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Asymptotic Field Operators in Quantum Field Theory*

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The formulation of the asymptotic condition in quantum field theory is viewed as a problem in asymptotic particle interpretation of a field theory and is solved in terms of a natural limiting process involving the retarded vacuum expectation values. Starting with the Haag-Brenig theorem translated into operator language we obtain the asymptotic fields for a nonrelativistic self-coupled field. Transcription of this expression for the asymptotic field in terms of a retarded two-point Green's function motivates the more general definition of the asymptotic field in the presence of external forces. The equations of motion as well as the commutation relations for these asymptotic fields are derived, and it is shown that in sufficiently simple cases this definition coincides with the usual definition. In addition to this conventional asymptotic field, in general, there are asymptotic fields which create particles in bound states. The definition generalizes in a natural manner to many-particle asymptotic fields and, in particular, to asymptotic fields which create and annihilate multiparticle bound states. This construction is facilitated by the standard analysis of the Bethe-Salpeter amplitudes. The familiar "contraction rules" are deduced within the present formalism. These developments are extended to relativistic field theory, again in terms of a limiting procedure involving the retarded Green's functions. As a particular application it is shown that the asymptotic electron field describes electrons which react to an external electromagnetic disturbance in the same manner as physical electrons, i.e., with their anomalous moment and vacuum polarization effects etc., in addition to the "normal" gauge invariant interaction. We briefly discuss the im-

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plications for theories with unstable quanta, and draw attention to the possibilities offered for a consistent theory of interacting particles involving unstable ghost fields.

INTRODUCTION

The asymptotic condition in quantum field theory has been the object of several investigations (*1*). In the axiomatic formulations of field theory which start from the concept of Heisenberg field, this condition is invoked to introduce a particle interpretation of the theory and thus to make contact with the experimental situation (*2*). In those formulations which initially define only in- and out-fields it proves necessary to introduce interpolating fields, of which these in- and out-fields are the asymptotic limits, in order to introduce the concept of causality into the theory (*3*). These asymptotic fields allow one to describe scattering experiments in a Lorentz invariant fashion. In the time-dependent description we may consider these experiments to consist in the preparation of specified physical particles at some initial time (usually taken as $t = -\infty$), allowing these particles to interact subsequently, and finally measuring the configuration of all the particles present after the interaction has taken place (taken as $t = +\infty$).

Usually the theory is formulated for situations in which no bound states can occur, i.e., in which all the particles in the initial and final states are free particles. Under these circumstances the mathematical formulation of the theory consists in introducing for each kind of particle (of mass m_i, spin s_i, type i) two fields, $\phi_{\text{in}}^{(i)}(x)$ and $\phi_{\text{out}}^{(i)}(x)$, which satisfy free field equations corresponding to the given mass and spin and satisfying free field commutation relations $[\phi_{\text{in}}^{(i)}(x), \phi_{\text{in}}^{(i)}(y)] = [\phi_{\text{out}}^{(i)}(x), \phi_{\text{out}}^{(i)}(y)] =$ free field singular function appropriate to the given mass and spin. The operators ϕ_{in} operating on the (unique) vacuum state create the Heisenberg vectors $|\Phi\rangle_{\text{in}}$ that describe the knowledge of the system implied in its preparation at time $t = -\infty$. The state $|\Phi\rangle_{\text{out}}$, obtained by operating with the fields ϕ_{out} on the vacuum state, represent eigenfunctions of the commuting set of operators corresponding to compatible measurements that can be made at $t = +\infty$. The S-matrix elements ${}_{\text{out}}\langle\Psi|\Phi\rangle_{\text{in}}$ are then the amplitudes for the transition from the initial state $|\Phi\rangle_{\text{in}}$ at $t = -\infty$ to the final state at $t = +\infty$ that has the properties described by $|\Psi\rangle_{\text{out}}$.

In the time-independent description we may consider a scattering experiment to consist in the preparation of an initial state labeled in one manner by free particle labels (the "in" states) and in the analysis of these states in terms of states labeled in a different manner again by free-particle labels (the "out" states). The free-particle labels correspond to the asymptotic particle labels in the time-dependent description; but the "particles" include both primitive particles and bound states. Needless to say any actual source or detector has a "response function" which may be arbitrarily peaked about some free-particle labels for the "in" or "out" states. The actual scattering states are, for this

reason, always normalizable. The use of nonnormalizable "states" is then only a computational shorthand. The asymptotic fields (at any fixed instant) may be defined as the operators which intertwine these asymptotic ("in" or "out") states; and their number and type ought to be determined by the number and type of the particles (primitive or bound) which label the asymptotic states.

It is the purpose of this paper to generalize in a systematic fashion the notion of asymptotic fields to encompass situations in which bound states occur (*4*). We have in mind both the case where the bound states may be due to prescribed external fields as well as that in which the bound states are the result of interparticle interactions. The asymptotic fields that we introduce are defined in terms of the vacuum expectation values of chronological products of Heisenberg field operators. The singularities of these Green's functions as a function of the total momentum determine the properties of the asymptotic field operators (*5*, *6*). The advantage of this procedure lies in the fact that it is a simple and consistent procedure for the definition of the asymptotic operators from which the Hilbert space of the physically realizable stable states can be constructed. The masses of the asymptotic fields are brought into the theory in a natural fashion and reflect the locations of the singularities of the Green's functions. We show that the asymptotic fields so defined describe the motion of a charged particle in a weak external electromagnetic field in terms of the electromagnetic form factors of the particle. This in fact was the motivation for the present definition of the asymptotic fields, since on intuitive grounds one would expect the asymptotic field describing a charged particle in a weak external electromagnetic field to satisfy an equation wherein the particle interacts with the external field through its electromagnetic form factors.

In Section I we briefly review the definition of in- and out-fields in nonrelativistic quantum mechanics. In Section II we introduce within the nonrelativistic framework our asymptotic operators and investigate their properties. Section III generalizes these definitions to encompass relativistic field theories. The last section summarizes the work of this paper and indicates certain new avenues opened by the present definitions.

I. NONRELATIVISTIC CASE

Before introducing the asymptotic fields mentioned in the introduction let us briefly review the customary way of defining the in- and out-fields in the nonrelativistic situation (*7*).

Our point of departure is the Haag-Brenig theorem (*8*, *9*). In the case of one particle moving in a potential $V(\mathbf{x})$ of finite range it asserts that to every normalizable (wave packet) solution of

$$i\hbar\partial_t|\Psi_+(t)\rangle = H|\Psi_+(t)\rangle \tag{1a}$$

$$H = H_0 + V \tag{1b}$$

which is orthogonal to the bound state solutions of (1a) there corresponds a unique solution of

$$i\hbar\partial_t|\Phi(t)\rangle = H_0|\Phi(t)\rangle \tag{2}$$

such that[1]

$$\lim_{t\to-\infty} \| \Psi_+(0) - e^{+iHt/\hbar}e^{-iH_0t/\hbar}\Phi(0) \| \to 0. \tag{3}$$

The theorem can also be interpreted as stating the existence of the Møller wave matrix

$$\Omega^{(+)} = \lim_{t\to-\infty} e^{iHt/\hbar}e^{-iH_0t/\hbar} \tag{4a}$$

which operator has the property that

$$H\Omega^{(+)} = \Omega^{(+)}H_0 \quad \text{(on the continuum eigenstates)} \tag{4b}$$

and

$$|\Psi_+(0)\rangle = \Omega^{(+)}|\Phi(0)\rangle. \tag{4c}$$

A similar correspondence can be established between solution $|\Psi_-(t)\rangle$ of Eq. (1a) and unperturbed states (i.e., solutions of $H_0|\Phi(t)\rangle = i\hbar\partial_t|\Phi(t)\rangle$) such that as $t \to +\infty$,

$$\lim_{t\to\infty} \| \Psi_-(0) - \exp(iHt) \exp(-iH_0t)\Phi(0) \| \to 0.$$

The generalization of this theorem to a system of nonrelativistic particles interacting with one another through repulsive two body forces of finite range is straightforward. We briefly outline here the steps involved in the transcription of Brenig and Haag's theorem into operator language. Consider a system of nonrelativistic particles described by a Hamiltonian:

$$\begin{aligned} H = &-\frac{\hbar^2}{2m}\int d^3x\psi^*(\mathbf{x}, t)\nabla^2\psi(\mathbf{x}, t) \\ &+ \frac{1}{2}\int d^3x \int d^3x'\psi^*(\mathbf{x}', t)\psi^*(\mathbf{x}, t)v(|\mathbf{x} - \mathbf{x}'|)\psi(\mathbf{x}, t)\psi(\mathbf{x}', t). \end{aligned} \tag{5}$$

We shall assume that the particles obey Bose statistics so that the equal time commutation rules for the operator $\psi(\mathbf{x}, t)$ and $\psi^*(\mathbf{x}, t)$ are[2]

$$[\psi(\mathbf{x}, t), \psi^*(\mathbf{x}', t)] = \delta(\mathbf{x} - \mathbf{x}') \tag{6a}$$

[1] $\| f \|$ denotes the norm of the vector $|f\rangle$: $\| f \|^2 = (f, f)$.

[2] As is clear from these commutation rules $\psi(\mathbf{x}, t)$ and $\psi^*(\mathbf{x}, t)$ are strictly speaking not well defined operators since operating on a normalizable vector they do not necessarily yield a normalizable vector. One should therefore consider operators of the form

$$\int \overline{f(\mathbf{x})}\psi(\mathbf{x}, t)\, d^3x$$

with $\int d^3x\, |f(\mathbf{x})|^2 < \infty$. We shall however not do so here.

$$[\psi(\mathbf{x}, t), \psi(\mathbf{x}', t)] = [\psi^*(\mathbf{x}, t), \psi^*(\mathbf{x}', t)] = 0. \tag{6b}$$

We fix the representation of these commutation rules asserting that there exists a *no*-particle state (*10*) defined by

$$\psi(\mathbf{x}, t)|0\rangle = 0. \tag{6c}$$

Since the number operator $N = \int d^3x\psi^*(\mathbf{x}, t)\psi(\mathbf{x}, t)$ commutes with H, there is no creation or annihilation of particles. A state vector describing an N particle system is therefore given by[3]

$$|\Psi_+(t)\rangle = \frac{1}{\sqrt{N!}} \int d^3x_1 \cdots \int d^3x_N \, \Psi_+^{(N)}(\mathbf{x}_1, \cdots, \mathbf{x}_N; t) \cdot \psi^*(\mathbf{x}_1) \cdots \psi^*(\mathbf{x}_N) \, |0\rangle. \tag{7}$$

This vector will satisfy the Schrödinger equation:

$$i\hbar\partial_t|\Psi_+(t)\rangle = H|\Psi_+(t)\rangle \tag{8}$$

if the amplitude

$$\Psi_+^{(N)}(\mathbf{x}_1, \cdots, \mathbf{x}_N; t) = \langle 0|\psi(\mathbf{x}_1) \cdots \psi(\mathbf{x}_N)|\Psi_+(t)\rangle \tag{9}$$

satisfies

$$H(\mathbf{x}_1, \cdots, \mathbf{x}_N; t)\Psi_+^{(N)}(\mathbf{x}_1, \cdots, \mathbf{x}_N; t) = i\hbar\partial_t\Psi_+^{(N)}(\mathbf{x}_1, \cdots, \mathbf{x}_N; t) \tag{10a}$$

where

$$H(\mathbf{x}_1, \cdots, \mathbf{x}_N) = \sum_{j=1}^{N} \left(\frac{-\hbar^2}{2m} \nabla_j^2\right) + \frac{1}{2} \sum_{i \neq j}^{N} v(|\mathbf{x}_i - \mathbf{x}_j|). \tag{10b}$$

