al.\textsuperscript{10) who find isotropy for large multiplicity and axial collimation for low multiplicity; these may be interpreted to take place in low and high orbital angular momentum channels respectively. The detailed predictions and numerical results do depend on the full matrix element; the situation is thus very similar to, say, nucleon-nucleon scattering results analysed in terms of partial wave phase shifts. In the latter case there is a limit to the number of partial waves one can reasonably include and the present problem is similar.

Table II

Typical total π-N and N-N cross-sections

<table>
<thead>
<tr>
<th>Energy (GeV)</th>
<th>Collision</th>
<th>Total cross-section (mb)</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>π−p</td>
<td>22.5 ± 2.4</td>
<td>Maenchen et al.\textsuperscript{a)}</td>
</tr>
<tr>
<td>7</td>
<td>π-N</td>
<td>23 ± 3</td>
<td>Petrzelka\textsuperscript{b)}</td>
</tr>
<tr>
<td>2.75</td>
<td>p-p</td>
<td>15</td>
<td>Block et al.\textsuperscript{c)}</td>
</tr>
<tr>
<td>5</td>
<td>p-p</td>
<td>43.7 ± 0.7</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>p-p</td>
<td>43.3 ± 0.4</td>
<td>von Dardel et al.\textsuperscript{d)}</td>
</tr>
<tr>
<td>10</td>
<td>p-p</td>
<td>42.1 ± 0.4</td>
<td></td>
</tr>
</tbody>
</table>

\textsuperscript{a)} Maenchen, Fowler, Powell and Wright, Phys.Rev. 108, 850 (1957).
\textsuperscript{b)} Petrzelka, Proc. of the 1960 Ann.Int.Conf. on High-Energy Physics, Session S1.
\textsuperscript{c)} Block, Harth, Cecconi, Hart, Fowler, Shutt, Thorndike and Whittemore, Phys.Rev. 102, 1484 (1956).
Table III
Proton-proton elastic scattering cross-section measurements from 0.81 to 6.2 GeV

<table>
<thead>
<tr>
<th>Energy (GeV)</th>
<th>Method</th>
<th>Cross-section (mb)</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.81</td>
<td>Cloud chamber</td>
<td>24 ± 2</td>
<td>Morris et al. a)</td>
</tr>
<tr>
<td>0.925</td>
<td>Emulsion</td>
<td>17 ± 3</td>
<td>Duke et al. b)</td>
</tr>
<tr>
<td>1.00</td>
<td>Counter</td>
<td>19 ± 3</td>
<td>Smith et al. c)</td>
</tr>
<tr>
<td>1.50</td>
<td>Cloud chamber</td>
<td>15 ± 2</td>
<td>Fowler et al. d)</td>
</tr>
<tr>
<td>2.24</td>
<td>Counter</td>
<td>17</td>
<td>Cork et al. e)</td>
</tr>
<tr>
<td>2.75</td>
<td>Cloud chamber</td>
<td>15 ± 2</td>
<td>Block et al. f)</td>
</tr>
<tr>
<td>2.85</td>
<td>Cloud chamber</td>
<td>17.3 ± 1.5</td>
<td>Smith et al. g)</td>
</tr>
<tr>
<td>3.00</td>
<td>Emulsion</td>
<td>8.9 ± 1.0</td>
<td>Cester et al. h)</td>
</tr>
<tr>
<td>4.4</td>
<td>Counter</td>
<td>10</td>
<td>Cork et al. e)</td>
</tr>
<tr>
<td>5.7</td>
<td>Emulsion</td>
<td>13 ± 6</td>
<td>Giles i)</td>
</tr>
<tr>
<td>6.15</td>
<td>Counter</td>
<td>8</td>
<td>Cork et al. e)</td>
</tr>
<tr>
<td>6.2</td>
<td>Emulsion</td>
<td>8.8 ± 2.8</td>
<td>Kalbach et al. j)</td>
</tr>
<tr>
<td>8.2</td>
<td>Emulsion</td>
<td>8.7 ± 0.4</td>
<td>Azimov et al. k)</td>
</tr>
</tbody>
</table>

