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Theory of Leptons - I (*).

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Summary. — A finite relativistic theory of four-fermion interactions is formulated; the theory involves as an essential ingredient the use of an indefinite metric. The problems of interpretation raised by the use of the indefinite metric are analyzed in relation to the observables and structure of many-particle states in quantum field theory; and the consistency of the interpretive postulate is demonstrated. The theory incidentally provides a *raison d'être* for the muon.

– Introduction.

Elementary particles exhibit an extended spectrum not only in their masses but also in their interaction properties. However, in the classification into the four groups of particles, namely photon, leptons, mesons and baryons the particles belonging to each group not only cover a well-defined range of the mass spectrum but also have fairly similar properties [1]. As far as is known at present the photon has a « universal interaction » (with all charged particles); the leptons on the other hand take part both in universal electromagnetic interaction and also universal weak interaction [2]. The mesons and baryons

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([1]) R. E. MARSHAK and E. C. G. SUDARSHAN: *Introduction to Elementary Particle Physics* (New York, 1961).

([2]) By universal we mean an « all-or-none » characterization, *i.e.*, all charged particles interact with the same strength, but no neutral particle interacts directly. Similarly, in the weak interactions only charged currents interact, the neutral do not. For a more systematic discussion, see ref. ([1]).

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take part in all interaction strong, weak and electromagnetic; but precisely because of the « strength » of their dominant interaction it is not possible to give a definitive answer to the question of a possible universal law of interaction for the strong interactions. In terms of our present understanding, the number of particles included in the elementary particle spectrum belonging to the baryon and meson classes is larger than is strictly necessary to account for the number of conserved quantities; and the question has often been raised whether some of these particles may be understood as composite objects so that the number of truly elementary particles is smaller. Stated more precisely, the question is whether one can introduce fewer elementary fields into the theory rather than introduce a distinct field associated with every particle; in principle, in a Lagrangian theory such questions can be meaningfully asked. But in practice there are two major difficulties: firstly in view of the « strength » of the interaction simple approximations are not adequate and one has to use rather ingenious computational tricks to get satisfactory results even in orthodox theory, and even more so in an attempt at answering fundamental questions. Secondly, the systematic development of the theory in a perturbation series gives divergent answers. In orthodox field theory there exists at least certain models in which these infinite quantities can be circumvented by a formal renormalization, but the mathematically ill-defined operations involved make it unclear whether the question as to which particles are elementary can still be meaningful in the renormalized theory.

However, as far the system of leptons and photon are concerned, though they have interactions with particles belonging to the baryon-meson system, thus forming only an « open system », for many purposes the lepton-photon system can be treated as if it is a « closed system ». The justification for such an approximate treatment derives mainly from the great success of quantum electrodynamics (not involving baryons and mesons) and of the weak universal $V-A$ interaction as applied to leptonic weak decays. In these cases the coupling strengths are sufficiently small for a perturbation expansion to be meaningful; so the first obstacle mentioned in the last paragraph does not arise for the photon-lepton system. But the standard divergences appear; and in a more troublesome manner as far as weak interactions are concerned since a four-fermion interaction is not « renormalizable ». Nevertheless, the calculations of the lowest order results involving weak interactions are in excellent agreement with experimental results just as in the case of electrodynamics. It then appears that there should be a formulation of the theory of weak and electromagnetic interactions of leptons somewhat different in principle in that the infinities no longer appear, but to lowest order the predictions are practically the same as given in the lowest order predictions of the usual theory.

The point of view that the formal introduction of an indefinite metric together with a corresponding interpretative postulate (defining the « physical states ») supplies the additional physical principle needed to construct a finite theory of elementary particle interactions has been emphasized in an earlier paper [3]. It is the purpose of this paper to implement the program for the weak interaction of leptons since no consistent theory of weak interactions exists at present.

In the following section the orthodox Lagrangian theory of leptonic weak four-fermion interaction is summarized with a view to establishing the notation and for easy comparison with the present theory which is introduced in Section **3**.

2. – Conventional theory of four-fermion interactions.

The conventional field theory for interacting leptons (with electromagnetic interactions omitted) is formulated in terms of the Lagrangian density [4]:

$$\mathscr{L}(x) \quad \mathscr{L}_0(x) + GJ^+_\lambda(x)J^\lambda(x)\,, \tag{1}$$

where $p = \gamma^\lambda p_\lambda$ and

$$\mathscr{L}_0(x) \quad \bar{\chi} p \chi + \overline{\psi}_1(p - m_1)\psi_1 + \overline{\psi}_2(p - m_2)\psi_2\,, \tag{2}$$

$$J_\lambda(x) \quad \overline{\psi}_1\gamma_\lambda(1+\gamma_5)\chi + \overline{\psi}_2\gamma_\lambda(1+\gamma_5)\chi \tag{3}$$

and the fields χ, ψ_1, ψ_2 refer to the neutrino, electron and muon fields and m_1, m_2 the observed electron and muon masses. We have included a four-component neutrino field (rather than a two-component field) since this gives a more uniform treatment of all the lepton fields, though in the conventional theory the negative chiral component has no interactions at all. These fields obey the anticommutation relations for equal times:

$$\begin{aligned} &\{\chi^+(\boldsymbol{x}, t),\, \chi(\boldsymbol{x}', t)\} = \delta(\boldsymbol{x} - \boldsymbol{x}')\,,\\ &\{\psi_1^+(\boldsymbol{x}, t),\, \psi_1(\boldsymbol{x}', t)\} = \delta(\boldsymbol{x} - \boldsymbol{x}')\,,\\ &\{\psi_2^+(\boldsymbol{x}, t),\, \psi_2(\boldsymbol{x}', t)\} = \delta(\boldsymbol{x} - \boldsymbol{x}')\,.\\ &\text{all other anticommutators} \quad 0\,. \end{aligned} \tag{4}$$

(3) E. C. G. SUDARSHAN: *Quantum-mechanical systems with indefinite metric* - I, submitted to *Phys. Rev.*

(4) E. C. G. SUDARSHAN and R. E. MARSHAK: *Proc. Padua-Venice Conference on Mesons and Newly Discovered Particles* (1957); *Phys. Rev.*, **109**, 1860 (1958); R. P. FEYNMAN and M. GELL-MANN: *Phys. Rev.*, **109**, 193 (1958); J. J. SAKURAI: *Nuovo Cimento*: **7**, 649 (1958).

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For this Lagrangian there are three constants of motion:

$$(5)\qquad \begin{cases} Q = \int \mathrm{d}^3x \, \{\psi_1^+(\boldsymbol{x}, t)\psi_1(\boldsymbol{x}, t) \quad \psi_2^+(\boldsymbol{x}, t)\psi_2(\boldsymbol{x}, t)\} \\ L = \int \mathrm{d}^3x \, \{\chi^+(\boldsymbol{x}, t)\chi(\boldsymbol{x}, t) + \psi_1^+(\boldsymbol{x}, t)\psi_1(\boldsymbol{x}, t) + \psi_2^+(\boldsymbol{x}, t)\psi_2(\boldsymbol{x}, t)\} \\ K = \int \mathrm{d}^3x \, \{\chi^+(\boldsymbol{x}, t)\gamma_5\chi(\boldsymbol{x}, t)\} \end{cases}$$

All of these have integral eigenvalues and refer to the total charge, the lepton number and the neutrino chiral number respectively. The weak coupling constant G has the dimensions of an area. We have, in accordance with present experimental results, included only the universal chirality-invariant $V-A$ interaction through charged currents.