The Haag-Brenig theorem in the present situation asserts that, to every solution of

$$i\hbar\partial_t\Psi_+^{(N)}(\mathbf{x}_1, \cdots, \mathbf{x}_N; t) = H(\mathbf{x}_1, \cdots, \mathbf{x}_N)\Psi_+^{(N)}(\mathbf{x}_1, \cdots, \mathbf{x}_N; t), \tag{11}$$

there corresponds a unique solution of

$$i\hbar\partial_t \Phi^{(N)}(\mathbf{x}_1, \cdots, \mathbf{x}_N; t) = H_0(\mathbf{x}_1, \cdots, \mathbf{x}_N)\Phi^{(N)}(\mathbf{x}_1, \cdots, \mathbf{x}_N; t) \tag{12a}$$

$$H_0(\mathbf{x}_1, \cdots, \mathbf{x}_N) = \sum_{j=1}^{N} \frac{-\hbar^2}{2m} \nabla_j^2 \tag{12b}$$

[3] The operators $\psi(\mathrm{x})$, $\psi^*(\mathrm{x})$ without any time label are Schrödinger picture operators; we assume the Heisenberg and Schrödinger pictures to coincide at time $t = 0$ so that $\psi(\mathbf{x}) = \psi(\mathbf{x}, 0)$, i.e. $\psi(\mathbf{x})$ is identical with the Heisenberg operator $\psi(\mathbf{x}, t)$ at time $t = 0$.

such that

$$\lim_{t\to-\infty} \int \cdots \int d^3x_1 \cdots d^3x_N \,|\, \Psi_+^{(N)}(\mathbf{x}_1, \cdots \mathbf{x}_N ; 0) - e^{+iH(\mathbf{x}_1,\cdots,\mathbf{x}_N)t/\hbar} e^{-iH_0(\mathbf{x}_1,\cdots,\mathbf{x}_N)t/\hbar} \Phi^{(N)}(\mathbf{x}_1, \cdots \mathbf{x}_N ; 0) \,|^2 \to 0 \tag{13a}$$

or equivalently

$$\Psi_+^{(N)}(\mathbf{x}_1, \cdots, \mathbf{x}_N ; 0) = \lim_{t\to-\infty} e^{+iH(\mathbf{x}_1,\cdots,\mathbf{x}_N)t/\hbar} e^{-iH_0(\mathbf{x}_1,\cdots,\mathbf{x}_N)t/\hbar} \Phi^{(N)}(\mathbf{x}_1, \cdots ; \mathbf{x}_N ; 0) \tag{13b}$$

In the proof it is assumed that at some time t_0 in the (remote) past

$$\Psi_+^{(N)}(\mathbf{x}_1, \cdots, \mathbf{x}_N ; t_0) = \Phi^{(N)}(\mathbf{x}_1, \cdots, \mathbf{x}_N ; t_0). \tag{14}$$

The relations (13a, b) can be translated into statements on state vectors by introducing the vector

$$|\Phi(t)\rangle = \frac{1}{\sqrt{N!}} \int d^3x_1 \cdots \int d^3x_N \, \Phi^{(N)}(\mathbf{x}_1, \cdots, \mathbf{x}_N ; t) \cdot \psi^*(\mathbf{x}_1) \cdots \psi^*(\mathbf{x}_N) \,|\, 0\rangle \tag{15a}$$

which, by virtue of Eqs. (12a, b), will obey the equation

$$H_0|\Phi(t)\rangle = i\hbar\partial_t|\Phi(t)\rangle \tag{15b}$$

In terms of $|\Psi_+(t)\rangle$ and $|\Phi(t)\rangle$, Eq. (13a, b) asserts that

$$|\Psi_+(0)\rangle = \lim_{t\to-\infty} e^{+iHt/\hbar} e^{-iH_0t/\hbar} |\Phi(0)\rangle = \Omega^{(+)}|\Phi(0)\rangle \tag{16}$$

with $\Omega^{(+)}H_0 = H\Omega^{(+)}$ and where the limit in Eq. (16) is to be understood in the sense of convergence in the norm.

We next ask whether it is possible to express the state vector $|\Psi(t)\rangle$ in the form

$$|\Psi_+(t)\rangle = \frac{1}{\sqrt{N!}} \int d^3x_1 \cdots \int d^3x_N \, \Phi^{(N)}(\mathbf{x}_1, \cdots, \mathbf{x}_N ; t) \cdot \psi_{\text{in}}^*(\mathbf{x}_1) \cdots \psi_{\text{in}}^*(\mathbf{x}_N) \,|\, 0\rangle \tag{17}$$

where $\Phi^{(N)}(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_N ; t)$ is the *unperturbed* amplitude satisfying Eq. (12a), and where the operators $\psi_{\text{in}}(\mathbf{x})$ which appear in Eq. (17) obey the same commutation rules as the ψ operators, i.e.,

$$[\psi_{\text{in}}(\mathbf{x}), \psi_{\text{in}}^*(\mathbf{x}')] = \delta(\mathbf{x} - \mathbf{x}') \tag{18a}$$

$$[\psi_{\text{in}}(\mathbf{x}), \psi_{\text{in}}(\mathbf{x}')] = [\psi_{\text{in}}^*(\mathbf{x}), \psi_{\text{in}}^*(\mathbf{x}')] = 0 \tag{18b}$$

and also have the property that

$$\psi_{\text{in}}(\mathbf{x})|0\rangle = 0. \tag{18c}$$

In other words the operators $\psi(\mathbf{x})$ and $\psi_{\text{in}}(\mathbf{x})$ are to form equivalent representation of the commutation relations (and hence are to be unitarily related). The fact that it is the *unperturbed* amplitude $\Phi^{(N)}(\mathbf{x}_1, \cdots, \mathbf{x}_N; t)$ which occurs in Eq. (17) for $|\Psi(t)\rangle$ is to be noted. To obtain the explicit form for the unitary transformation connecting $\psi(\mathbf{x})$ to $\psi_{\text{in}}(\mathbf{x})$ we proceed as follows:

Upon substituting into the left-hand side of Eq. (16) the expression

$$|\Psi_+(0)\rangle = e^{iHt_0/\hbar}|\Psi_+(t_0)\rangle \tag{19}$$

with $|\Psi_+(t_0)\rangle$ expressed in the form (17), and similarly for the right-hand side member of that equation, writing

$$|\Phi(0)\rangle = e^{iH_0t_0/\hbar}|\Phi(t_0)\rangle \tag{20}$$

with $|\Phi(t_0)\rangle$ in the form (15) and then using the completeness of the amplitudes $\Phi^{(N)}(\mathbf{x}_1, \cdots, \mathbf{x}_n; t)$, one deduces that

$$\psi_{\text{in}}(\mathbf{x}, t) = V_+(t)\psi(\mathbf{x}, t)V_+^{-1}(t) \tag{21}$$

where[4]

$$V_+(t) = e^{iHt/\hbar}\Omega^{(+)}e^{iHt/\hbar}. \tag{22}$$

A more explicit form for the "in" operators can be obtained from Eq. (22) by explicitly evaluating the right-hand side

$$\begin{aligned}\psi_{\text{in}}(\mathbf{x}, t) &= e^{iHt/\hbar}\Omega^{(+)}\psi(\mathbf{x}, 0)\Omega^{(+)-1}e^{-iHt/\hbar} \\ &= \lim_{t_0\to-\infty} e^{iH(t+t_0)/\hbar}e^{-iH_0t_0/\hbar}\psi(\mathbf{x}, 0)e^{+iH_0t_0/\hbar}e^{-iH(t+t_0)/\hbar} \qquad (23)\\ &= \lim_{t_0\to-\infty} e^{iH(t+t_0)/\hbar}\{e^{(i\hbar t_0\nabla_x^2/2m)}\psi(\mathbf{x}, 0)\}e^{-iH(t+t_0)/\hbar}\end{aligned}$$

$$= \lim_{t_0\to-\infty} \exp\left(\frac{i\hbar}{2m}t_0\nabla_x^2\right)\cdot\psi(\mathbf{x}, t+t_0) \tag{24}$$

$$= \lim_{t'\to-\infty}\int d^3x'\mathfrak{D}(\mathbf{x}-\mathbf{x}', t-t')\psi(\mathbf{x}', t')$$

where

$$\mathfrak{D}(\mathbf{x}-\mathbf{x}', t-t') = (2\pi\hbar)^{-3}\int d^3p e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')/\hbar}e^{-(i\mathbf{p}^2/2m)(t-t')/\hbar} \tag{25}$$

[4] The absence of bound states is reflected in the fact that under these circumstances the isometric operator $\Omega^{(+)}$ has the reciprocal $\Omega^{(+)*}$, i.e., $V_+(t)$ will be unitary.

is the Green's function for the unperturbed Schrödinger equation, satisfying the equation

$$+i\hbar\partial_t\mathfrak{D}(\mathbf{x}, t) = -\frac{\hbar^2}{2m}\nabla^2\mathfrak{D}(\mathbf{x}, t) \tag{26a}$$

and the boundary condition

$$\mathfrak{D}(\mathbf{x}, 0) = \delta(\mathbf{x}). \tag{26b}$$

In obtaining Eq. (24) it should be noted that we have defined ψ_{in} in terms of the single limit

$$\psi_{\text{in}}(\mathbf{x}, t) = \lim_{t_0\to-\infty} e^{iHt/\hbar}e^{iHt_0/\hbar}e^{-iH_0t_0/\hbar}\psi(\mathbf{x}, 0)\, e^{iH_0t_0/\hbar}e^{-iHt_0/\hbar}e^{iHt/\hbar}. \tag{27a}$$

rather than in terms of the double limit

$$\lim_{t_0\to-\infty}\lim_{t_0'\to-\infty} e^{iHt/\hbar}e^{iHt_0/\hbar}e^{-iH_0t_0/\hbar}\psi(\mathbf{x}, 0)e^{iH_0t_0'/\hbar}e^{-iHt_0'/\hbar}e^{iHt/\hbar}. \tag{27b}$$

These definitions may in fact be different in the presence of bound states.

From the above definition of the in operator, the usual properties of these operators can be deduced (*9*), namely, that they satisfy free field equations

$$\begin{aligned} i\hbar\partial_t\psi_{\text{in}}(\mathbf{x}, t) &= [\psi_{\text{in}}(\mathbf{x}, t), H] = [\psi_{\text{in}}(\mathbf{x}, t), H_{0\text{in}}] \\ &= -\frac{\hbar^2}{2m}\nabla^2\psi_{\text{in}}(\mathbf{x}, t) \end{aligned} \tag{28}$$

and free field commutation rules

$$\begin{aligned} &[\psi_{\text{in}}(\mathbf{x}, t), \psi^*_{\text{in}}(\mathbf{x}', t')] = \mathfrak{D}(\mathbf{x} - \mathbf{x}', t - t'); \\ &[\psi_{\text{in}}(\mathbf{x}, t), \psi_{\text{in}}(\mathbf{x}', t')] = [\psi^*_{\text{in}}(\mathbf{x}, t), \psi^*_{\text{in}}(\mathbf{x}', t')] = 0. \end{aligned} \tag{29a}$$

These definitions of the "in" operators (and the analogous ones for $t = +\infty$ for the "out" operators) are satisfactory in the absence of bound states. It can for example be shown that the set of states

$$\sum_{j=1}^{m}\int d^3x_1 \cdots \int d^3x_j f_j(\mathbf{x}_1, \cdots, \mathbf{x}_j)\, \psi^*_{\text{in}}(\mathbf{x}_1)\, \psi^*_{\text{in}}(\mathbf{x}_j)\, |\, 0\rangle \tag{30}$$

for any integer m and any square integrable functions f_j are everywhere dense in the smallest Hilbert space which contains all physical state vectors.

II. DEFINITION OF ASYMPTOTIC FIELDS IN NONRELATIVISTIC SITUATION

In the presence of bound states (e.g., in the presence of an external potential which can bind particles) it is no longer true that the states of the form (30)

span the Hilbert space of physical states. At most these states span the "scattering" subspace of the Hilbert space of physical states.