i) P.C. Giles, UCRL-3223.
Table IV

Typical antiproton total cross-sections

<table>
<thead>
<tr>
<th>Momentum (GeV/c)</th>
<th>Cross-section (mb)</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$49 \pm 6$</td>
<td>Armenteros et al. a)</td>
</tr>
<tr>
<td>5</td>
<td>$67.0 \pm 2.1$</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>$63.0 \pm 2.3$</td>
<td>von Dardel et al. b)</td>
</tr>
<tr>
<td>10</td>
<td>$46.0 \pm 2.0$</td>
<td></td>
</tr>
<tr>
<td>10.7</td>
<td>$53.0 \pm 1.1$</td>
<td></td>
</tr>
</tbody>
</table>


In the nucleon-nucleon scattering problem there is a possibility of including the higher angular momentum contribution without expanding it into partial waves by explicitly taking account of the one-pion exchange contribution. In terms of analytic functions this corresponds to taking the nearest singularities in the $(\cos \Theta)$ plane (where $\Theta$ is the scattering angle); but this may be restated in a potential language as taking account of the longest range ("peripheral") interactions. This allows one to take fewer phenomenological phase shifts to achieve the same fit with experiments.

A similar procedure is suggested here also. The higher partial waves may be taken phenomenologically in terms of one-particle poles, so that we consider two subgroups of the final particles with partial statistical matrix elements connected by a single pion propagator. The magnitude of the matrix element will depend on the magnitude of the propagator and the dominant contributions will come from almost real pions. With such a modification, retaining a few partial waves in the purely phenomenological term would be expected to be a fair approximation. Thus we are led to the single-pion exchange models as a simple refinement of the primitive statistical theory to bring it in closer agreement with the actual situation.
The verifies that the one-pion exchange contribution generates higher partial waves and axial collimation of the final state particles and provides through total cross-section.

To be complete one must take more complicated groupings, but it to be expected that the primitive matrix element along with the one-pion change contribution and any isobars is sufficiently detailed to provide a satisfactory model.

IV. ANTINUCLEON ANNIHILATION

The multiple production of mesons in antinucleon annihilation is perhaps the most ideal case for the application of a statistical theory. The final state consists only of pions; and the isotopic spin and extended charge conjugation selection rules can be easily accommodated. The initial nucleon pair can annihilate at essentially zero kinetic energy. This phenomenon lends itself to a careful experimental study since the c.m. frame is practically the same as the laboratory frame. And a great deal of experimental results are available on this process.

The earliest calculations\(^4\),\(^5\),\(^10\) using the Fermi model gave multiplicities of the order of 3, too small compared with the observed\(^10\) multiplicity of about 5. In these calculations the symmetry factors were properly included and the standard Fermi volume \(\Omega_0\) was used. Recalculation of these results using the covariant model seems to make no great difference\(^8\). To obtain agreement with observed multiplicities it was necessary to take \(\Omega_0\) which seems difficult to justify directly. There are a great number of selection rules applicable to this reaction (see below) but their inclusion did not improve the simple statistical model calculations. On the other hand, we already know that the primitive statistical matrix element is inadequate; we must include appropriate channel spin and isotopic spin as well as include resonant interactions. The introduction of resonances by introducing isobars has been tried by various authors. Thus Cras,\(^6\) introduced an \(I = 1, J = 1, G = +1\) pion isobar with a mass of 3 pion masses and was able to fit the multiplicity. A simpler method involves taking the Pomeranchuk version of the Fermi model, i.e., \(\Omega = n\Omega_0\)
and this gives the correct multiplicity\(^a\). But from our previous section this treatment of the annihilation matrix element is particularly justified.