Unlike the other three-field (Yukawa) interactions, the four-fermion interaction can lead to physical scattering and decay processes in the first order (first Born approximation):

$$(6)\qquad \begin{array}{ll} e + \nu \to e + \nu & e + \bar{\nu} \to e + \bar{\nu} \\ \mu + \nu \to \mu + \nu & \mu + \bar{\nu} \to \mu + \bar{\nu} \\ \mu + \nu \rightleftarrows e + \nu & \mu + \bar{\nu} \rightleftarrows e + \bar{\nu} \\ \mu + \bar{e} \rightleftarrows \nu + {}^{-} \rightleftarrows e + \bar{e} & e + \bar{\mu} \rightleftarrows \nu + \bar{\nu} \rightleftarrows \mu + \bar{\mu} \end{array}$$

$$(7)\qquad \mu \to e + \nu + \bar{\nu}$$

Of these interactions only the weak decay process (7) is experimentally tested; and here the detailed predictions regarding the energy spectrum, polarization correlations, etc., are in excellent agreement with the results of the lowest order calculations, except for small corrections. Most of these corrections have been quantitatively analyzed in terms of electromagnetic effects (5); the remainder, if any, has been the subject of several speculations concerning intermediate vector mesons (6). Since the calculation of the decay spectrum, etc., from the covariant transition matrix element for the decay reaction is available elsewhere in the literature we shall not present them here. The scattering processes (6) have not been experimentally investigated so far; in most cases

(5) Quantitative calculations have been made by S. Berman and T. Kinoshita. For a comprehensive review, see R. P. Feynman: *Proc. Tenth Annual Rochester Conference on High Energy Physics*, University of Rochester, (Rochester, 1960).

(6) See, for example, T. D. Lee: *Proc. Tenth Annual Rochester Conference on High Energy Physics*, University of Rochester, (Rochester, 1960).

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they are masked by other dominant reactions (7). By the same token « all available experimental results » are consistent with the interaction introduced above.

It is interesting to note that with the specific choice of the $V-A$ interaction there are no self-energy effects in lowest order; the only possible fermion loop contribution shown in Fig. 1 vanishes by reasons of invariance.

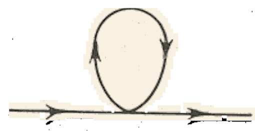

Fig. 1. – The first-order fermion self-energy diagram.

The agreement of the lowest order predictions with experimental results is gratifying especially in view of the smallness (8) of the coupling constant. However in any consistent theory one must be able to show that the higher order corrections are small; and one must be able, with sufficient labour, to compute these corrections. This is all the more important since there are small but significant deviations from the lowest order calculations (with electromagnetic corrections included) in the experimental results. These results may be summarized by saying that the effective four-fermion interaction is « slightly » non-local. Now a non-local effective interaction always results from any local interaction taken in higher orders. Some authors have considered this non-locality of the effective interaction as sufficient ground to interpret the four-fermion interaction as being mediated by the exchange of a vector meson, the basic interaction then being taken as the direct coupling of the vector meson field with the weak interaction current (3). We shall not do so, but rather consider the four-fermion interaction (1) itself to be the basic interaction; the problem then arises as to the magnitude of the higher order corrections stemming from this interaction.

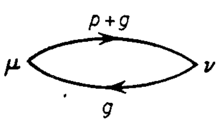

Fig. 2. – The two-vertex fermion loop.

It is however immediately verified that the expressions for higher order corrections are in general divergent and therefore meaningless as they stand. This divergence has its origin in the divergence of the two-vertex fermion loop illustrated in Fig. 2 with the corresponding matrix element proportional to the expression

$$\int d^4q \, \mathrm{Tr}\,\{\gamma^\mu(1+\gamma_5)(p+q-m)^{-1}\gamma^\nu(1+\gamma_5)(q-m')^{-1}\}\,.$$

(7) Recently H. M. Chiu has pointed out the possible relevance of the reaction $e+\bar{e}\to\nu+\bar{\nu}$ in stellar evolution; but no *experimental* information about the details of these processes (or even about their existence) is available to date.

(8) When we say that a coupling constant with the dimensions of an area is « small » we must include some natural unit of length; for this we take the inverse of the total energy in the center of mass for the process considered.

which is quadratically divergent. The lowest non-vanishing correction to the fermion self-energy as well as the lowest correction to the two-fermion scattering (or muon decay) processes involve this divergent quantity and hence cannot be computed without an essential change in the interpretative postulates of the theory. Such divergences are typical of all local interactions and have been handled in a covariant fashion through the renormalization technique in electrodynamics (and other renormalizable theories) by a redefinition of the physical parameters of renormalized coupling constant and masses. However such an elimination of the divergences is not possible for this theory in higher orders since new infinities appear in the higher orders and there are an infinite number of such terms. We shall see later that in the theory discussed in this paper also it is necessary to carry through a renormalization of the physical quantities but not for the purpose of elimination of the divergences. We must find then some other method of circumventing the divergences. A method of constructing such a finite theory was outlined in a previous paper (3); and we shall discuss the theory of four-fermion interactions based on this method.

3. – Theory of coupled lepton fields.

The Lagrangian density (1) involves three different fermion fields χ, ψ_1, ψ_2. We now generalize this to contain eight fields, four charged fields and four neutral fields represented by $\psi^{(j)}$, $\chi^{(j)}$ respectively and write the analogues of (2) and (3) in the form

$$\mathscr{L}_0(x) = \sum_{j=1}^{4} \overline{\chi}^{(j)}(p - \mu^{(j)})\chi^{(j)} + \sum_{j=1}^{4} \overline{\psi}^{(j)}(p - m^{(j)})\psi^{(j)} , \tag{8}$$

$$J_\lambda(x) = \sum_{j=1}^{4} \sum_{j'=1}^{4} \overline{\psi}^{(j)}\gamma_\lambda(1 + \gamma_5)\chi^{(j')} . \tag{9}$$

The eight parameters $\mu^{(j)}$ and $m^{(j)}$ have the dimensions of a mass but are not to be identified with any observed masses. These fields obey the following anticommutation relations for equal times:

$$\begin{aligned} &\{\chi^{+(j)}(\boldsymbol{x}, t),\ \chi^{(j')}(\boldsymbol{x}', t)\} = (-1)^{j+1}\delta_{jj'}\,\delta(\boldsymbol{x} \quad \boldsymbol{x}') , \\ &\{\psi^{+(j)}(\boldsymbol{x}, t),\ \psi^{(j')}(\boldsymbol{x}', t)\} = (-1)^{j+1}\delta_{jj'}\,\delta(\boldsymbol{x} \quad \boldsymbol{x}') , \\ &\text{all other anticommutators} \quad 0 \end{aligned} \tag{10}$$