In order to generalize the definition of the asymptotic fields in the nonrelativistic case, so as to encompass bound states, we define the one particle retarded asymptotic operator as

$$\psi_{\mathrm{R}}(\mathbf{x}, t) = i \lim_{t' \to -\infty} \int d^3x' < 0 \mid R(\psi(\mathbf{x}, t)\psi^*(\mathbf{x}', t')) \mid 0 > \psi(\mathbf{x}', t') \tag{31}$$

where R denotes the chronological retardation operation:

$$R(\psi(\mathbf{x}, t)\psi^*(\mathbf{x}', t')) = -i\theta(t - t')[\psi(\mathbf{x}, t), \psi^*(\mathbf{x}', t')] \tag{32}$$

and the state $|0\rangle$ is the no particle state. Since in a nonrelativistic theory

$$\psi(\mathbf{x}, t)|0\rangle = 0,$$

only the first term of the commutator contributes, so that we could also have defined the retarded asymptotic one particle operator as

$$\psi_{\mathrm{R}}(\mathbf{x}, t) = \lim_{t' \to -\infty} \int d^3x' \langle 0 \mid T(\psi(\mathbf{x}, t)\psi^*(\mathbf{x}', t')) \mid 0\rangle \psi(\mathbf{x}', t') \tag{33}$$

where T is the Wick chronological operator, or more simply as

$$\psi_{\mathrm{R}}(\mathbf{x}, t) = \lim_{t' \to -\infty} \int d^3x' \langle 0 \mid \psi(\mathbf{x}, t)\psi^*(\mathbf{x}', t' \mid 0\rangle \psi(\mathbf{x}', t'). \tag{34}$$

For a system described by the Hamiltonian (5) the state $\psi^*(\mathbf{x}, t)|0\rangle$ obeys the equation of motion

$$\begin{aligned} i\hbar\partial_t\psi^*(\mathbf{x}, t) \mid 0\rangle &= [\psi^*(\mathbf{x}, t), H] \mid 0\rangle \\ &= + \frac{\hbar^2}{2m} \nabla^2\psi^*(\mathbf{x}, t) \mid 0\rangle. \end{aligned} \tag{35}$$

Hence $\psi^*(\mathbf{x}, t)|0\rangle$ is a "free" one particle state. If we denote by $|\mathbf{p}\rangle$ the one-particle state of momentum $\mathbf{p}$ and energy $E_{\mathbf{p}} = \mathbf{p}^2/2m$ then:

$$\langle \mathbf{p} \mid \psi^*(\mathbf{x}, t) \mid 0\rangle = (2\pi\hbar)^{-3/2} \exp\left\{\frac{i}{\hbar}(E_{\mathbf{p}}t - \mathbf{p}\cdot\mathbf{x})\right\} \tag{36}$$

and

$$\begin{aligned} i\langle 0 \mid R(\psi(\mathbf{x}, t)\psi^*(\mathbf{x}', t)) \mid 0\rangle &= \theta(t - t')\langle 0 \mid \psi(\mathbf{x}, t)\psi^*(\mathbf{x}', t') \mid 0\rangle \\ &= \theta(t - t') \sum_{|\mathbf{p}\rangle} \langle 0 \mid \psi(\mathbf{x}, t) \mid \mathbf{p}\rangle\langle \mathbf{p} \mid \psi^*(\mathbf{x}', t') \mid 0\rangle \\ &= \theta(t - t')\mathfrak{D}(\mathbf{x} - \mathbf{x}', t - t') \end{aligned} \tag{37}$$

so that the definition (31) of $\psi_R(\mathbf{x}, t)$ reduces to the previous definition of $\psi_{in}(\mathbf{x}, t)$, Eq. (24). However, in the case that bound one-particle states exist, the above definition is more general, and the operator $\psi_R(\mathbf{x}, t)$ includes a part which creates particles in a bound state. To verify this assertion consider a system of nonrelativistic bosons interacting with a time-independent potential $V(\mathbf{x})$, as well as interacting with one another through a potential $v(|\mathbf{x} - \mathbf{x}'|)$ which is of finite range. The Hamiltonian for the second quantized theory is now given by

$$H = \int d^3x \psi^*(\mathbf{x}, t) \left\{ -\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{x}) \right\} \psi(\mathbf{x}, t) + \frac{1}{2} \int d^3x \int d^3x' \psi^*(\mathbf{x}', t) \psi^*(\mathbf{x}, t) v(|\mathbf{x} - \mathbf{x}'|) \psi(\mathbf{x}, t) \psi(\mathbf{x}', t). \tag{38}$$

Explicitly, we shall assume that the potential $V(\mathbf{x})$ is of short range and that it has the property that in addition to the usual (one-particle) scattering states, $f_\alpha^{(\pm)}(\mathbf{x})$, which are solutions of

$$f_\alpha^{(\pm)}(\mathbf{x}) = e^{i\mathbf{k}_\alpha \cdot \mathbf{x}} - \frac{m}{2\pi\hbar^2} \int \frac{e^{\pm ik_\alpha |\mathbf{x} - \mathbf{x}'|}}{|\mathbf{x} - \mathbf{x}'|} V(\mathbf{x}') f_\alpha^{(\pm)}(\mathbf{x}') \, d^3x' \tag{39}$$

$V(\mathbf{x})$ allows a finite number of bound states, $f_b(\mathbf{x})$, normalizable solutions of

$$\left\{ -\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{x}) \right\} f_b(\mathbf{x}) = E_b f_b(\mathbf{x}) \tag{40}$$

with $E_b < 0$.

We shall assume that these solutions are complete in the sense that

$$\int d\alpha \, f_\alpha^{(\pm)}(\mathbf{x}) \overline{f_\alpha^{(\pm)}(\mathbf{x}')} + \sum_b f_b(\mathbf{x}) \overline{f_b(\mathbf{x}')} = \delta(\mathbf{x} - \mathbf{x}') \tag{41}$$

and that they satisfy the following orthonormality relations:

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$$\int d^3x \overline{f_\alpha(\mathbf{x}, t)} f_\beta(\mathbf{x}, t) = \delta(\alpha - \beta) \tag{42a}$$

$$\int d^3x \overline{f_b(\mathbf{x}, t)} f_{b'}(\mathbf{x}, t) = \delta_{bb'} \tag{42b}$$

$$\int d^3x \overline{f_\alpha(\mathbf{x}, t)} f_b(\mathbf{x}, t) = 0. \tag{42c}$$

Under the above circumstances, the one-particle state $\psi^*(\mathbf{x}, t)|0\rangle$ obeys the equation

$$-i\hbar\partial_t\,\psi^*(\mathbf{x},t)\,|\,0\rangle = \left\{-\frac{\hbar^2}{2m}\,\nabla^2 + V(\mathbf{x})\right\}\psi^*(\mathbf{x},t)\,|\,0\rangle \tag{43}$$

so that if $|f\rangle$ is a one-particle state, $\int\psi^*(\mathbf{x},t)\psi(\mathbf{x},t)\,d^3x\,|f\rangle = |f\rangle$, the amplitude

$$f(\mathbf{x},t) = \langle 0\,|\,\psi(\mathbf{x},t)\,|f\rangle \tag{44}$$

satisfies the Schrödinger equation

$$i\hbar\partial_t\,f(\mathbf{x},t) = \left\{-\frac{\hbar^2}{2m}\,\nabla^2 + V(\mathbf{x})\right\}f(\mathbf{x},t). \tag{45}$$

Within the one-particle subspace, the states $|f_\alpha^{(\pm)}\rangle$, $|f_b\rangle$ where

$$\langle 0\,|\,\psi(\mathbf{x},t)\,|f_\alpha^{(\pm)}\rangle = f_\alpha^{(\pm)}(\mathbf{x},t) = e^{-iE_\alpha t/\hbar}f_\alpha^{(\pm)}(\mathbf{x}); \qquad E_\alpha = \frac{\mathbf{p}_\alpha^{\,2}}{2m}, \tag{46a}$$

$$\langle 0\,|\,\psi(\mathbf{x},t)\,|f_b\rangle = f_b(\mathbf{x},t) = e^{-iE_b t/\hbar}f_b(\mathbf{x}) \tag{46b}$$

are therefore complete, in the sense that

$$\int d\alpha\,|f_\alpha^{(\pm)}\rangle\langle f_\alpha^{(\pm)}| + \sum_b |f_b\rangle\langle f_b| = \text{unit operator in one-particle subspace.} \tag{47}$$

Hence for a system described by the Hamiltonian (38), we have:

$$\begin{aligned}\langle 0\,|\,\psi(\mathbf{x},t)\psi^*(\mathbf{x}',t')\,|\,0\rangle &= \int d\alpha\langle 0\,|\,\psi(\mathbf{x},t)\,|f_\alpha^{(\pm)}\rangle\langle f_\alpha^{(\pm)}\,|\,\psi^*(\mathbf{x}',t')\,|\,0\rangle \\ &\quad + \sum_b \langle 0\,|\,\psi(\mathbf{x},t)\,|f_b\rangle\langle f_b\,|\,\psi^*(\mathbf{x}',t')\,|\,0\rangle \\ &= \int d\alpha f_\alpha^{(\pm)}(\mathbf{x},t)\overline{f_\alpha^{(\pm)}(\mathbf{x}',t')} + \sum_b f_b(\mathbf{x},t)\overline{f_b(\mathbf{x}',t')} \\ &= \mathfrak{D}^{\mathrm{V}}(\mathbf{x},\mathbf{x}';t-t')\end{aligned} \tag{48}$$

where $\mathfrak{D}^{\mathrm{V}}(\mathbf{x},\mathbf{x}';t)$ is the singular function which satisfies

$$i\hbar\partial_t\,\mathfrak{D}^{\mathrm{V}}(\mathbf{x},\mathbf{x}';t) = \left\{-\frac{\hbar^2}{2m}\,\nabla_x^{\,2} + V(\mathbf{x})\right\}\mathfrak{D}^{\mathrm{V}}(\mathbf{x},\mathbf{x}';t) \tag{49a}$$

and the boundary condition

$$\mathfrak{D}^{\mathrm{V}}(\mathbf{x},\mathbf{x}';0) = \delta(\mathbf{x}-\mathbf{x}'). \tag{49b}$$

The retarded operator $\psi_{\mathrm{R}}(\mathbf{x},t)$ therefore can be written in the form

$$\begin{aligned}\psi_{\mathrm{R}}(\mathbf{x},t) = \lim_{t'\to-\infty}\Bigg\{&\int d\alpha f_\alpha^{(+)}(\mathbf{x},t)\int d^3x'\overline{f_\alpha^{(+)}(\mathbf{x}',t')}\psi(\mathbf{x}',t') \\ &+ \sum_b f_b(\mathbf{x},t)\int d^3x'\overline{f_b(\mathbf{x}',t')}\psi(\mathbf{x}',t')\Bigg\}.\end{aligned} \tag{50}$$

The orthonomality relations (42) allow one to define the operators

$$\psi_{\mathrm{R}}^{f_\alpha^{(+)}}(t) = \lim_{t'\to-\infty} \int d^3x' \overline{f_\alpha^{(+)}(\mathbf{x}', t')}\psi(\mathbf{x}', t') \tag{51a}$$

and

$$\psi_{\mathrm{R}}^{f_b}(t) = \lim_{t'\to-\infty} \int d^3x' \overline{f_b(\mathbf{x}', t')}\psi(\mathbf{x}', t'). \tag{51b}$$

Equations (51) together with Eq. (43) allow us to infer that the vectors

$$\psi_{\mathrm{R}}^{f_\alpha *}(t)\,|\,0\rangle \qquad \text{and} \qquad \psi_{\mathrm{R}}^{f_b *}(t)\,|\,0\rangle$$

are time independent

$$i\hbar\partial_t\psi_{\mathrm{R}}^{f_b *}(t)\,|\,0\rangle = i\hbar\partial_t\psi_{\mathrm{R}}^{f_\alpha *}(t)\,|\,0\rangle = 0. \tag{52}$$

In fact, one readily verifies that the state $\psi_{\mathrm{R}}^{f_b *}|\,0\rangle$ is a bound one-particle state: it is an eigenstate of H with eigenvalue E_b. Similarly $\psi^{f_\alpha *}|\,0\rangle$ is a one-particle eigenstate of H with eigenvalue E_α. By virtue of the choice of amplitude $f_\alpha^{(+)}$ (rather than $f_\alpha^{(-)}$) it can furthermore be shown that the state $\psi_{\mathrm{R}}^{f_\alpha(+)*}|\,0\rangle$ coincides with the state $\psi_{\mathrm{in}}^{f_\alpha{}^0 *}|\,0\rangle$ where

$$\psi_{\mathrm{in}}^{f_\alpha{}^0}(t) = \lim_{t'\to-\infty} \int d^3x' \overline{f_\alpha{}^0(\mathbf{x}', t')}\psi(\mathbf{x}', t'), \tag{53a}$$

with $f_\alpha{}^0(\mathbf{x}, t)$ a normalizable solution of

$$-\frac{\hbar^2}{2m}\nabla^2 f_\alpha{}^0(\mathbf{x}, t) = i\hbar\partial_t f_\alpha{}^0(\mathbf{x}, t) \tag{53b}$$

in the limit as $f_\alpha{}^0(\mathbf{x}, t) \to (2\pi\hbar)^{-3/2}\exp\left(i\mathbf{k}_\alpha\mathbf{x} - \frac{iE_{\mathbf{k}_\alpha}}{\hbar}t\right)$. Hence, if we write

$$\psi_{\mathrm{R}}(\mathbf{x}, t) = \psi_S(\mathbf{x}, t) + \psi_B(\mathbf{x}, t), \tag{54}$$

$\psi_S(\mathbf{x}, t)$ as given by the first term in Eq. (50) (arising from the sum over the scattering state α) is a destruction operator for particles in scattering states, and $\psi_B(\mathbf{x}, t)$, the second term in Eq. (54), is a destruction operator for particles bound in the potential $V(\mathbf{x})$.