This is most clearly indicated by the observation by and co-workers\(^{12}\) that the angular correlations of the annihilation are different for pions of like and unlike charges, the like pions being more strongly correlated. The forward-to-backward ratio for pions was 1.23 ± 0.10 and for unlike pions 2.18 ± 0.12 which is to be compared with the value 1.80 predicted by the covariant model. A relation is the very sharp maximum in the c.m. momentum distribution of charged pions (which is equivalent to a maximum in the two-pion mass distribution around 4.7 ± 0.2 pion masses), the maximum being pronounced in the \(\pi^+\pi^0\) and \(\pi^-\pi^0\) distributions for the three-body state in the reaction

\[
p + \bar{p} \rightarrow \pi^+ + \pi^- + \pi^0.
\]

These considerations suggest that the pion-pion interaction is spin dependent. The calculations of Cerulus\(^6\) and of Pinsker\(^*\) took such an interaction. Maglić has also observed a marked correlation of the pions with the direction of the incident antiprotons at 1.65 GeV/c. He finds that the negative pions tend to go forward, while the positive pions show the opposite tendency. The degree of correlation is given in the following table.

<table>
<thead>
<tr>
<th>Type</th>
<th>Experimental correlation</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\bar{p} \pi^-)</td>
<td>(1 + (0.25 \pm 0.08) \cos \Theta + (0.415 \pm 0.12) \cos 2 \Theta)</td>
</tr>
<tr>
<td>(\bar{p} \pi^+)</td>
<td>(1 - (0.25 \pm 0.08) \cos \Theta + (0.415 \pm 0.12) \cos 2 \Theta)</td>
</tr>
</tbody>
</table>

As usual, the statistical theory (or any of its modifications designed to decrease the multiplicity) will have no explanation for this effect.
In passing we note that there were two rather detailed theories for explaining the large multiplicity and the large antiproton total cross-sections, one by Ball and Chew \(^{21}\) and the other by Koba and Takeda \(^{22}\). Both these theories provide detailed mechanisms for the reactions. The first one is aimed at explaining the interaction cross-sections rather than the multiplicity. The latter explains the multiplicity by a two-stage mechanism, a "core-annihilation" and a "cloud dispersion". In this case again while the multiplicity is explained the dependence of the angular correlations, etc., cannot be explained; but these authors note the necessity to introduce higher partial waves to explain the total cross-sections.

Let us now briefly recall the selection rules in nucleon-antinucleon annihilation. The invariance of the strong interactions under charge conjugation and isotopic spin rotations lead to the conservation of the extended charge conjugation parity (G-parity). For a state of the nucleon-antinucleon system with orbital angular momentum L, spin S, and isotopic spin I, the G-parity is \((-1)^{L+S+I}\) while for n pions it is \((-1)^n\). Hence we must have \(L+S+I = n\) (modulo 2). On the other hand, a two-spin system with angular momentum \(J\) has space parity \((-1)^J\) so that a state with \(J = L\) of the nucleon-antinucleon system cannot decay into two pions. (Recall that a Dirac particle-antiparticle system has odd intrinsic parity.) In addition the \(^3P_0\) state cannot decay into three pions from parity conservation. Consequently the several states cannot decay into less than four pions (see Table VI).

<table>
<thead>
<tr>
<th>Table VI</th>
</tr>
</thead>
</table>

**Minimum number of pions in antinucleon annihilation**

<table>
<thead>
<tr>
<th>State</th>
<th>(I = 0)</th>
<th>(I = 1)</th>
<th>Minimum no. of pions</th>
</tr>
</thead>
<tbody>
<tr>
<td>(^1S_0)</td>
<td>+</td>
<td>-</td>
<td>2 3</td>
</tr>
<tr>
<td>(^3S_1)</td>
<td>-</td>
<td>+</td>
<td>3 2</td>
</tr>
<tr>
<td>(^1P_1)</td>
<td>-</td>
<td>+</td>
<td>3 4</td>
</tr>
<tr>
<td>(^3P_0)</td>
<td>+</td>
<td>-</td>
<td>4 4</td>
</tr>
<tr>
<td>(^3P_1)</td>
<td>+</td>
<td>-</td>
<td>4 3</td>
</tr>
<tr>
<td>(^3P_2)</td>
<td>+</td>
<td>-</td>
<td>2 3</td>
</tr>
</tbody>
</table>
Note that we must use the complete channel specification to fully utilize these selection rules. In general one does not know in antiproton annihilation the labels for the state from which capture takes place. But the important role played by the Stark effect shows that the capture will take place predominantly from the $S$ states, according to the calculations of Desai \(^{23}\); both $^3S_1$ and $^1S_0$ contribute. And the $pp$ state is an equal mixture of $I = 0$ and $I = 1$.