One notes that there are two constants of motion for this interaction

$$\left|\begin{aligned} Q &= \int \mathrm{d}^3x \sum_{j=1}^{4} \psi^{(j)+}(\boldsymbol{x}, t)\,\psi^{(j)}(\boldsymbol{x}, t) , \\ L &= \int \mathrm{d}^3x \sum_{j=1}^{4} [\psi^{(j)+}(\boldsymbol{x}, t)\,\psi^{(j)}(\boldsymbol{x}, t) + \chi^{(j)+}(\boldsymbol{x}, t)\,\chi^{(j)}(\boldsymbol{x}, t)] . \end{aligned}\right.$$

The indefinite sign for the anticommutators in (10) shows that the metric in the space cannot be positive definite: more specifically, the fields $\chi^{(2)}$, $\chi^{(4)}$, $\psi^{(2)}$, $\psi^{(4)}$ all have the « wrong » sign for their anticommutators. The metric appropriate to this quantization ([9]) is given by

$$\eta = \exp\left[\pi i\int \mathrm{d}^3x\,[\psi^{(2)+}\psi^{(2)} + \psi^{(4)+}\psi^{(4)} - \chi^{(2)+}\chi^{(2)} + \chi^{(4)+}\chi^{(4)}]\right]$$

Here the definitions of χ^+ and $\overline{\chi}$, etc., are made in accordance with this metric; more specifically χ^+ is the pseudo-hermitian adjoint of χ. With these definitions if G is chosen real it then follows that $\mathscr{L}$ is pseudohermitian; and all expectation values of the pseudohermitian quantities being real, the Hamiltonian density constructed from the new Lagrangian density (as well as all hermitian functionals of it) would have real expectation values.

We now proceed in the usual fashion ([10]) to construct a perturbation series which is manifestly covariant, which is appropriate to the above interaction. Most of the steps in the derivation of the perturbation series are identical to the corresponding derivation for a theory with positive definite metric. We omit this derivation but merely state the intermediate result giving the S-matrix as a power series in the time-ordered products:

$$S = \sum_{n=0}^{\infty}(-i)^n(n!)^{-1}\int \mathrm{d}^4x_1 \ldots \int \mathrm{d}^4x_n\, T\,\{\mathscr{H}(x_1)\ldots\mathscr{H}(x_n)\} \tag{13}$$

where the co-ordinate space integrations run over the entire range $-\infty$ to $+\infty$, T is the time-ordering symbol, and $\mathscr{H}(x)$ is the interaction density in the interaction representation. For expressing S in terms of particle amplitudes, it is necessary to express the time-ordered product in terms of normal products and contraction functions (« propagators »). At this point the indefinite metric shows itself in giving a different expression for some of the contraction functions; one finds in fact that the « ghost fields » $\chi^{(2)}$, $\chi^{(4)}$, $\psi^{(2)}$, $\psi^{(4)}$ give rise to propagators with the opposite sign as compared to a normal field with the same mass.

The origin of this negative sign may be seen most clearly by working with a single Fermi oscillator with the « wrong » sign for the anticommutator; the creation and destruction operators a^+, a satisfy:

$$\{a, a^+\} = -1 \qquad \{a, a\} = \{a^+, a^+\} = 0$$

([9]) Compare S. N. Gupta: *Proc. Phys. Soc.*, **63**, 681 (1950); **64**, 850 (1951); K. Bleuler: *Helv. Phys. Acta*, **23**, 567 (1950).

([10]) F. J. Dyson: *Phys. Rev.*, **75**, 1736 (1949).

The states $|0\rangle$ and have the scalar products

$$\langle 0|0\rangle \qquad \langle 0|1 \qquad 1|0\rangle$$

Consequently the contraction function is given

$$\langle 0|aa^+|0\rangle \qquad \langle 0|\{a, a^+\} \qquad a^+a|0\rangle$$

which is of the opposite sign to the usual case. Note that this result (like the corresponding result for the « normal » case) is independent of the representation. However, in writing down the complete matrix element it is also necessary to specify the representation chosen for the creation and destruction operators; for the « normal » case the usual choice is to take a, a^+ to be real and we shall choose the same convention for the present theory also. The « abnormal » fields would have a relative sign change between the emission and absorption matrix elements.

The rules for calculating transition amplitudes may then be written down using Feynman diagrams and the Feynman rules with two modifications:

i) all contraction functions involving the « abnormal » fields are to be taken with an additional negative sign;

ii) all emission matrix elements involving « abnormal » fields acquire an additional negative sign: all absorption matrix elements are unchanged.

As in conventional theory it is advantageous to work in momentum space; in the usual theory the fermion contraction function ([11]) is of the form $(p-m+i\varepsilon)^{-1}$. We mentioned in the last section that the abnormal fields acquired an additional negative sign to their contraction functions. But from (9) it follows that to every Feynman diagram there corresponds another Feynman diagram in which any neutral lepton line corresponding to any type is replaced by a neutral lepton line corresponding to any of the other three types; and similarly for the charged lepton lines. It then follows that, as far as the *internal* fermion lines are concerned only an effective neutral lepton contraction function

$$S_\chi(p) \quad \sum^{4} \quad 1 \qquad u^{(}$$

and an effectiv charged lepton contraction function

$$S_w(p \quad \sum^{4} \quad 1) \qquad m^{(}$$

([11]) This contraction function differs from the usual expression by a factor $(2\pi)^4$ we shall use this definition throughout this paper.

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enter the theory. We may now choose the mass parameters such that

$$\mu^{(1)} \quad \mu^{(2)} + \mu^{(3)} \quad \mu^{(4)} \quad m^{(1)} \quad m^{(2)} + m^{(3)} \quad m^{(4)} = 0 \tag{15}$$

since they are so far undetermined. With this condition satisfied, the effective contraction functions decrease as fast as p^{-3} for large values of the momentum ([12]). This decrease is sufficient to make the two-vertex fermion loop contribution (Fig. 2) finite; we shall discuss their precise evaluation later. We note in passing that this replacement of individual lines by an effective contribution applies to internal lines only. (The external lines have to be considered in a somewhat different fashion and involves a new interpretive postulate. These matters are discussed in Section **4**.) We shall assume in the rest of this paper that the requirement (15) is fulfilled by the masses.

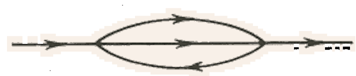

Fig. 3. – The second-order fermion self-energy diagram.