Let us next derive the equation of motion that ψ_{R} obeys. To that end we note that

$$\begin{aligned}\psi_{\mathrm{R}}(\mathbf{x}, t) = i \lim_{t'\to+\infty} \int d^3x' \langle 0\,|\,R(\psi(\mathbf{x}, t)\psi^*(\mathbf{x}', t'))\,|\,0\rangle\psi(x', t') \\ -i\int d^3x' \int_{-\infty}^{+\infty} dt' \frac{\partial}{\partial t'}\{\langle 0\,|\,R(\psi(\mathbf{x}, t)\psi^*(\mathbf{x}', t'))\,|\,0\rangle\psi(x', t')\}.\end{aligned} \tag{55}$$

The first term vanishes by virtue of the retarded character of $R(\psi(\mathbf{x}, t)\psi^*(\mathbf{x}', t'))$. Next, by virtue of the canonical commutation rules,

$$i\partial_{t'}R(\psi(\mathbf{x}, t)\psi^*(\mathbf{x}', t')) = -\delta(\mathbf{x} - \mathbf{x}')\delta(t - t') + iR(\psi(\mathbf{x}, t)\partial_{t'}\psi^*(\mathbf{x}', t')) \quad (56)$$

so that

$$\begin{aligned}\psi_{\mathrm{R}}(\mathbf{x}, t) = \psi(\mathbf{x}, t) &- \int d^3x \int_{-\infty}^{+\infty} dt' \langle 0 \mid R(\psi(\mathbf{x}, t)\psi^*(\mathbf{x}', t)) \mid 0\rangle i\partial_{t'}\psi(\mathbf{x}', t') \\ &- \int d^3x' \int_{-\infty}^{+\infty} dt' \langle 0 \mid R(\psi(\mathbf{x}, t)i\partial_{t'}\psi^*(\mathbf{x}', t') \mid 0\rangle \psi(\mathbf{x}', t').\end{aligned} \quad (57)$$

Since

$$\begin{aligned}-i\hbar\partial_t \psi^*(\mathbf{x}, t) &= \left\{-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{x})\right\}\psi^*(\mathbf{x}, t) + J^*(\mathbf{x}, t), \\ J^*(\mathbf{x}, t) &= \int d^3y \psi^*(\mathbf{x}, t)\psi^*(\mathbf{y}, t)v(|\mathbf{x} - \mathbf{y}|)\psi(\mathbf{y}, t),\end{aligned} \quad (58)$$

upon noting that $\psi(\mathbf{y}, t)|0\rangle = 0$, we find that

$$\begin{aligned}\psi_{\mathrm{R}}(\mathbf{x}, t) &= \psi(\mathbf{x}, t) - \frac{1}{\hbar}\int d^3x \int_{-\infty}^{\infty} dt' \langle 0 \mid R(\psi(\mathbf{x}, t)\psi^*(\mathbf{x}', t')) \mid 0\rangle \\ &\quad \cdot \left(i\hbar\partial_{t'} + \frac{\hbar^2}{2m}\nabla_{x'}^2 - V(\mathbf{x}')\right)\psi(\mathbf{x}', t') \\ &= \psi(\mathbf{x}, t) - \frac{i}{\hbar}\int d^3x' \int_{-\infty}^{\infty} dt' \mathfrak{D}_{\mathrm{R}}{}^{V}(\mathbf{x}, \mathbf{x}', t - t')J(\mathbf{x}', t);\end{aligned} \quad (59)$$

where

$$\mathfrak{D}_{\mathrm{R}}{}^{V}(\mathbf{x}, \mathbf{x}', t) = -\theta(t)\mathfrak{D}^{V}(\mathbf{x}, \mathbf{x}'; t). \quad (60)$$

Equation (59) again indicates that in the absence of a one-particle potential (i.e., when $V(\mathbf{x}) = 0$ and $\mathfrak{D}^{V}(\mathbf{x}, \mathbf{x}'; t)|_{V=0} = \mathfrak{D}(\mathbf{x} - \mathbf{x}'; t))\psi_{\mathrm{R}}$ satisfies the same equation of motion as the conventionally defined $\psi_{\mathrm{in}}(\mathbf{x}, t)$.

Consider next the commutation rules which these "retarted" asymptotic fields satisfy:

$$\begin{aligned}[\psi_{\mathrm{R}}(\mathbf{x}, t), \psi_{\mathrm{R}}{}^*(\mathbf{x}', t')] = \lim_{t''\to-\infty} \lim_{t'''\to-\infty} \int d^3y \int d^3y' [\psi(\mathbf{y}, t''), \psi^*(\mathbf{y}', t''')] \\ \cdot \langle 0 \mid \psi(\mathbf{x}, t)\psi^*(\mathbf{y}, t'') \mid 0\rangle \overline{\langle 0 \mid \psi(\mathbf{x}', t')\psi^*(\mathbf{y}', t''') \mid 0\rangle}\end{aligned} \quad (61)$$

Since the amplitudes $\Psi_{\pm}^{(N)}(\mathbf{x}_1, \cdots, \mathbf{x}_N; t)$ converge strongly to the amplitudes

$$\Phi^{(N)}(\mathbf{x}_1, \cdots, \mathbf{x}_N; t)$$

(the norm of the states are conserved) the convergence of $\psi(\mathbf{x}, t)$ to $\psi_R(\mathbf{x}, t)$ (or more accurately that of $\psi^f(t)$ to $\psi_R{}^f$) is likewise strong. The two limits in Eq. (61) can therefore be replaced by the single limit, $\lim t'' = t''' \to -\infty$. Hence

$$
\begin{aligned}
&[\psi_R(\mathbf{x}, t), \psi_R{}^*(\mathbf{x}', t')] \\
&\quad = \lim_{t''\to-\infty} \int d^3y \int d^3y' \langle 0 \mid \psi(\mathbf{x}, t)\psi^*(\mathbf{y}, t'') \mid 0\rangle \\
&\qquad\qquad \cdot \overline{\langle 0 \mid \psi(\mathbf{x}', t')\psi^*(\mathbf{y}', t'') \mid 0\rangle}[\psi(\mathbf{y}, t''), \psi^*(\mathbf{y}', t'')] \\
&\quad = \lim_{t''\to-\infty} \int d^3y\, d^3y' \langle 0 \mid \psi(\mathbf{x}, t)\psi^*(\mathbf{y}, t'') \mid 0\rangle \\
&\qquad\qquad \cdot \overline{\langle 0 \mid \psi(\mathbf{x}', t')\psi^*(\mathbf{y}', t'') \mid 0\rangle}\delta(\mathbf{y} - \mathbf{y}') \\
&\quad = \lim_{t''\to-\infty} \int d^3y \langle 0 \mid \psi(\mathbf{x}, t)\psi^*(\mathbf{y}, t'') \mid 0\rangle\langle 0 \mid \psi(\mathbf{y}, t'')\psi^*(\mathbf{x}', t') \mid 0\rangle.
\end{aligned}
\tag{62}
$$

Now in the nonrelativistic situation under discussion

$$
\begin{aligned}
&\langle 0 \mid \psi(\mathbf{x}, t)\psi^*(\mathbf{y}, t'') \mid 0\rangle\langle 0 \mid \psi(\mathbf{y}, t'')\psi^*(\mathbf{x}', t') \mid 0\rangle \\
&\qquad = \langle 0 \mid \psi(\mathbf{x}, t)\psi^*(\mathbf{y}, t'')\psi(\mathbf{y}, t'')\psi^*(\mathbf{x}', t') \mid 0\rangle
\end{aligned}
\tag{63}
$$

since $\langle n \mid \psi(\mathbf{y}, t)\psi^*(\mathbf{x}, t) \mid 0\rangle = 0$ if $\mid n\rangle \neq \mid 0\rangle$. Furthermore, since

$$\int d^3y\, \psi^*(\mathbf{y}, t)\psi(\mathbf{y}, t)$$

is the number operator, we finally conclude that

$$[\psi_R(\mathbf{x}, t), \psi_R{}^*(\mathbf{x}', t')] = \langle 0 \mid \psi(\mathbf{x}, t)\psi^*(\mathbf{x}', t') \mid 0\rangle = \mathfrak{D}^+(\mathbf{x}, \mathbf{x}', t - t'). \tag{64}$$

We next define the advanced one-particle asymptotic operator as

$$\psi_A(\mathbf{x}, t) = -i \lim_{t'\to+\infty} \int d^3x' \langle 0 \mid A(\psi(\mathbf{x}, t)\psi^*(\mathbf{x}', t')) \mid 0\rangle\psi(\mathbf{x}', t') \tag{65a}$$

where

$$A(\psi(\mathbf{x}, t)\psi^*(\mathbf{x}', t')) = i\theta(t' - t)[\psi(\mathbf{x}, t), \psi^*(\mathbf{x}', t')], \tag{65b}$$

or equivalently

$$\psi_A(\mathbf{x}, t) = \lim_{t'\to+\infty} \int d^3x' \theta(t' - t)\langle 0 \mid \psi(\mathbf{x}, t)\psi^*(\mathbf{x}', t) \mid 0\rangle\psi(\mathbf{x}', t'). \tag{65c}$$

This advanced operator satisfies the equation of motion

$$\psi_A(\mathbf{x}, t) = \psi(\mathbf{x}, t) - \frac{i}{\hbar}\int d^3x' \int_{-\infty}^{+\infty} dt'\theta(t' - t)\langle 0 \mid \psi(\mathbf{x}, t)\psi^*(\mathbf{x}', t') \mid 0\rangle$$

$$\cdot \left(i\hbar\partial_{t'} + \frac{\hbar^2}{2m}\nabla_{x'}^2 - V(\mathbf{x}')\right)\psi(\mathbf{x}', t') \quad (66)$$

$$= \psi(\mathbf{x}, t) - \frac{i}{\hbar}\int d^3x' \int_{-\infty}^{+\infty} dt'\mathfrak{D}_A{}^V(\mathbf{x}, \mathbf{x}', t - t')J(\mathbf{x}', t')$$

where

$$\mathfrak{D}_A{}^V(\mathbf{x}, \mathbf{x}' \; t - t') = \theta(t' - t)\mathfrak{D}^V(\mathbf{x}, \mathbf{x}', t - t') \quad (67)$$

so that

$$\psi_A(\mathbf{x}, t) = \psi_R(\mathbf{x}, t) - \frac{i}{\hbar}\int d^3x' \int dt'\langle 0 \mid \psi(\mathbf{x}, t)\psi^*(\mathbf{x}', t') \mid 0\rangle J(\mathbf{x}', t')$$

$$= \psi_R(\mathbf{x}, t) - \frac{i}{\hbar}\int d^3\mathbf{x}' \int dt'\mathfrak{D}^V(\mathbf{x}, \mathbf{x}', t - t')J(\mathbf{x}', t'). \quad (68)$$