We have remarked before the need for taking a large "volume" explaining the observed multiplicities by a statistical theory. Since covariant model is more convenient for calculations it has sometimes been considered significant to introduce a volume parameter into the covariant model by the correspondence

$$2m \pi^2 \leftrightarrow \frac{(2\pi)^3}{\Omega}.$$  

The calculated values (for various choices of this adjusted volume $\Omega'$) of the charged pion multiplicities in antinucleon annihilation are compared with the latest available experimental data by Lynch \(^{24}\) (Fig. 3) and show very good agreement for a choice

$$\Omega' \approx 5\Omega_0$$

which is significantly smaller than previously quoted values. More data specification of the annihilation at 1.61 GeV/c in terms of charged pion branching ratio is also given \(^{24}\). (See Table VII)

**Table VII**

<table>
<thead>
<tr>
<th>Type of event</th>
<th>Measured fraction of pion annihilations in per cent</th>
<th>Values of $\Omega'$ needed to fit the measurement</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-prong</td>
<td>$1.0 \pm 0.3$</td>
<td>$4.6 \pm 1.3$</td>
</tr>
<tr>
<td></td>
<td>$-0.6$</td>
<td></td>
</tr>
<tr>
<td>2-prong</td>
<td>$36.0 \pm 5.2$</td>
<td>$5.1 \pm 0.9$</td>
</tr>
<tr>
<td>4-prong</td>
<td>$54.6 \pm 1.3$</td>
<td>$4.6 \pm 0.3$</td>
</tr>
<tr>
<td>6-prong</td>
<td>$8.4 \pm 0.3$</td>
<td>$5.8 \pm 0.7$</td>
</tr>
<tr>
<td>8-prong</td>
<td>$0.15 \pm 0.4$</td>
<td>$4.8 \pm 0.4$</td>
</tr>
<tr>
<td>$\pi^+\pi^-$</td>
<td>$0.20 \pm 0.05$</td>
<td></td>
</tr>
</tbody>
</table>
We shall now report briefly on a calculation by G. Pinski in progress at Rochester. The model aims at using the multiplicity, angular correlations and effective mass distributions as a method of finding evidence for the pion-pion interaction and to determine its nature. The basic idea is similar to that of Cerulus but all the calculations are done covariantly. The usual license for using probabilities rather than matrix elements is invoked and the partial spins have been treated only approximately, but isotopic spin weights have been carefully included. Rather than calculate energy spectra of particles weighted over many multiplicities, the distribution in energy and two-particle masses has been treated for charge-identified prongs in events with small multiplicity in terms of appropriate phase space integrals.

The relative transition probability for annihilation into n pions in a configuration \( K(n) \) (including the momentum and isotopic spin labels) is given by

\[
R(K(n)) = \int \Lambda(K(n)) \prod_{i=1}^{n} d^4k_i \Theta(k_i^0) \delta(k_i^2 - m_i^2) \times \delta(k - p_1 - p_2) f(p_1, p_2, k, \xi)
\]

where \( \Lambda \) is a projection operator to the configuration \( K(n) \), \( f(p_1, p_2, k, \xi) \) is an arbitrary invariant function of the momenta and of the isotopic spin labels determined by the dynamics of the problem. The covariant statistical theory implies that \( f(p_1, p_2, k, \xi) \) is a constant; the "resonant" model obtained by choosing

\[
f = 1 + \lambda^2 \sum_{i>j} \mathcal{F} \left( (p_i + p_j)^2 - M^2 \right) \Lambda_{ij}
\]

can be used to represent the suggested \( T = J = 1 \) resonance by requiring \( \Lambda_{ij} \) to vanish unless the partial "spin" and isotopic spin are both unity. Here both \( \lambda^2 \) and \( M^2 \) are parameters with the dimensions of the square of a mass; and \( \mathcal{F} \) is a sharply peaked function and approximated by the smeared-out \( \delta \)-function

\[
\mathcal{F}(\mu^2 - M^2) = \frac{1}{2M} \frac{\epsilon / \pi}{(\mu - M)^2 + \epsilon^2}
\]
where $\epsilon$ is the half-width. In our calculation we took $\epsilon$ to be half a mass and $M$ to be four pion masses.