Let us now consider some higher order effects. The simplest is the fermion self-energy and there is only one type of diagram which contributes to this which is shown in Fig. 3. By a procedure completely analogous to the one adopted in the usual theory ([13]) we can write for the self-energy of the charged and neutral leptons the matrices:

$$\begin{aligned} \Sigma_\psi(p) &= (2\pi)^{-8} G^2 \int \mathrm{d}^4 q\, \gamma_\mu (1+\gamma_5) S_\psi(p \quad q)\, \gamma_\nu (1+\gamma_5)\, C_\chi^{\mu\nu}(q)\,, \\ \Sigma_\chi(p) &= (2\pi)^{-8} G^2 \int \mathrm{d}^4 q\, \gamma_\mu (1+\gamma_5) S_\chi(p \quad q)\, \gamma_\nu (1+\gamma_5)\, C_\psi^{\mu\nu}(q)\,, \end{aligned} \tag{16}$$

$$C_\chi^{\mu\nu}(q) = \int \mathrm{d}^4 k\, \mathrm{Tr}\,\{\gamma^\mu (1+\gamma_5) S_\chi(q+k)\, \gamma^\nu (1+\gamma_5) S_\chi(k)\}$$

and similarly for $C_\psi^{\mu\nu}(q)$. Making use of the identity

$$(z \quad m_1)^{-1} + (z \quad m_2)^{-1} \quad \left(z + k - \frac{m_1+m_2}{2}\right)^{-1} - \left(z - k \quad \frac{m_1+m_2}{2}\right)^{-1} = 2 \int_{(3m_1+m_2+k)/4}^{(m_1+3m-k)/4} \mathrm{d}\alpha \int_{(m_1-m_2-k)/2}^{(m_2-m_1+k)/2} \mathrm{d}\beta\, (z-\alpha-\beta)^{-3}\,, \tag{18}$$

([12]) With only four fields with contraction functions of unit weight it is not possible to make the effective contraction functions fall off faster.

([13]) See, for example, J. M. Jauch and F. Rohrlich: *Theory of Photons and Electrons* (Cambridge, Mass., 1955).

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we can rewrite the expression for $C^{\mu\nu}(q)$ in the form

$$C_\chi^{\mu\nu} = 16 \iint \mathrm{d}\alpha_1\,\mathrm{d}\alpha_2 \iint \mathrm{d}\beta_1\,\mathrm{d}\beta_2\, B^{\mu\nu}_{\alpha_1+\beta_1,\,\alpha_2+\beta_2}(q) \tag{19}$$

where the limits of the α and β integrations are as given above; and

$$B^{\mu\nu}_{\lambda_1,\lambda_2}(q) \quad \int \mathrm{d}^4k\, \frac{1}{4}\,\mathrm{Tr}\,\{\gamma^\mu(1 \quad k_5)(q+k \quad \lambda_1 \quad i\varepsilon)^{-3}\gamma^\nu(1 \quad \gamma_5)(k \quad \lambda_2 \quad i\varepsilon)^{-3}\}$$

$$= \pi^2 i \int_0^1 \mathrm{d}x\, \{[\lambda_1^2 x \quad \lambda_2^2(1-x) \quad x(1 \quad x)q^2]g^{\mu\nu} + 3x(1 \quad x)[q^{\mu\nu} - q^\mu q^\nu]\}$$

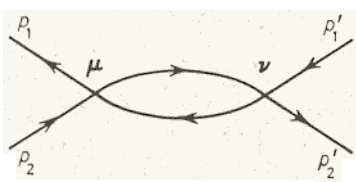
$$x^2(1 \quad x)^2[\lambda_1^2 x + \lambda_2^2(1-x) - x(1-x)q^2]^{-4}$$

The important point to note is that these quantities are finite; and for « reasonable » values of the masses, in view of the smallness of the coupling constant G, these expressions are « small ».

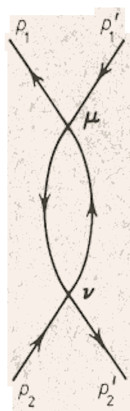
Similarly for the basic scattering process involving four external fermion lines the lowest order correction arises through the bubble diagram illustrated in Fig. 4. There are two such diagrams; the corresponding « reduced matrix element » (which replaces $Gg^{\mu\nu}$) is given by

$$\Lambda^{\mu\nu}(p_1, p_2, p_1', p_2') = (2\pi)^{-4} i G^2 C^{\mu\nu}(p_1 \quad p_2)$$

for the diagram Fig. 4a (and a similar expression for Fig. 4b). Here $C^{\mu\nu}$ is $C_\chi^{\mu\nu}$ for the choice of particles with momenta p_1, p_2 to be two charged leptons of opposite charge; and similar expressions for the other cases ([14]).



a)



b)

Fig. 4a, b. – Lowest-order corrections to scattering.

([14]) In case p_1, p_2 are respectively charged and neutral, then the defining expressi for $C^{\mu\nu}$ contains one S_ψ and one S_χ in an obvious manner.

One notes that these corrections are also finite. It is also of interest to note that this correction, eq. (21) leads to non-locality in the effective interaction; and the two-vertex loop, Fig. 2, here plays the role of an « intermediate vector meson ». There are two important differences however: the intermediate mesons are both charged and neutral, so that the theory does not correspond to a theory involving only charged vector mesons. The quantity $C^{\mu\nu}$ which plays the role of a vector meson contraction function is a non-trivial tensor in its indices; and corresponds to vector mesons with a continuum of masses. A consequence of this formal analogy is that the predictions of the present theory will be qualitatively similar to those of an intermediate vector meson theory.

One might calculate higher order effects in a similar fashion. Examples of some higher corrections in the next order are illustrated in Fig. 5. In a systematic analysis of these higher order corrections it is necessary to note

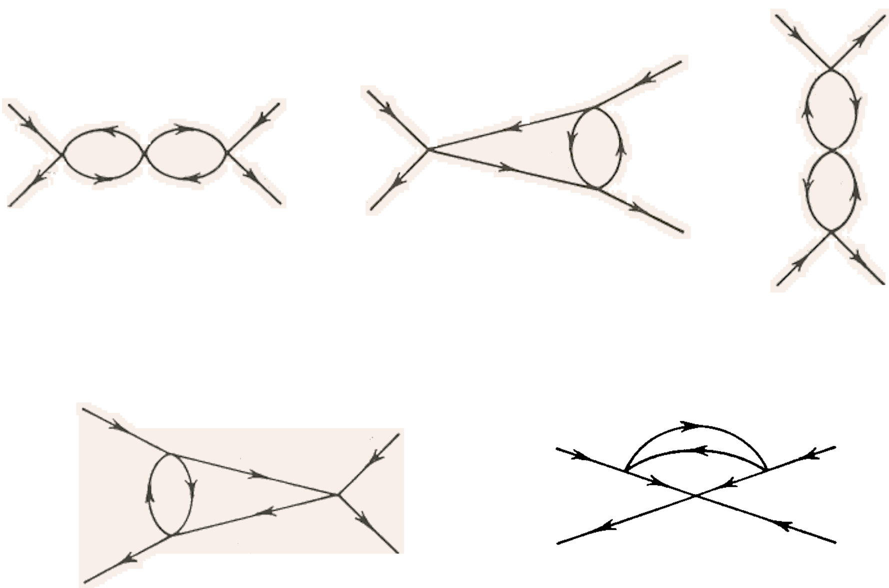


Fig. 5. – Some higher-order corrections to scattering.

that the « physical » lepton masses are shifted by varying amounts according to (16) and (17). In addition the physical particles are associated with « modified » fields; this feature is a consequence of the existence of several fields with all their « kinematical quantum numbers » the same and differing only in their mass parameters. To exhibit the theory in a form which is completely analogous to the conventional renormalized perturbation expansions ([10]) it is necessary to rewrite the perturbation series in terms of the « observed » masses, the « modified » fields and « observed » coupling constants. This renormali-

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zation programme has of course nothing to do with the elimination of any infinite quantities; the renormalizations (or at least the coefficients of the power series expansion) are *finite* in the present theory. The manipulations are complicated and are relegated to the Appendix.