The commutation rules which the one-particle advanced asymptotic field satisfies are verified to be

$$[\psi_A(\mathbf{x}, t), \psi_A{}^*(\mathbf{x}', t')] = \langle 0 \mid \psi(\mathbf{x}, t)\psi^*(\mathbf{x}', t') \mid 0\rangle. \quad (69)$$

We conclude these remarks concerning the one-particle asymptotic operators by noting that their definitions allow the derivation of contraction rules similar to the usual case. Consider, for example, the matrix element ${}_{\text{out}}\langle b, \alpha' \mid b, \alpha\rangle_{\text{in}}$ corresponding to the elastic scattering of two bosons, one of them being in a bound state:

$${}_{\text{out}}\langle b, \alpha' \mid b, \alpha\rangle_{\text{in}} = \langle 0 \mid \psi_A^{f_b}\psi_A^{f_{\alpha'}^{(-)}}\psi_R^{f_\alpha^{(+)}*}\psi_R^{f_b*} \mid 0\rangle. \quad (70)$$

Using the defining relations one readily deduces that

$${}_{\text{out}}\langle b, \alpha' \mid b, \alpha\rangle_{\text{in}} = \delta_{\alpha\alpha'} - \frac{i}{\hbar}\int d^3x' \int_{-\infty}^{+\infty} dt'$$

$$\cdot\; {}_{\text{out}}\langle b, \alpha' \mid \psi^*(\mathbf{x}', t') \mid \alpha\rangle_{\text{in}}\overleftarrow{S}^*_{\mathbf{x}',t'}f_b(\mathbf{x}', t') \quad (71)$$

with

$$S_{\mathbf{x},t} = i\hbar\partial_t + \frac{\hbar^2}{2m}\nabla_x{}^2 - V(\mathbf{x}) \quad (72)$$

and the arrow on top of S implies that it is to operate on the expression standing to the left of it. Similarly upon contracting the bound particle b in the final state we deduce that

$$
\begin{aligned}
{}_{\text{out}}\langle b, \alpha' \mid b, \alpha\rangle_{\text{in}} &= \delta_{\alpha\alpha'} + \frac{1}{\hbar^2}\int d^3x' \int d^3x'' \int_{-\infty}^{+\infty} dt' \int_{-\infty}^{+\infty} dt'' \\
&\cdot \overline{f_b(\mathbf{x}'', t'')}\, \vec{S}_{\mathbf{x}'', t''}\, {}_{\text{out}}\langle \alpha' \mid T(\psi(\mathbf{x}', t')\psi^*(\mathbf{x}'', t'')) \mid \alpha\rangle_{\text{in}} \overleftarrow{S}^*_{\mathbf{x}', t'} f_b(\mathbf{x}', t').
\end{aligned} \tag{73}
$$

We could of course also contract the particles α' and α, in which case we would be led to an expression involving the Green's function

$$\langle 0 \mid T(\psi(\mathbf{x}_1, t_1)\psi(\mathbf{x}_2, t_2)\psi^*(\mathbf{x}_1', t_1')\psi^*(\mathbf{x}_2', t_2')) \mid 0\rangle.$$

It should be noted that the matrix element (70) can directly be written in term of this latter Green's function as follows

$$
\begin{aligned}
{}_{\text{out}}\langle b, \alpha' \mid b, \alpha\rangle_{\text{in}} &= \lim_{t\to+\infty} \lim_{t'\to-\infty} \int d^3x_1 \int d^3x_2 \int d^3x_1' \int d^3x_2' \\
&\cdot \overline{f_b(\mathbf{x}_1, t) f_\alpha^{(-)}(\mathbf{x}_2, t)} \langle 0 \mid T(\psi(\mathbf{x}_1, t')\psi(\mathbf{x}_2, t')\psi^*(\mathbf{x}_1', t')\psi^*(\mathbf{x}_2', t')) \mid 0\rangle \\
&\cdot f_b(\mathbf{x}_1', t') f_\alpha^{(+)}(\mathbf{x}_2', t').
\end{aligned} \tag{74}
$$

Such considerations, as well as the definition (33) for the retarded one-particle operator suggest that we define the retarded asymptotic two-particle operator $\psi_{\text{R}}^{(2)}(\mathbf{x}_1, \mathbf{x}_2; t)$ as

$$
\begin{aligned}
\psi_{\text{R}}^{(2)}(\mathbf{x}_1, \mathbf{x}_2; t) &= i \lim_{t'\to-\infty} \int d^3x_1 \int d^3x_2 \\
&\cdot \langle 0 \mid R(\psi(\mathbf{x}_1, t)\psi(\mathbf{x}_2, t)\psi^*(\mathbf{x}_1', t')\psi^*(\mathbf{x}_2', t') \mid 0\rangle \psi(\mathbf{x}_1', t')\psi\mathbf{x}_2', t')
\end{aligned} \tag{75a}
$$

where, if $A(t)$ and $B(t')$ are products of operators labeled by the same time, we define the retarded product by the rule:

$$R(A(t)B(t')) = -i\theta(t - t')[A(t), B(t')]. \tag{75b}$$

In the nonrelativistic situation we can once again replace in the defining equation (75a) the R product by a T product. Actually a more general definition of the two-particle retarded operator is possible involving two times, one for each particle. As the desired properties of the two-particle retarded asymptotic operator is already incorporated in the operator defined above, we shall not consider these more general operators in the nonrelativistic situation. Also, for the sake of simplicity, we shall hereafter only consider the case that $V = 0$ so that the system is described by the Hamiltonian (5).

Since we are to take the limit $t' \to -\infty$ in Eq. (75), we first study the properties of the kernel $\langle 0 \mid T(\psi\psi\psi^*\psi^*) \mid 0\rangle$ when $t > t'$. For $t > t'$

$$
\begin{aligned}
&G^{(2)}(\mathbf{x}_1, \mathbf{x}_2, t; \mathbf{x}_1', \mathbf{x}_2', t') \\
&\quad = \langle 0 \mid T(\psi(\mathbf{x}_1, t)\psi(\mathbf{x}_2, t)\psi^*(\mathbf{x}_1', t')\psi^*(\mathbf{x}_2', t')) \mid 0\rangle \\
&\quad = \sum_{\{X_n\}} \langle 0 \mid \psi(\mathbf{x}_1, t)\psi(\mathbf{x}_2, t) \mid X_n\rangle\langle X_n \mid (\mathbf{x}_1', t')\, \psi^*(\mathbf{x}_2', t')) \mid 0\rangle
\end{aligned} \tag{76}
$$

where $\{|\mathrm{X}_n\rangle\}$ is a complete set of two-particle states. The amplitudes

$$\begin{aligned}\chi_n(\mathbf{x}_1, \mathbf{x}_2; t) &= \langle 0 \mid \psi(\mathbf{x}_1, t)\psi(\mathbf{x}_2, t) \mid \mathrm{X}_n\rangle \\ &= \chi_n(\mathbf{x}_2, \mathbf{x}_1; t)\end{aligned} \tag{77}$$

are the usual two-particle Schrödinger amplitudes satisfying the equation

$$i\hbar\partial_t\chi_n(\mathbf{x}_1, \mathbf{x}_2; t) = \left\{-\frac{\hbar^2}{2m}(\nabla_1^2 + \nabla_2^2) + v(|\mathbf{x}_1 - \mathbf{x}_2|)\right\}\chi_n(\mathbf{x}_1, \mathbf{x}_2; t). \tag{78}$$

If we denote by E_n and $\mathbf{p}_n$ the eigenvalues of the Hamiltonian and (total) momentum operators in the state $|\chi_n\rangle$:

$$H \mid \mathrm{X}_n\rangle = E_n \mid \mathrm{X}_n\rangle \tag{79a}$$

$$\mathbf{P} \mid \mathrm{X}_n\rangle = \mathbf{p}_n \mid \mathrm{X}_n\rangle \tag{79b}$$

then

$$i\hbar\partial_t\chi_n(\mathbf{x}_1, \mathbf{x}_2; t) = E_n\chi_n(\mathbf{x}_1, \mathbf{x}_2; t) \tag{80a}$$

$$-i\hbar(\nabla_1 + \nabla_2)\chi_n(\mathbf{x}_1, \mathbf{x}_2; t) = \mathbf{p}_n\chi_n(\mathbf{x}_1, \mathbf{x}_2; t). \tag{80b}$$

From the completeness of $|\mathrm{X}_n\rangle$ in the two particle subspace it follows that the amplitudes $\chi_n(\mathbf{x}_1, \mathbf{x}_2, t)$ satisfy the following relations

$$\begin{aligned}&\sum_{\{\chi_n\}} \langle \mathbf{x}_1, \mathbf{x}_2 \mid \mathrm{X}_n(t)\rangle\langle \mathrm{X}_n(t) \mid \mathbf{x}_1', \mathbf{x}_2'\rangle \\ &\quad = \sum_n \chi_n(\mathbf{x}_1, \mathbf{x}_2, t)\overline{\chi_n(\mathbf{x}_1', \mathbf{x}_2', t)} \\ &\quad = \langle \mathbf{x}_1, \mathbf{x}_2 \mid \mathbf{x}_1', \mathbf{x}_2'\rangle \\ &\quad = \frac{1}{2!}\{\delta(\mathbf{x}_1 - \mathbf{x}_1')\delta(\mathbf{x}_2 - \mathbf{x}_2') + \delta(\mathbf{x}_1 - \mathbf{x}_2')\delta(\mathbf{x}_2 - \mathbf{x}_1')\}.\end{aligned} \tag{81}$$

Upon introducing relative and center-of-mass coordinates

$$\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{P}; \qquad \tfrac{1}{2}(\mathbf{x}_1 + \mathbf{x}_2) = \mathbf{R}; \tag{82a}$$

$$\tfrac{1}{2}(\mathbf{p}_1 - \mathbf{p}_2) = \mathbf{p}; \qquad \mathbf{x}_1 - \mathbf{x}_2 = \mathbf{r} \tag{82b}$$

we can write

$$\chi_n(\mathbf{x}_1, \mathbf{x}_2, t) = \exp\left\{\frac{i}{\hbar}(\mathbf{P}_n\cdot\mathbf{R} - E_n t)\right\} f(\mathbf{r}, \epsilon_n, \alpha) \tag{83}$$

where E_n is the total energy:

$$E_n = \frac{\mathbf{P}_n^2}{4m} + \epsilon_n \tag{84}$$

and ϵ_n the energy of the relative motion. The $f(\mathbf{r}, \epsilon_n, \alpha)$ are solutions of

$$\left\{-\frac{\hbar^2}{m}\nabla_r^2 + v(|\mathbf{r}|)\right\} f(\mathbf{r}, \epsilon_n, \alpha) = \epsilon_n f(\mathbf{r}, \epsilon_n, \alpha) \tag{85}$$

which, due to the Bose character of the particles, are symmetric under the interchange $\mathbf{r} \to -\mathbf{r}$:

$$f(\mathbf{r}, \epsilon_n, \alpha) = f(-\mathbf{r}, \epsilon_n, \alpha). \tag{86}$$