We have already remarked about the complications introduced by channel spin; to make progress we had to ignore the restriction $J = 1$ proceed to do the numerical calculations. Hence the results of our calculation do not test the $J = 1$ restriction; but the qualitatively inter predictions made must survive in the more complicated exact treatment. reservation is nevertheless to be borne in mind.

Let us now compute the effective mass of a pair of particles; distribution can be separated into two parts, one coming through the non resonant part of $f(p_1, p_2, k, \zeta)$ and the other through the resonant terms proportional to $\lambda^2$. The non-resonant term is identical with what has been calculated in Section I; so we need to compute only the coefficient of in the (unnormalized) effective mass distribution. There are three distinct cases according to whether the particles are chosen both resonant only one resonant, or neither resonant. We denote these terms by $P_{rr}$, $P_{rf}$ and $P_{ff}$ respectively; they are proportional to

$$P_{rr} \sim \int \prod_{i=1}^{n} \left[ d^4 k_i \delta(k_i^2 - m_i^2) \Theta(k_i^2) \right] \delta(\Sigma k_i - p_1 - p_2) \times$$

$$\times \mathcal{H}(\mu^2, \lambda^2) \delta([k_{n-1} + k_n]^2 - \mu^2)$$

$$P_{rf} \sim \int \prod_{i=1}^{n} \left[ d^4 k_i \delta(k_i^2 - m_i^2) \Theta(k_i^2) \right] \delta(\Sigma k_i - p_1 - p_2) \times$$

$$\times \mathcal{H}([k_{n-1} + k_{n-2}]^2 - \mu^2) \delta([k_{n-1} + k_n]^2 - \mu^2)$$

$$P_{ff} \sim \int \prod_{i=1}^{n} \left[ d^4 k_i \delta(k_i^2 - m_i^2) \Theta(k_i^2) \right] \delta(\Sigma k_i - p_1 - p_2) \times$$

$$\times \mathcal{H}([k_{n-3} + k_{n-2}]^2 - \mu^2) \delta([k_{n-3} + k_n]^2 - \mu^2)$$

respectively. All these covariant integrals can be computed (using appropriate recurrence relations) by numerical methods. To complete the calculation it is necessary to determine the relative weights of these three contributions in the resonant part of the effective mass distributions; and the contributions correspond to the factor $A_{n-1, n}$ in the resonance function.
to compute the relative weights of $P_{rr}$, $P_{rf}$, and $P_{ff}$ for two particles (of
known charges) we compute the expectation value of $A(n,I)A_{n',n-1}$ for the iso-
topic spin wave function of the $n$ pions where the factor $A(n,I)$ is the pro-
jection operator to a state of total isotopic spin $I$. These projection
operators can be written down as matrices in the charge labels $\zeta$, but in
actual practice it seems more efficient to construct actual isotopic spin
wave functions with total isotopic spin $I$ and two particles coupled to iso-
topic spin $1$. This is particularly so in the case of small multiplicities
$n = 3$ and $n = 4$. The results of the calculation of relative weights are
summarized in Table VIII. The computed squared mass distributions for
three-pion final states are illustrated in Fig. 4 and are to be compared
with the experimental results of Maglić et al. in Fig. 5.