Two points are to be mentioned here. Firstly the weakness of the coupling and the consequent smallness of the expansion parameter makes most of these higher order corrections somewhat academic; but it is interesting to demonstrate how the theory could be consistently renormalized. Secondly, (as seen from (A.12)), the renormalized coupling constants are no longer equal; such a departure from universality of effective coupling strengths is of course expected, but again, in view of the smallness of the interaction constants, these departures will be small.

These remarks, then, dispose of the problem of higher order corrections to transition amplitudes; the transition amplitudes are now completely described once the external lines are specified. In view of the underlying indefinite metric, the transition amplitudes would correspond to a pseudounitary S-matrix. Hence the physical interpretation of such a theory requires a new interpretive postulate consistent with the present formalism as well as with general quantum mechanical principles. The nature of this new postulate was briefly discussed in two earlier papers. A more systematic discussion is given in the following sections.

4. Measurable quantities in many-particle states.

In usual forms of quantum mechanics the states consist of vectors of unit length in a Hilbert space; if the dynamics is invariant under a group of transformations the states furnish a representation of the group which is in general reducible. These statements are true in all forms of quantum mechanics; in particular in all relativistic quantum theories the set of all states furnish a reducible representation of the proper inhomogeneous Lorentz group ([15]). If the relativistic quantum theory is a consistent theory and has an underlying Hilbert space (with a positive definite metric) these representations are unitary. For the inhomogeneous Lorentz group one can prove ([16]) that all reducible representations are the direct sum (or direct integral) of irreducible representations. So, for the time being, one may confine attention to either irreducible unitary representations or to appropriate direct integrals of these.

So far nothing has been said about particle concepts related to these states. It is generally assumed that physically interesting systems are capable of a particle interpretation and that all dynamical effects can be described in terms

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([15]) See, for example, A. S. WIGHTMAN: *Phys. Rev.*, **101**, 860 (1956).
([16]) E. P. WIGNER: *Ann. Math.*, **40**, 149 (1939).

of particle interactions and scatterings. All the irreducible unitary representations of the inhomogeneous Lorentz group are known and at present it appears that the only irreducible representations needed are the trivial no-particle (vacuum) representation and the one-particle representations appropriate to particles of zero mass and spin $\frac{1}{2}$ or 1 and finite mass and arbitrary (finite) spin. It is, of course, by no means clear that any interacting field theory could be constructed to give only these states (or can be constructed at all consistently!) within the present formalism.

The vacuum state is invariant under all Lorentz transformations and the one-particle states (belonging to a definite particle type) form an irreducible manifold in the sense that any state can be obtained from any other by means of an appropriate Lorentz transformation. These states are non-degenerate and are « steady » in the sense that a state with a definite value of energy, momentum and helicity belongs only to one such irreducible manifold. The two-particle states on the other hand are neither irreducible nor steady in general; consequent on this fact one can expect scattering processes in two-particle systems.

All these comments are equally true for any many-particle system. Yet there is intuitively a well-defined distinction between two-particle states and all other many-particle states. To make these notions precise we introduce the following characterization. Consider a quantum-mechanical state consisting of two-particles; then all dynamical variables of the system are functionals of the two sets of « particle variables » and the wave function of the system is the product of two one-particle wave functions. This last statement is perhaps made more specific by saying that the (reducible) representation furnished by the two-particle states (which is a direct integral of irreducible one-particle states) is isomorphic to the direct product of two irreducible representations. We now say that a certain state of an *interacting* system belongs to a two-particle manifold if there exists a manifold of states of the interacting system containing the state in question and closed under all Lorentz transformations which is isomorphic to a manifold of states (with the same values of the momentum and helicity) of a non-interacting system containing its two-particle states but to no smaller manifold. (Thus defined a « bound state » of two interacting particles also belongs to a two-particle manifold.) The need for such an elaborate definition is that in a quantum field theory the states are defined in terms of a Hamiltonian operator, etc., as suitable realizations of the Lorentz group but not directly in terms of particle observables. So the particle concepts have to be introduced from outside, and are to some extent arbitrary though restricted by consistency requirements [17]; this point arises

[17] Compare A. S. WIGHTMAN and S. S. SCHWEBER: *Phys. Rev.*, **98**, 812 (1955); R. ACHARYA and E. C. G. SUDARSHAN: *Journ. Math. Phys.*, **1**, 532 (1960).

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even in connection with « one-particle » states for zero mass finite spin cases. It is then clear that the particle interpretation of a field theory is not uniquely defined by the field theory alone but depends on the choice of the construction of particle variables.

We shall now consider the configurational notions associated with a two-particle state. There is first of all the concept of the « distorted » two-particle wave function which is nothing but the expansion coefficients for the states of the two-particle system in terms of the states of the two-particle states of the non-interacting comparison system (chosen so that the continuous energy spectra of the two systems coincide). There is also an intuitive notion of measurement of one-particle properties « when the other particle is far away »; this notion is not sufficiently precise to enable us to proceed with the construction of particle variables. To sharpen this postulate of particle measurement we demand that there exist interactions involving the two-particle states and classical apparatus which converts the two-particle system into a one-particle system; the measurement of one-particle properties can then be made in the standard manner on the states so prepared. The detection of one-particle properties thus consists of a sequence of two operations on the system; the first stage of the compound operation may consist of absorption of the « other » particle (or of a suitably devised « shield »). The important point is that *the other particle may no longer belong to the quantum mechanical state on which the one-particle measurements are performed.* While no non-trivial completely solvable relativistic example is available, for simple non-relativistic models these principles can be illustrated ([18]).

When we consider the more general framework involving the indefinite metric some of these characterizations have to be modified. The no-particle, one-particle, two-particle and many-particle classifications are still valid though the representations furnished for the Lorentz group are not necessarily unitary. However the no-particle and one-particle states do furnish representations; and in general in any representation which is a direct integral of irreducible representations, if one can exhibit a vector of each irreducible manifold which has a definite norm the reducible representation can be used to furnish a unitary representation, in spite of the underlying indefinite metric in the theory. Needless to say there may be manifolds of states which do not furnish a unitary representation; these states can then not be interpreted as physical states. The particle interpretation must then be in accordance with this limitation.