In Eq. (83), α denotes the eigenvalues of whatever other observables specify the relative motion of the two-particle system. The orthonomality conditions satisfied by the χ_n are

$$\int d^3R\, \overline{\chi_{n'}(\mathbf{x}_1, \mathbf{x}_2, t)}\chi_n(\mathbf{x}_1, \mathbf{x}_2, t) = (2\pi\hbar)^3\delta(\mathbf{P}_n - \mathbf{P}_{n'})\overline{f(\mathbf{r}, \epsilon_n, \alpha^1)}f(\mathbf{r}, \epsilon_n, \alpha): \tag{87a}$$

$$\int d^3R \int d^3r\, \overline{\chi_{n'}(\mathbf{x}_1, \mathbf{x}_2, t)}\chi_n(\mathbf{x}_1, \mathbf{x}_2, t) = (2\pi\hbar)^3\delta(\mathbf{P}_n - \mathbf{P}_{n'})\delta_{\epsilon_n, \epsilon_{n'}}\delta_{\alpha\alpha'}. \tag{87b}$$

The Green's function (76) occurring in the definition of $\psi_R(\mathbf{x}_1, \mathbf{x}_2; t)$, upon introducing the explicit representation (83) for χ_n, can therefore be written as

$$\begin{aligned}\langle 0 \mid T(\psi(\mathbf{x}_1, t)\psi(\mathbf{x}_2, t)\psi^*(\mathbf{x}_1', t')\psi^*(\mathbf{x}_2', t)) \mid 0\rangle &= \theta(t - t')\sum_n \chi_n(\mathbf{x}_1, \mathbf{x}_2, t)\overline{\chi_n(\mathbf{x}_1', \mathbf{x}_2', t)} \\ &= \int dE \int d^3P \sum_{\epsilon_n, \alpha} \delta\left(E - \epsilon_n - \frac{\mathbf{P}^2}{4m}\right)\theta(t - t')\exp\left(-\frac{iE(t - t')}{\hbar}\right) \\ &\quad \cdot \exp\left(+\frac{i\mathbf{P}_n\cdot(\mathbf{R} - \mathbf{R}')}{\hbar}\right) f(\mathbf{r}, \epsilon_n, \alpha)\overline{f(\mathbf{r}', \epsilon_n, \alpha)}.\end{aligned} \tag{88}$$

The sum $\sum_{\epsilon_n, \alpha}$ can be further split into 2 parts: a discrete sum over the bound states of energy $\epsilon_n = -B_n$ of the two-particle system and an integral over the continuum states. For $t > t'$ the Green's function is therefore given by

$$\begin{aligned}G^{(2)}(\mathbf{r}, \mathbf{R}, t; \mathbf{r}', \mathbf{R}', t') &= \sum_{B_n, \alpha} f(\mathbf{r}, B_n, \alpha)\overline{f(\mathbf{r}', B_n, \alpha)} \\ &\quad \cdot \int d^3P \exp\left(-\frac{i\mathbf{P}\cdot(\mathbf{R} - \mathbf{R}')}{\hbar}\right)\exp\left(-\frac{i}{\hbar}\left(\frac{\mathbf{P}^2}{4m} - B_n\right)(t - t')\right) \\ &\quad + \int_0^\infty d\epsilon \sum_\alpha f^{(+)}(\mathbf{r}, \epsilon, \alpha)\overline{f^{(+)}(\mathbf{r}', \epsilon, \alpha)} \\ &\quad \cdot \int d^3P \exp\left(-\frac{i\mathbf{P}\cdot(\mathbf{R} - \mathbf{R}')}{\hbar}\right)\exp\left(-\frac{i}{\hbar}\left(\frac{\mathbf{P}^2}{4m} + \epsilon\right)(t - t')\right).\end{aligned} \tag{89}$$

Considered as a function of the total energy and momentum the Green's function $G^{(2)}$ can be exhibited as follows:

$$G^{(2)}(\mathbf{r}, \mathbf{r}'; \mathbf{P}, E) = \int d^3(R - R') \int_{-\infty}^{+\infty} d(t - t') \exp\left(\frac{iE(t - t')}{\hbar}\right)$$
$$\cdot \exp\left(-\frac{i\mathbf{P}\cdot(\mathbf{R} - \mathbf{R}')}{\hbar}\right) \theta(t - t') \sum_n \chi_n(\mathbf{x}_1, \mathbf{x}_2, t)\overline{\chi_n(\mathbf{x}_1, \mathbf{x}_2, t')}$$
$$= \frac{1}{2\pi i\hbar} \sum_{B_n,\alpha} \frac{f(\mathbf{r}, B_n, \alpha)\overline{f(\mathbf{r}', B_n, \alpha)}}{E - \dfrac{\mathbf{P}^2}{4m} + B_n + i\eta} \tag{90}$$
$$+ \frac{1}{2\pi i\hbar} \sum_\alpha \int_0^\infty d\epsilon \frac{f^{(+)}(\mathbf{r}, \epsilon, \alpha)\overline{f^{(+)}(\mathbf{r}, \epsilon, \alpha)}}{E - \dfrac{\mathbf{P}^2}{4m} - \epsilon + i\eta}$$

where η is a positive infinitesimal. Hence for a fixed total momentum, $\mathbf{P}_n$, $G^{(2)}$ has poles at $E = (\mathbf{P}^2/4m) - B_n$ and a branch point at $E = \mathbf{P}^2/4m$, with a branch line running from $E = \mathbf{P}^2/4m$ to $+\infty$. It is precisely the location and character of these singularities which determine the properties of the asymptotic operators. To see this more clearly consider for example the matrix element of $\psi_{\mathrm{R}}^{(2)}(\mathbf{x}_1, \mathbf{x}_2, t)$ between an $N + 2$ and an N particle state of momentum $\mathbf{P}_{N+2}$ and $\mathbf{P}_N$, energy E_{N+2} and E_N respectively:

$$\langle N \mid \psi_{\mathrm{R}}^{(2)}(\mathbf{x}_1, \mathbf{x}_2, t) \mid N + 2\rangle = \lim_{t'\to-\infty} \int d^3x_1' \int d^3x_2' \int \frac{dE}{(2\pi\hbar)} \int \frac{d^3P}{(2\pi\hbar)^3}$$
$$\cdot \exp\left(\frac{i\mathbf{P}\cdot(\mathbf{R} - \mathbf{R}')}{\hbar}\right) \exp\left(-\frac{iE(t - t')}{\hbar}\right)$$
$$\cdot \frac{1}{i}\left\{\sum_{B_n,\alpha} \frac{f(\mathbf{r}, B_n, \alpha)\overline{f(\mathbf{r}', B_n, \alpha)}}{E - \dfrac{\mathbf{P}^2}{4m} + B_n + i\eta}\right. \tag{91}$$
$$\left. + \sum_\alpha \int_0^\infty d\epsilon \frac{f^{(+)}(\mathbf{r}, \epsilon, \alpha)\overline{f^{(+)}(\mathbf{r}', \epsilon, \alpha)}}{E - \dfrac{\mathbf{P}^2}{4m} - \epsilon + i\eta}\right\} \cdot \langle N \mid \mathbf{x}_1', t')\psi(\mathbf{x}_2', t') \mid N + 2\rangle.$$

Making use of the translational invariance of the theory we can write

$$\langle N \mid \psi(\mathbf{x}_1', t')\psi(\mathbf{x}_2', t') \mid N + 2\rangle$$
$$= \exp\left(-\frac{i(E_{N+2} - E_N)t'}{\hbar}\right) \exp\left(\frac{i(\mathbf{P}_{N+2} - \mathbf{P}_N)\cdot\mathbf{R}'}{\hbar}\right) \tag{92}$$
$$\cdot \langle N \mid \psi(\tfrac{1}{2}\mathbf{r}', 0)\psi(-\tfrac{1}{2}\mathbf{r}', 0) \mid N + 2\rangle$$

where $\mathbf{r}' = \mathbf{x}_1' - \mathbf{x}_2'$ is the relative coordinate. The limit $t' \to -\infty$ implies that

$$\lim_{t'\to-\infty} \frac{\exp(i/\hbar)(E - \Delta E_{N+2,N})t'}{E - E' + i\eta} = \lim_{t'\to-\infty} 2\pi i\delta(E - E') \exp\left\{\frac{i}{\hbar}(E' - \Delta E_{N,N+2})t\right\} \tag{93}$$

(in the sense that both the sides of (93) inserted into an integral with respect to E' give the same result). Hence, upon carrying out the R' and the E, P integrations, we obtain

$$\begin{aligned}\langle N \mid \psi_{\mathrm{R}}^{(2)}(\mathbf{x}_1, \mathbf{x}_2, t) \mid N+2\rangle = \lim_{t'\to-\infty} \int d^3r' \exp\left(\frac{i}{\hbar}\Delta\mathbf{P}_{N+2,N}\cdot\mathbf{R}\right) \\ \cdot\exp\left(-\frac{i}{\hbar}\Delta E_{N+2,N}\, t\right)\Bigg\{\sum_{B_n,\alpha} f(\mathbf{r}, B_n, \alpha)\overline{f(\mathbf{r}', B_n, \alpha)} \\ \cdot\exp\left\{\frac{it'}{\hbar}\left(\frac{(\mathbf{P}_{N+2} - \mathbf{P}_N)^2}{4m} - B_n - \Delta E_{N+2,N}\right)\right\} \\ + \sum_\alpha \int_0^\infty d\epsilon f^{(+)}(\mathbf{r}, \epsilon, \alpha)\overline{f^{(+)}(\mathbf{r}', \epsilon, \alpha)} \\ \cdot\exp\left\{\frac{it'}{\hbar}\left(\frac{(\mathbf{P}_{N+2} - \mathbf{P}_N)^2}{4m} + \epsilon - \Delta E_{N+2,N}\right)\right\} \\ \cdot\langle N \mid \psi(\tfrac{1}{2}\mathbf{r}', 0)\psi(-\tfrac{1}{2}\mathbf{r}', 0) \mid N+2\rangle.\end{aligned} \tag{94}$$

The energies and momenta of the states $\mid N\rangle$ and $\mid N+2\rangle$ now determine whether the contribution comes from the bound states $\sum_{B_n} \cdots$ or from the continuum states $\int d\epsilon \cdots$. It is the bound states which can contribute if

$$-B_0 < E_{N+2} - E_N - \frac{(\mathbf{P}_{N+2} - \mathbf{P}_N)^2}{4m} < 0$$

(where B_0 is the binding energy of the ground state of the bound two-particle system), whereas the contribution to the matrix element comes from the continuum states if

$$E_{N+2} - E_N - \frac{(\mathbf{P}_{N+2} - \mathbf{P}_N)^2}{4m} > 0.$$

For the bound states to contribute it is of course necessary that $-\Delta E_{N+1,N} + (\mathbf{P}_{N+2} - \mathbf{P}_N)^2/4m$ be exactly equal to one of the two-particle bound states energies. These considerations becomes obvious for the particular case that we consider the matrix element of ψ_{R} between a two-particle state and the vacuum.