In the resonant case we see that in $\bar{p}p$ annihilation for the $(++)$
case the normalized distribution is concentrated around the resonance mass
and the difference between $(+-)$ and $(++)$ or $(-+)$ curves is unmistakable.
for the $p\bar{p}$ annihilation into $\pi^+\pi^-\pi^0$ the resonance must show up most promi-
nently in the $\pi^+\pi^0$ distribution in view of the isotopic spin weight fac-
tors. For the four-pion modes the situation is more complicated since the
number of independent isotopic spin functions is considerably larger. (How-
ever, compare Table VI.) In general, as the number of annihilation pions
increases, the number of ways of picking the two particles also increases.
Therefore the effect of a pion-pion resonance is more prominent for smaller
multiplicities and it may be worth while to confine our attention to events
with few pions only.

A corresponding analysis can be carried out for three-pion reso-
nances also. The computations are more involved than for the two-particle
resonance but no new principle is involved. In this case again one ex-
pects to find a peaked distribution for the three-particle effective mass;
for the four-pion channel this would experimentally be seen as a peak in
the momentum distribution of pions, and should be easy to detect if it
exists.

More complete investigations including further refinements appro-
priate to a more realistic matrix element including pion-pion interactions
are suggested, particularly as regards the treatment of partial spins, exten-
ded charge conjugation, and multiple resonances. Only a complete fit with all
data is significant since it is clear that correct multiplicities can be ob-
tained by a variety of models.
<table>
<thead>
<tr>
<th>Reaction</th>
<th>Correlation</th>
<th>rr</th>
<th>rf</th>
<th>ff</th>
</tr>
</thead>
<tbody>
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<td>$\pi^+\pi^-$</td>
<td>(+-)</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\pi^+\pi^-\pi^0$</td>
<td>(+0), (0-)</td>
<td>$\frac{3}{12}$</td>
<td>$\frac{7}{12}$</td>
<td>0</td>
</tr>
<tr>
<td>$\pi^+\pi^-\pi^0$</td>
<td>(+-)</td>
<td>$\frac{1}{6}$</td>
<td>$\frac{1}{6}$</td>
<td>0</td>
</tr>
<tr>
<td>$\pi^+\pi^-\pi^0$</td>
<td>(00)</td>
<td>0</td>
<td>$\frac{1}{7}$</td>
<td>$\frac{1}{7}$</td>
</tr>
<tr>
<td>$\pi^+\pi^-\pi^0$</td>
<td>(0-)</td>
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<td>$\frac{1}{2}$</td>
<td>$\frac{1}{4}$</td>
</tr>
<tr>
<td>$\pi^0\pi^-$</td>
<td>(0-)</td>
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<td>0</td>
<td>0</td>
</tr>
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<td>$\pi^-\pi^+$</td>
<td>(+-)</td>
<td>$\frac{1}{2}$</td>
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</tr>
<tr>
<td>$\pi^-\pi^+$</td>
<td>(--)</td>
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<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$\pi^0\pi^-\pi^0$</td>
<td>(0-)</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
</tr>
<tr>
<td>$\pi^0\pi^-\pi^0$</td>
<td>(00)</td>
<td>0</td>
<td>1</td>
<td>0</td>
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<tr>
<td>$\pi^+\pi^-\pi^0$</td>
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<td>$\frac{1}{3}$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{3}$</td>
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<tr>
<td>$\pi^+\pi^-\pi^0$</td>
<td>(+-)</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{3}$</td>
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<tr>
<td>$\pi^+\pi^-\pi^0$</td>
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<td>$\frac{1}{3}$</td>
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<tr>
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<td>(00)</td>
<td>0</td>
<td>$\frac{1}{3}$</td>
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* * *


Fig. 1. Comparison between isobar calculations and experiments for the probabilities \( W_n \) of 2-, 4- and 6-prog star production and the average number \( \bar{n} \) of produced charged particles in \((\pi^-p)\) collisions.

Fig. 2. Comparison between isobar calculations and experiments for the probabilities \( W_n \) of 2-, 4- and 6-prog star production and the average number \( \bar{n} \) of produced charged particles in \((pp)\) collisions.
Fig. 3. Calculated values for charged pion multiplicities for various values of the adjusted volume $\Omega'$ compared to experiments.

Fig. 4. Computed squared mass distribution for three-pion final states.
Fig. 5. CM momentum distribution of pions from annihilation.