Let us now consider a two-particle manifold of an interacting quantum-mechanical system involving an indefinite metric and consider the scattering

([18]) H. J. SCHNITZER and E. C. G. SUDARSHAN: *Quantum-mechanical systems with indefinite metric* - II, submitted to *Phys. Rev.*

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matrix. If there is only one kind of energy eigenstate belonging to the continuous eigenstate, the S-matrix is either unitary (corresponding to real phase shifts) or the states are null vectors. In the first case there is no problem; the second case does not describe a physical system. On the other hand the energy eigenstates may be degenerate and there may be different « kinds » of two-particle states. In this case the S-matrix is in general not diagonal in the label specifying the « composition »; but we may construct the eigenstates of the S-matrix. It is then again clear that only states with normalizable vectors are to be considered; and from the discussion in ref. [3] it is clear that only states with a definite sign of the norm are to be allowed. The totality of all such states (which may be taken to be eigenstates of the S-matrix with positive norm without any loss of generality) appear as candidates for being physical states. It is clear that in the case of a positive definite metric, these conditions imply no restrictions whatever.

There is one more condition to be satisfied by a two-particle state if it is to conform to intuitive notions as defined precisely in an earlier paragraph; namely there should exist the possibility of producing a *physical* one-particle state from this state by removing the « other » particle. This restricts physical states to be the scattering eigenstates with positive definite norm and which map on to two-particle states of the non-interacting system which consists of two physical particles under the defining isomorphism. This is the « asymptotic condition » in our quantum field theory.

One might argue that the « physical » two-particle states so defined are really not pure in composition and contain really different kinds of particles [19]. But such a statement implies that it is possible to analyze a given two-particle system and extract from it a negative norm single-particle state; we have seen above that any detection of one-particle properties presupposes an operation of conversion of a two-particle state into a one-particle state. Now all physical operations should, within the postulational framework, connect only physical states. Hence an operation which converts a physical two-particle state into a non-physical one-particle state is not possible and consequently the analysis of the composition of the irreducible physical states is not possible. *This non-analyzability is fundamental to the framework and distinguishes these states from the analyzable eigenamplitudes for scattering of two coupled physical channels.*

The treatment of the external lines thus developed is the following: compute all relevant transition amplitudes in the theory and thus construct the generalized S-matrix of the theory to any desired degree of approximation. No infinities are encountered at any stage of the perturbation series calculation.

[19] The author is indebted to C. J. GOEBEL for emphasizing that the consistency of the interpretive postulate enunciated in ref. [3] had not been explicitly demonstrated; this demonstration is given in ref. [18].

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The S-matrix is now diagonalized and a correspondence is established between these eigenstates of the S-matrix and non-interacting states of a comparison system [18]. The physical states are chosen from amongst those states which are mapped onto many-particle states involving only physical particles. The set of physical states so formed describe physical particles undergoing mutual scattering. An illustration of these ideas for a simple exactly solvable model is given in ref. [18].

In the following section some simple predictions of the theory are mentioned.

5. – Applications.

In the discussion so far we have dealt with general questions and made no specific choices for the masses or specific identifications of the physical particles. We know that there are *two* physical charged leptons namely the electron and the muon, the masses of these particles then fix two of the three independent charged lepton mass parameters. On the other hand for the neutral leptons there is only one neutral lepton known and it has zero mass; so two neutral lepton mass parameters are left undetermined. But the other physical neutral lepton mass must be so high that it does not appear in the decays of pions, kaons or muons, (or it should be practically the same as a neutrino as to be indistinguishable in experiments). This still leaves one parameter undetermined.

We may now consider the decay of the muon in terms of the transition matrix element involving one initial physical particle and three final particles. There are many final states allowed by selection rules but only one physical three-particle state. But considered as a function of the momenta the matrix element would deviate from the predictions of the lowest order calculation by a small amount; we shall not attempt to calculate this correction here but simply mention that the quantitative analysis of these deviations (after the electromagnetic effects have been properly taken into account) should provide some estimates of the undetermined lepton mass parameters [20].

Strictly speaking we cannot discuss any of the observed meson or baryon decays in any quantitative manner since our theory as presented in this paper does not deal with strongly interacting particles. But the qualitative features of the weak interaction process can be displayed by considering a classical source (with a strength given by the vector or axial vector matrix element of the strongly interacting system) to be coupled to the leptons. The extraor-

(20) Such an analysis is now being made by S. HATSUKADE. With regard to the doubling of fermions, compare M. GOLDHABER: *Phys. Rev. Lett.*, 1, 467 (1958).

dinarily good fit of the predictions of the $V-A$ theory for the $(\pi\to e+\nu)/(\pi\to\mu+\nu)$ ratio suggests that the damping of high energy transitions is still unappreciable at this magnitude of the momentum; and this is consistent with the absence of evidence for a neutral counterpart to the muon. On the other hand one expects some damping at the energy appropriate to kaon decay and hence the comparable rates of kaon and pion decays (in spite of the larger phase space in kaon decay) is at least in part due to this. If this is true then it is of great interest to measure precisely the $(K\to e+\nu)/(K\to\mu+\nu)$ ratio and compare it with the unique prediction given by the lowest order $V-A$ calculation. The leptonic hyperon decay problem as well as the other leptonic kaon decay modes are considerably more difficult to discuss because of the three-particle final states involved.

6. – Concluding remarks.

In the previous sections we have presented a finite relativistic quantum theory of leptons. The theory as it stands is incomplete since it does not include other particles and interactions, especially the interaction with photons; this will be remedied in subsequent papers of this series. The essential result of this paper is to show that it is possible to construct a consistent theory of four-fermion interactions without having to introduce intermediate vector mesons. The theory also gives a natural *raison d'être* for the muon; and *requires the muon to be degenerate with the electron in its interaction properties.*

The theory does predict two neutral leptons; and in the present state of our experimental knowledge such a prediction is undesirable. But it should be stressed that such a particle is natural in the present formalism; and it is possible that it may have a mass high enough to be not observed in low energy reactions or low enough to be confused for the « usual » neutrino. If the latter circumstance prevails then the measurements usually made would not distinguish this case from the usual assumption of only one kind of neutrino. But it is necessary to stress that since the definition of the « physical particle » in many-particle states involves a certain amount of freedom as explained in Section **4** and in ref. [18], it is possible to arrange the « comparison amplitude » so that the neutrino is the only physical particle which is coupled [21]. However this requirement appears somewhat arbitrary and unsymmetric; the question must ultimately be answered only by experiment.

[21] It was mentioned in ref. [3] that in indefinite metric theories a particle interpretation requires the asymptotic condition selecting a subset of positive norm scattering states, (see also Sect. **4**) the possibility mentioned in the text corresponds to additional restrictions and implies a further selection among the positive norm steady one-particle states.

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The question of the arbitrariness of the details of the present theory invites comment on another aspect, namely the choice of four charged lepton fields and four neutral lepton fields. In earlier studies involving modified propogators, particularly in the outstanding contribution by PAIS and UHLENBECK ([22]), the several independent fields were brought in by working with Lagrangians involving higher order derivatives. The question naturally arises as to whether our theory can be reformulated in that form; and whether it would be more desirable to start from such a formulation. Straightforward algebraic manipulations enable us to rewrite the theory of *uncoupled* fields in this manner (provided the two « abnormal » fields have equal masses), but the coupling term is no longer simple; this circumstance is not accidental but due to the desire to incorporate the « observed » *universality* of the interaction between *particles* ([23]). Nor are the commutation relations of the primitive field of standard type but involve explicitly the self-same masses which we have introduced into our primitive Lagrangian ([24]). In view of this it appears that there is no advantage in insisting that the theory be reformulated in terms of a field satisfying higher order equations.