If we revert to the original definition (75) for the two-particle retarded

asymptotic operator, and note that for the two-particle scattering states

$$\chi_n(\mathbf{x}_1, \mathbf{x}_2, t) = \exp\left\{-\frac{it}{\hbar}\left(\frac{\mathbf{P}_n^2}{4m} + \epsilon_n\right)\right\} f^{(+)}(\mathbf{r}, \epsilon_n, \alpha); \tag{95a}$$

$$\epsilon_n = \frac{\mathbf{p}_n^2}{m} \tag{95b}$$

($\mathbf{p}_n$ is the relative momentum), the large time behavior is such that

$$\lim_{t\to-\infty} \chi_n(\mathbf{x}_1, \mathbf{x}_2, t) = \frac{1}{\sqrt{2!}}\left\{f_\alpha^0(\mathbf{x}_1, t) f_\beta^0(\mathbf{x}_2, t) + f_\alpha^0(\mathbf{x}_2, t) f_\beta^0(\mathbf{x}_1, t)\right\} \tag{96}$$

where $f_\alpha^0(\mathbf{x}, t)$ are solutions of Eq. (53a), it then becomes clear that if we again split $\psi_{\mathrm{R}}^{(2)}$ in terms of contributions from the two-particle scattering states and a contribution from bound states

$$\psi_{\mathrm{R}}^{(2)}(\mathbf{x}_1, \mathbf{x}_2; t) = \psi_{S,\mathrm{R}}^{(2)}(\mathbf{x}_1, \mathbf{x}_2; t) + \psi_{B,\mathrm{R}}^{(2)}(\mathbf{x}_1, \mathbf{x}_2; t) \tag{97}$$

then

$$\begin{aligned}\psi_{S,\mathrm{R}}^{(2)}(\mathbf{x}_1, \mathbf{x}_2; t) &= \lim_{t'\to-\infty} \sum_{\substack{\text{scattering}\\ \text{states}}} \chi_n(\mathbf{x}_1, \mathbf{x}_2, t) \\ &\cdot \int d^3x_1' \int d^3x_2' \overline{\chi_n(\mathbf{x}_1', \mathbf{x}_2', t')} \psi(\mathbf{x}_1, t')\psi(\mathbf{x}_2, t') = \psi_{S,\mathrm{R}}(\mathbf{x}_1, t)\psi_{S,\mathrm{R}}(\mathbf{x}_2, t).\end{aligned} \tag{98}$$

Hence only the operator $\psi_{B,\mathrm{R}}^{(2)*}(\mathbf{x}_1, \mathbf{x}_2, t)$, where

$$\begin{aligned}\psi_{B,\mathrm{R}}^{(2)}(\mathbf{x}_1, \mathbf{x}_2, t) &= \sum_{\substack{\text{Bound}\\ \text{states}}} \chi_n(\mathbf{x}_1, \mathbf{x}_2, t) \\ &\cdot \lim_{t'\to-\infty} \int d^3x_2 \int d^3x_2 \overline{\chi_n(\mathbf{x}_1, \mathbf{x}_2, t')} \psi(\mathbf{x}_1, t')\psi(\mathbf{x}_2, t')\end{aligned} \tag{99}$$

operating on states of the form $\prod_{j=1}^N \psi_{\mathrm{R}}^{j*}|0\rangle$, introduces new vectors into the Hilbert space. Note incidentally that by virtue of the fact that the $\chi_n(\mathbf{x}_1, \mathbf{x}_2, t)$ satisfy Eq. (78), the operator $\psi_{\mathrm{R}}^{(2)}(\mathbf{x}_1, \mathbf{x}_2, t)$ will satisfy this same equation

$$\left\{i\hbar\partial_t + \frac{\hbar^2}{2m}(\nabla_1^2 + \nabla_2^2) + v(|\mathbf{x}_1 - \mathbf{x}_2|)\right\} \psi_{\mathrm{R}}^{(2)}(\mathbf{x}_1, \mathbf{x}_2, t) = 0. \tag{100}$$

The operator

$$\begin{aligned}\psi_{\mathrm{R}}^{(2)}(\chi_n) &= \int d^3x_1 \int d^3x_2\, \overline{\chi_n(\mathbf{x}_1, \mathbf{x}_2, t)} \psi_{\mathrm{R}}(\mathbf{x}_1, \mathbf{x}_2, t) \\ &= \lim_{t'\to-\infty} \int d^3x_1' \int d^3x_2'\, \overline{\chi_n(\mathbf{x}_1', \mathbf{x}_2', t')} \psi(\mathbf{x}_1', t')\psi(\mathbf{x}_2', t')\end{aligned} \tag{101}$$

is therefore time independent. The operator $\psi_{\mathrm{R}}^{*}(\chi_n)$ creates a two-particle state of total momentum $\mathbf{P}_n$, total energy E_n, and other quantum numbers α, i.e., a two-particle system characterized by a Schrödinger amplitude χ_n.

Clearly the above considerations can be extended to the two-particle advanced asymptotic operator, which can be defined as

$$\psi_{\mathrm{A}}^{(2)}(\mathbf{x}_1, \mathbf{x}_2, t) = \lim_{t'\to+\infty} \int d^3x_1' \int d^3x_2' \cdot \langle 0 \mid \psi(\mathbf{x}_1, t)\psi(\mathbf{x}_2, t)\psi^*(\mathbf{x}_1', t')\psi^*(\mathbf{x}_2', t') \mid 0\rangle\psi)\mathbf{x}_1, t')\psi(\mathbf{x}_2, t'). \tag{102}$$

We note, in passing, that the commutation rules for the two-particle asymptotic operators are somewhat complicated since care must now be exercised in performing the double limiting process required in its evaluation. We shall evaluate these commutation rules, as well as those for $\psi_{\mathrm{R}}^{(2)}$ and $\psi_{\mathrm{R}}^{(1)}$, in a sequel to this paper dealing with the application of the present formalism to rearrangement collisions.

Finally, in concluding this section on the nonrelativistic theory, we note that an N-particle asymptotic retarded and (advanced) operator can be defined by

$$\begin{aligned}\psi_{\substack{\mathrm{R}\\ \mathrm{A}}}^{(N)}(\mathbf{x}_1, \cdots, \mathbf{x}_N; t) &= \lim_{t'\to\pm\infty} \int d^3x_1' \cdots \int d^3x_N' \langle 0 \mid \psi(\mathbf{x}_1, t) \cdots \psi(\mathbf{x}_N, t)\psi^*(\mathbf{x}_1', t') \\ &\quad \cdots \psi^*(\mathbf{x}_N', t') \mid 0\rangle\psi(\mathbf{x}_1', t') \cdots \psi(\mathbf{x}_N', t') \qquad (103)\\ &= \lim_{t'\to\pm\infty} \int d^3x_1' \cdots \int d^3x_N' G^{(N)}(\mathbf{x}_1, \cdots, \mathbf{x}_N; t; \mathbf{x}_1', \cdots, \mathbf{x}_N', t') \\ &\quad \cdot\psi(\mathbf{x}_1', t') \cdots \psi(\mathbf{x}_N', t'). \qquad (104)\end{aligned}$$

An analysis similar to that carried out to arrive at Eq. (90) now indicates that the N-particle Green's function $G^{(N)}$ considered as a function of the total energy and momentum now has the form

$$\begin{aligned}G^{(N)}(\mathbf{r}^{(N)}, \mathbf{r}'^{(N)}; \mathbf{P}, E) &= \sum_{B_n,\alpha} \frac{f^{(N)}(\mathbf{r}^{(N)}, B_n^{N}, \alpha_n)\overline{f^{(N)}(\mathbf{r}'^{(N)}, B_n, \alpha_n)}}{E - (\mathbf{P}^2/2Nm) + B_n^{(N)} + i\eta} \\ &\quad + \text{terms regular at } E = \frac{\mathbf{P}^2}{2Nm} - B_m^{(N)} \text{ for fixed } \mathbf{P}.\end{aligned} \tag{105}$$

The $B_m^{(N)}$ are the binding energies for N-particle bound configurations, which are configurations in which N particles are bound together (and which asymptotically "remain and move together"). The total energy of such a bound N-particle system is equal to $-B_m^{(N)}$ plus the kinetic energy of the center-of-mass

motion which is equal to $\mathbf{P}^2/2Nm$. In Eq. (105), the $\mathbf{r}^{(N)}$ stands for a specification of the $N - 1$ relative coordinates and the $f^{(N)}$ are the wave functions describing the relative motion of the bound N-particle system. The singularities of $G^{(N)}$ at $E = (\mathbf{P}^2/2Nm) - B_m^{(N)}$ define for us the bound state part of the N-particle asymptotic operators $\psi_{B,\mathrm{R}}^{(N)}$. As in the two-particle case it can be shown that the operator $\psi_{\mathrm{R}}^{(N)} - \psi_{B,\mathrm{R}}^{(N)}$ can be constructed from products of $\psi_{B,\mathrm{R}}^{(N-1)}$, $\cdots$, $\psi_{B,\mathrm{R}}^{(3)}$, $\psi_{B,\mathrm{R}}^{(2)}$, $\psi_{\mathrm{R}}^{(1)} \equiv \psi_{\mathrm{R}}$ operators.

Summarizing, the Hilbert space of physically realizable states can be constructed from the product of operators $\psi_{S,\mathrm{R}}^{(1)*}$, $\psi_{B,\mathrm{R}}^{(1)*}$, $\psi_{B,\mathrm{R}}^{(2)*}$, $\cdots$, $\psi_{B,\mathrm{R}}^{(N)}$, $\cdots$ operating on the no-particle state; or, stated differently, the set of operators $\psi_{S,\mathrm{R}}^{(1)}$, $\psi_{B,\mathrm{R}}^{(1)}$, $\psi_{B,\mathrm{R}}^{(2)}$, $\cdots$, $\psi_{B,\mathrm{R}}^{(N)}$, $\cdots$, together with their adjoint constitute an irreducible operator ring $(1 \leqq N < \infty)$.

III. RELATIVISTIC CASE

The usual statement of the asymptotic conditions within the framework of relativistic field theories is the requirement that the field theory have an interpretation in terms of asymptotic observables corresponding to particles of definite mass and charge. For the case of scalar particles of mass μ (we shall henceforth choose units such that $\hbar = c = 1$) described by a renormalized Heisenberg field operator $\phi(x)$, the asymptotic condition requires that the weak operator limit

$$\lim_{t\to\pm\infty} \langle \Phi \mid \phi^f(t) \mid \Psi \rangle = \langle \Phi \mid \phi^f_{\substack{\mathrm{in}\\ \mathrm{out}}} \mid \Psi \rangle \tag{106}$$

exist, where

$$\begin{aligned} \phi^f(t) &= i \int \left\{ \phi(x) \frac{\partial \overline{f(x)}}{\partial x_0} - \frac{\partial \phi}{\partial x_0} \overline{f(x)} \right\} d^3x \\ &= i \int \phi(x) \overleftrightarrow{\frac{\partial}{\partial x_0}} \overline{f(x)}\, d^3x \end{aligned} \tag{107}$$

and f is a normalizable solution of the Klein-Gordon equation with mass μ:

$$(\Box + \mu^2) f(x) = 0. \tag{108}$$

Since the only stable one-particle state which the theory is to describe is that of a particle of mass μ, it is also assumed that $\lim_{t\to\pm\infty} \langle \Phi \mid \phi^{f'}(t) \mid \Psi \rangle$, where f' is any normalizable solution of the Klein-Gordon equation with mass $\mu' \neq \mu$, vanishes. The asymptotic condition as usually stated further requires that the asymptotic fields $\phi_{\mathrm{in}}(x)$ and $\phi_{\mathrm{out}}(x)$ satisfy the equations of motion and commutation relations of the free field operators for particles of mass μ.

Although it is necessary for the sake of mathematical precision to consider the operators $\phi^f(t)$ rather than the operator $\phi(x)$ we shall nonetheless deal in

the following with the field operators $\phi(x)$. In terms of these operators the "heuristic" formulation of the asymptotic condition requires that

$$\phi_{\text{in}}(x) = \lim_{x_0' \to -\infty} \int d^3x' \, \Delta(x - x'; \mu^2) \frac{\overleftrightarrow{\partial}}{\partial x_0'} \phi(x'). \tag{109}$$

Our previous work in the nonrelativistic situation suggests that we consider the following definition for the "retarded" operator:[5]

$$\phi_{\text{R}}(x) = \lim_{x_0' \to -\infty} i \int d^3x' \langle 0 \mid R(\phi(x)\phi(x')) \mid 0 \rangle \frac{\overleftrightarrow{\partial}}{\partial x_0'} \phi(x') \tag{110}$$

where

$$R(\phi(x)\phi(x')) = -i\theta(x_0 - x_0')[\phi(x), \phi(x')] \tag{111}$$

is the retarded product of $\phi(x)$ and $\phi(x')$ and $\mid 0\rangle$ denotes the (unique) invariant vacuum state.