It is also to be pointed out that unlike the motivation of certain earlier speculations involving an additional neutral particle, here there is no attempt to trace a symmetry between the numerical masses of the charged and neutral leptons. Nor is any sanctity attached to the zero mass; anyway, the success of the chiral $V-A$ interaction for the different kinds of four-fermion couplings make any principle directly related to the zero mass of the neutrino somewhat irrelevant.

Similar concepts are certainly applicable to the strongly interacting particles and one might attempt the construction of a theory of fundamental fields, say the Sakata model, as a quantitative theory. But while the framework discussed here gets rid of the infinities the strength of the interaction makes it necessary to study more suitable approximation procedures for these problems than a straight-forward application of perturbation theory. The treatment of strong interactions thus requires additional tools; but these difficulties are not present for the quantum electrodynamics of leptons.

One might argue that the method developed here (or any such involving an indefinite metric and a related interpretive postulate) is *only* a covariant

([22]) A. PAIS and G. E. UHLENBECK: *Phys. Rev.*, **79**, 145 (1950).

([23]) This is to be contrasted with the situation arising from coupling the primitive field (obeying the higher order equation) *directly*. In this latter case the particles will not be coupled universally; compare eq. (54), (57) and (58) of ref. ([22]).

([24]) The « unaesthetic arbitrariness » of this Lagrangian (as well as of the Lagrangians of current theories!) has been strongly criticized by I. BIALYNICKI-BIRULA in discussions with the author.

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form-factor [25]. Such an argument would certainly be valid; in fact, as argued in an earlier paper, the use of the indefinite metric is only as an aid to simplicity in constructing a finite relativistic quantum theory.

The author is indebted to I. BIALYNICKI-BIRULA, P. CZIFFRA, B. P. NIGAM and H. SCHNITZER for critical comments.

APPEND

Renormalization.

We remarked in Sect. 3 that while the perturbation theory developed there contains no infinite terms, there is still a difference between the « physical » mass of the various leptons and the « bare » mass parameter occurring in the original equation; similarly there will be higher order corrections to the various four-fermion reactions. The renormalization program for the present theory is essentially the same as in the case of the usual « renormalizable theories » except that in the present theory we have only finite renormalizations to carry out. The discussion below follows the pattern of JAUCH and ROHRLICH [13].

Consider an arbitrary diagram; it is made up of an arbitrary number of charged lepton lines, neutral lepton lines and four-fermion vertices; there are 8 kinds of lepton lines and 256 kinds of vertices. According to the choice of the universal $V-A$ four-fermion interaction all these vertices correspond to the same coupling constant and coupling type. We define a « self-energy part » as any part of a diagram which is connected to the rest of the diagram by exactly two lines. In the present theory it is necessary that both the lines are either charged lepton lines or neutral lepton lines, but the two lines need not belong to the same type of leptons; consequently the general self-energy part is a second rank tensor in the indices j, j' referring to the two lines in which the self-energy part is terminated. Similarly we define a « vertex part » as any part of a diagram which is connected to the rest of the diagram by exactly two charged lepton lines and two neutral lepton lines. The vertex part so defined is a tensor of the fourth rank in the two sets of two indices. (Note that the four-legged parts involving all charged or all neutral lines is *not* a vertex part according to this definition.) To every diagram there corresponds a « skeleton diagram » obtained by replacing all inserted self-energy

([25]) Note that the form-factor involved here is a *dynamical* form-factor and manifests itself differently in different states; the question whether a theory involving a non-dynamical form-factor can be consistent remains open.

parts by lepton lines [26] and all inserted vertex parts by simple four-fermion vertices. A diagram identical with its own skeleton is called an « irreducible » diagram. As in the usual case we see that the only irreducible self-energy diagram is the second order diagram (Fig. 3) (though there are such self-energy parts for the eight leptons), since any self-energy diagram starts with one four-fermion vertex and the remaining part of the diagram is a vertex part according to the above definition; hence in any skeleton diagram the remaining part is also a simple four-fermion vertex. But there are an infinite number of irreducible vertex parts.

Let $\Sigma_\psi^{jj'}(p)$ and $\Sigma_\chi^{jj'}(p)$ refer to the charged and neutral lepton self-energy tensors summed over all diagrams and let $S_\psi^{jj'}(p) = S_\psi(p; j)\delta_{jj'}$ and $S_\chi^{jj'}(p) = S_\chi(p; j)\delta_{jj'}$ be the bare propagation functions. The modified propagation functions $S'^{jj'}(p)$ are then given by

$$\left|\begin{array}{lll} S_\psi'^{jj'}(p) & S_\psi(p; j)\delta_{jj} & S_\psi(p)\Sigma_\psi^{jj'}(p)S_\psi(p)\,, \\ S_\chi'^{jj'}(p) & S_\chi(p; j)\delta_{jj} & S_\chi(p)\Sigma_\psi^{jj'}(p)S_\chi(p)\,. \end{array}\right.$$

And the modified vertex part is given by

$$\text{(A.2)}\qquad \Gamma^{j_1j_2j_3j_4}(p_1, p_2, p_1', p_2') \quad \gamma_\mu(1 \quad \gamma_5)\times\gamma^\mu(1 \quad \gamma_5) \quad \Lambda^{j_1j_2j_3j_4}(p_1, p_2, p_1', p_2')$$

where the first term on the right-hand side corresponds to the primitive chiral $V-A$ coupling and the second term represents the sum over all *proper* vertex parts, a « proper » diagram being defined as one which cannot be separated into two disjoint diagrams by opening a single line; the contributions from improper diagrams have already been included in the modification of the propagation functions. In view of the chiral $V-A$ interaction under recouplings the order of coupling the four fields in (A.2) is irrelevant.

We now separate the contribution from all the diagrams into « renormalization » and « physical » parts. For this purpose we rewrite the inertia term in (8) in the form

$$\sum_{j,j'} M^{jj'}\overline{\psi}^{(j)}\psi^{(j')} \qquad \sum_j m^{(j)}\overline{\psi}^{(j)}\psi^{(j)} \qquad \sum_{r=1}^{4} m_0^{(r)}\overline{\psi}_{(r)}\psi_{(r)} \qquad \sum_{j,j'} A^{jj'}\overline{\psi}^{(j)}\psi^{(j')}$$

where the modified field $\psi_{(r)}$ is given by

$$\psi_{(r)} = \sum_j v_\psi^j(r)\,\psi^{(j)}$$

The unitary matrix $v^j(r)$ and the numerical matrix $A^{jj'}$ are as yet undetermined and are to be determined as follows. Using the new masses $m_0^{(r)}$ and the new fields $\psi_{(r)}$ we may define the propagators $S_\psi^{(r)}(p)$. Let us first consider

[26] There is a slight difference in this operation from the usual case because of the tensor character od the self-energy parts; but no troubles arise from omitting an explicit recognition of this in our work.

the irreducible self-energy parts (Fig. 3) which are of the second order. We note that according to (A.1) the self-energy part is a tensor in the indices j, j' and may be written in the form

$$\text{(A.5)} \qquad \Sigma_{\psi}^{jj'}(p) = A_{\psi}^{jj'} - \sum_{r=1}^{4} B_{\psi}^{(r)} S_{\psi}^{-1(r)}(p)\, v_{\psi}^{j*}(r)\, v_{\psi}^{j'}(r) + \\ + \sum_{r,r'=1}^{4} S_{\psi}^{-1(r)}(p)\, M^{(rr')}(p)\, S_{\psi}^{-1(r')}\, v_{\psi}^{j*}(r)\, v_{\psi}^{j'}(r')\,,$$

and we require this to be computed using the new propagators $S_{\psi}^{(r)}(p)$. A completely analogous expression can be obtained for the neutral lepton self-energy part. Similarly we write the vertex modification by all the irreducible contributions as the sum of the renormalization and physical parts:

$$\text{(A.6)} \qquad \Lambda'^{j_1j_2j_3j_4}(p_1, p_2, p_1', p_2') = L^{j_1j_2j_3j_4}\gamma_{\mu}(1+\gamma_5)\times\gamma^{\mu}(1+\gamma_5) + \\ + \sum_{r_1r_2r_3r_4} \Lambda^{(r_1r_2r_1'r_2')}(p_1, p_2, p_1', p_2')\, v_{\psi}^{j_1*}(r_1)\, v_{\chi}^{j_2}(r_2)\, v_{\psi}^{j_3}(r_1')\, v_{\chi}^{j_4*}(r_2')\,.$$

In (A.5) and (A.6) the separation into the two parts is to be made definite by the requirement that for all the momenta p_1, p_2, p_1', p_2' on the mass-shell for the modified « particles » denoted by r_1, r_2, r_1', r_2' and the primed and unprimed quantities being taken equal and at the threshold for the reaction $p_1 + p_2 \to p_1' + p_2'$ the physical correction part of (A.6) vanishes.

We now proceed to the separation of the physical parts of proper reducible diagrams by a method of induction. For this purpose we take any diagram of arbitrary order and assume that diagrams of all lower orders have been separated into the renormalization and physical parts. We now replace each vertex part and each propagator by the sum of the unmodified part plus the physical part *i.e.*, drop all renormalization parts of the modifications; separate the physical part of the proper diagram so computed and sum the contributions so obtained from all the proper (reducible and irreducible) diagrams to obtain expressions formally identical with (A.1) and (A.2). These equations indicate a shift in mass of the modified fermion « r ». To define « physical » particles we now determine $v_{\psi}^{j}(r)$ so that by treating the term $m_0^{(r)}\overline{\psi}_{(r)}\psi_{(r)}$ as the inertia term in the calculation both (A.3) and (A.5) can be consistent with each other. When such a choice is made the mass renormalization is complete and in all further calculations the mass shift of the (modified) « free » particles may be taken to vanish. It is important to note that in the present theory the « mass renormalization » not only redefines the mass of the physical particles but also defines the physical particles in terms of the modified fields.

Let us now complete the renormalization by introducing the physical propagators, vertex functions and renormalized coupling constants. Define

$$\text{(A.7)} \qquad S_{0\psi}^{-1(r,r')}(p) = S_{\psi}^{-1(r)}(p)\,\delta_{rr'} + M_{\psi}^{(rr')}(p)\,,$$

$$\text{(A.8)} \qquad S_{0\chi}^{-1(r,r')}(p) = S_{\chi}^{-1(r)}(p)\,\delta_{rr'} + M_{\chi}^{(rr')}(p)\,,$$

$$\text{(A.9)} \qquad \Gamma_0^{(r_1r_2r_1'r_2')}(p_1p_2p_1'p_2') = \gamma_{\mu}(1+\gamma_5)\times\gamma^{\mu}(1+\gamma_5) + \Lambda^{(r_1r_2r_1'r_2')}(p_1p_2p_1'p_2')\,.$$

We now assert that

$$\text{(A.10)} \qquad \begin{cases} S_{0\psi}^{(jj')}(\{G_0\}) = \{Z_\psi^{(j)} Z_\psi^{(j')}\}^{-\frac{1}{2}} S_\psi^{\prime jj'}(\{G\}) \,, \\ S_{0\chi}^{(jj')}(\{G_0\}) = \{Z_\chi^{(j)} Z_\chi^{(j')}\}^{-\frac{1}{2}} S_\chi^{\prime jj'}(\{G\}) \,, \end{cases}$$

$$\text{A.11)} \qquad \Gamma_0^{(j_1 j_2 j'_1 j'_2)}(\{G_0\}) = \{Y_\psi^{(j_1)} Y_\chi^{(j_2)} Y_{\psi'}^{(j'_1)} Y_\chi^{(j'_2)}\}^{+\frac{1}{2}} \Gamma^{j_1 j_2 j'_1 j}(\{G\})$$

with $\{G_0\}$ denoting a new set of 256 renormalized coupling constants

$$\text{(A.12)} \qquad G_0(j_1 j_2 j'_1 j'_2) \qquad \sqrt{\frac{Z_\psi^{(j_1)} Z_\chi^{(j_2)} Z_\psi^{(j'_1)} Z_\chi^{(j'_2)}}{Y^{(j_1 j_2 j'_1 j'_2)}}}\, G$$

Along with the propagator renormalization there is also a wave function renormalization. The proof of the assertions embodied in (A.10) to (A.12) consists in showing that matrix elements computed in the usual fashion from the original Feynman graphs are identical with the result of computing only the physical part but with the renormalized coupling constants (A.12), provided the mass renormalization (including the definition of the physical combinations) is already carried out and provided that the renormalization constants are defined by the equations:

$$\text{(A.13)} \qquad Z^{(j)} = 1 - B^{(j)}(\{G_0\}) \,,$$

$$\text{A.14} \qquad {}^{(j_1 j_2 j'_1 j'_2)} = 1 - L^{(j_1 j_2 j'_1 j'_2)}(\{G_0\})$$

This may be verified in a straightforward fashion.

The renormalization constants (A.13) and (A.14) are thus given as power series in the renormalized coupling constants. The question as to whether these quantities are finite or not would then depend upon the convergence of these power series. There is no reason why the series should converge, though in the present treatment the coefficients of the power series are all finite quantities and an estimate of the expansion parameter made in the previous section shows it to be very small. Further, the study of certain simple models exhibits the analyticity of the functions (of the coupling constants) defined by the power series ([18]). In the absence of any proof to the contrary we take the renormalization constants to be finite and the theory to be well-defined.

RIASSUNTO (*)

Formulo una teoria relativistica finita delle interazioni dei quadrifermioni; la teoria comporta come componente essenziale l'uso di una metrica indefinita. I problemi di interpretazione che sorgono con l'uso di una metrica indefinita vengono analizzati in relazione agli osservabili ed alla struttura degli stati a molte particelle nella teoria quantistica dei campi; e si dimostra la coerenza dei postulati interpretativi. La teoria fornisce incidentalmente una « ragion d'essere » per il muone.

(*) *Traduzione a cura della Redazione.*

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