Now for a theory satisfying the usual assumptions of relativistic invariance, spectral conditions, and uniqueness of the vacuum it follows that *(10)*

$$\begin{aligned} \langle 0 \mid R(\phi(x)\phi(x')) \mid 0 \rangle &= -\int_0^\infty d(m^2)\rho(m^2)\Delta_{\text{R}}(x - x'; m^2) \\ &= -\Delta_{\text{R}}^{\rho}(x - x') \end{aligned} \tag{112}$$

where $\rho(m^2)$ is the Lehmann weight for the twofold vacuum expectation value which, for a theory containing a single stable particle of mass μ, has the following form:

$$\rho(m^2) = \delta(m^2 - \mu^2) + \sigma(m^2)\theta(m^2 - (2\mu)^2). \tag{113}$$

The equation of motion that $\phi_{\text{R}}(x)$ satisfies can be obtained as follows: We

[5] We could have defined a different retarded asymptotic operator by the equation

$$\phi_{\text{in}}(x) = \phi_{\text{R}}'(x) - \int d^4x' \langle 0 \mid R(\phi(x)\phi(x')) \mid 0 \rangle (\Box_{x'} + \mu^2)\phi(x')$$

which is related to the in-field as usually defined by the equation

$$\phi_{\text{in}}(x) = \phi_{\text{R}}'(x) + \int d^4x' \int_{(2\mu)^2}^{\infty} d(m^2)\sigma(m^2)\Delta_{\text{R}}(x - x'; m^2)(\Box_{x'} + \mu^2)\phi(x')$$

where $\sigma(m^2)$ is defined by Eqs. (112)–(113). This retarded operator however does not obey simple commutation rules. The retarded operator defined by Eq. (110) will in the absence of external fields be the same as the in-field.

rewrite Eq. (110) so

$$\phi_{\text{R}}(x) = \lim_{x_0' \to \infty} \int d^3x'$$

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$$\phi_{\text{R}}(x) = -\int d^4x' \{($$

Since (for renormaliz

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$$\phi_{\text{R}}(x) = \phi(x) - \int d$$

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rewrite Eq. (110) so as to read

$$\phi_R(x) = \lim_{x_0' \to \infty} \int d^3x' \Delta_R^{\rho}(x - x') \frac{\overleftrightarrow{\partial}}{\partial x_0'} \phi(x') - \int d^3x' \int dx_0' \frac{\partial}{\partial x_0'} \left\{ \Delta_R^{\rho}(x - x') \frac{\overleftrightarrow{\partial}}{\partial x_0'} \phi(x') \right\}. \tag{114}$$

The first term on the right-hand side of Eq. (114) vanishes by virtue of the retarded character of $\Delta_R^{\rho}(x - x')$, hence

$$\phi_R(x) = - \int d^4x' \{ (\Box_{x'} + \mu^2) \Delta_R^{\rho}(x - x') \phi(x') - \Delta_R^{\rho}(x - x') (\Box_{x'} + \mu^2) \phi(x') \}. \tag{115}$$

Since (for renormalized fields)

$$\begin{aligned} \Delta_R^{\rho}(x - x') &= \Delta_R(x - x'; \mu^2) + \int_{4\mu^2}^{\infty} d(m^2) \sigma(m^2) \Delta_R(x - x'; m^2) \\ &= \Delta_R(x - x'; \mu^2) + \Delta_R^{\sigma}(x - x') \end{aligned} \tag{116}$$

it follows that

$$(\Box_{x'} + \mu^2) \Delta_R^{\rho}(x - x') = - \delta(x - x') + \int_{4\mu^2}^{\infty} d(m^2) \sigma(m^2) (\mu^2 - m^2) \Delta_R(x - x'; m^2) \tag{117}$$

so that

$$\phi_R(x) = \phi(x) - \int d^4x' \{ (\Box_{x'} + \mu^2) \Delta_R^{\sigma}(x - x') \} \phi(x') + \int d^4x' \Delta_R^{\rho}(x - x') (\Box_{x'} + \mu^2) \phi(x'). \tag{118}$$

The integration by parts of the second term on the right-hand side of Eq. (118) is allowed so that we finally obtain

$$\phi_R(x) = \phi(x) + \int d^4x' \Delta_R(x - x'; \mu^2) (\Box_{x'} + \mu^2) \phi(x'). \tag{119}$$

Hence, in the absence of external fields (so that Eq. (112) is valid), $\phi_R(x)$ is defined by the identical equation as $\phi_{in}(x)$ and is therefore identical to it. It satisfies by virtue of Eq. (119) free field equations

$$(\Box + \mu^2) \phi_R(x) = 0. \tag{120}$$

On the assumption that the *unrenormalized* Heisenberg field $\phi_u(x)$ satisfies canonical commutation rules, i.e., $[\phi_u(x), \partial_0{}^y\phi_u(y)]|_{x_0=y_0} = i\delta^3(\mathbf{x} - \mathbf{y})$, we deduce that

$$[\phi_R(x), \phi_R(x')] = \lim_{y_0=y_0'\to-\infty} \int d^3y \int d^3y' \langle 0 \mid [\phi(x), \phi(y)] \mid 0\rangle \cdot \langle 0 \mid [\phi(x'), \phi(y')] \mid 0\rangle \frac{\overleftrightarrow{\partial}}{\partial y_0} \frac{\overleftrightarrow{\partial}}{\partial y_0'} [\phi(y), \phi(y')] \tag{121a}$$

$$= \lim_{y_0\to-\infty} \int d^3y \{ \langle 0 \mid [\phi(x), \phi(y)] \mid 0\rangle \frac{\overleftrightarrow{\partial}}{\partial y_0} \langle 0 \mid [\phi(x'), \phi(y)] \mid 0\rangle \tag{121b}$$

$$= i\Delta(x - x'; \mu^2),$$

only the $\delta(m^2 - \mu^2)$ contribution to Δ^ρ surviving in the limit as $y_0 \to -\infty$. However, since the asymptotic convergence in relativistic theories is only weak, it is not legitimate to let the times y_0 and y_0' go together simultaneously to $-\infty$ without further justification. Under reasonable assumptions concerning the singularities of the matrix elements of $\tilde{\phi}(p)\tilde{\phi}(q)$ ($\tilde{\phi}$ is the Fourier transform of ϕ) and assuming that the canonical commutation rules are valid (actually only the weaker form $[[\phi(x)\ \ \phi(y)]\ \ \phi(z)]|_{x_0=y_0} = 0$ for all z is required) it can be shown (*2*) that the commutator $[\phi_R(x)\ \ \phi_R(y)]$ is a c-number and hence equal to its vacuum expectation value. From the fact that this vacuum expectation value must be relativistically invariant, must be odd under the interchange of x and x', and must satisfy the Klein-Gordon equation for the mass μ by virtue of Eq. (120), one immediately deduces that the commutator must be proportional to $\Delta(x - x'; \mu^2)$. The proportionality constant is then fixed by the normalization of the one-particle states.

The "advanced" asymptotic operator $\phi_A(x)$ is defined by the equation

$$\phi_A(x) = \lim_{x_0'\to+\infty} \int d^3x' \langle 0 \mid A(\phi(x), \phi(x')) \mid 0\rangle \frac{\overleftrightarrow{\partial}}{\partial x_0'} \phi(x') \tag{122}$$

where

$$A(\phi(x)\phi(x')) = i\theta(x' - x)[\phi(x), \phi(x')] \tag{123}$$

denotes the advanced product of $\phi(x)$ and $\phi(x')$. One verifies that, in the absence of external fields,

$$\phi_A(x) = \phi(x) + \int \Delta_A(x - x'; \mu^2)(\Box_{x'} + \mu^2)\phi(x')\, d^4x' \tag{124a}$$

$$= \phi_R(x) + \int \Delta(x - x'; \mu^2)(\Box_{x'} + \mu^2)\phi(x')\, d^4x';$$

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$$(\Box + \mu^2)\phi_A(x) = 0; \tag{124b}$$

$$[\phi_A(x), \phi_A(x')] = i\Delta(x - x'; \mu^2); \tag{124c}$$

so that under these circumstances $\phi_A(x)$ coincides with the conventionally defined out-field.

The advantage of the definitions (110) and (122) lies in the fact that they avoid the somewhat artificial way of introducing the mass of the particle that is encountered in the usual definition of the asymptotic field: the mass is introduced through the Lehmann weight.[6] They also give a description of charged particles in the presence of weak external (prescribed) electromagnetic fields in terms of asymptotic operators which describe properly dressed particles (in contradistinction to the in- and out-field as usually defined under these circumstances (*II*)).

Let us consider this situation for the case of fermions in somewhat greater detail. In the absence of external fields we define the retarded operator as

$$\psi_R(x) = \lim_{x_0' \to -\infty} \int d^3x' \langle 0 \mid R(\psi(x)\tilde{\psi}(x')) \mid 0\rangle \gamma^0 \psi(x') \tag{125}$$

where $\tilde{\psi} = \psi^* \gamma^0$ and for Fermion operators

$$R(\psi(x)\tilde{\psi}(x')) = \theta(x - x')\{\psi(x), \tilde{\psi}(x')\}. \tag{126}$$

One verifies that the so defined retarded operator satisfies the equation of motion

$$\psi_R(x) = \psi(x) - \int S_R(x - x'; M)(-i\gamma\cdot\partial' + M)\psi(x')\, d^4x' \tag{127}$$

and the following commutation rules

$$\{\psi_R(x), \tilde{\psi}_R(x')\} = -iS(x - x'; M) \tag{128}$$

where M is the mass of the particle, as reflected in the δ-function contribution in the Lehmann weight for the twofold vacuum expectation value

$$\langle 0 \mid \psi(x)\tilde{\psi}(x') \mid 0\rangle.$$

In the presence of an external electromagnetic field we define the retarded operator as

$$\psi_R{}^e(x) = \lim_{x_0' \to -\infty} \int d^3x' \langle \Psi_0{}^e \mid R(\psi^e(x)\tilde{\psi}^e(x')) \mid \Psi_0{}^e\rangle \gamma^0 \psi^e(x') \tag{129}$$

[6] Note therefore that if the field theory had two stable particles of mass μ_1 and μ_2, the operator $\phi_R(x)$ as defined by Eq. (110) would be equal to $\phi_{in}(x; \mu_1) + \phi_{in}(x; \mu_2)$. To isolate operators corresponding to particles of definite mass one would have to consider the operators $\phi_R^{f(\mu_1)}$ and $\phi_R^{f(\mu_2)}$. The masses μ_1 and μ_2 are however known from the Lehmann weight.

in contradistinction to the in-field operator which is defined as

$$\psi_{\text{in}}(x) = \lim_{x_0' \to -\infty} \int d^3x' S_R(x - x'; M)\gamma^0 \psi^e(x') \tag{130}$$

and which satisfies the equation of motion

$$\psi(x) = \psi_{\text{in}}(x) + \int S_{\text{R}}(x - x'; M)(-i\gamma\cdot\partial' + M)\psi(x')\, d^4x'. \tag{131}$$

In Eq. (129) $|\Psi_0^e\rangle$ denotes the vacuum state, i.e., the state of lowest energy, in the presence of the external field. We shall assume the external field to be sufficiently weak and slowly varying so that this state is steady so that

$$|\Psi_0^e\rangle_{\text{out}} = |\Psi_0^e\rangle_{\text{in}} = |\Psi_0^e\rangle.$$

For definiteness we shall consider the case of quantum electrodynamics so that apart from renormalization counter terms

$$(-i\gamma\cdot\partial + m)\psi^e(x) = e\gamma\cdot(A(x) + A^e(x))\psi^e(x) + \delta m\psi^e(x). \tag{132}$$

Under these circumstances the retarded Green's function

$$r^e(x, x') = \langle\Psi_0 | R(\psi^e(x)\bar{\psi}^e(x') | \Psi_0\rangle \tag{133}$$

satisfies the equation (*12*)

$$\gamma\cdot(-i\partial + e\mathfrak{A}^e)r^e(x, x') - \int M^e(x, y)r^e(y, x')\, d^4y = \delta(x - x') \tag{134}$$

where $M^e(x, y)$ is the mass operator:

$$M^e(x, y) = m\delta(x - y) - ie^2 \int d^4\xi \int d^4x' \gamma_\mu G_+^e(x, x')\Gamma_\nu^e(x', y; \xi)g^{\mu\nu}D_+^e(\xi, x). \tag{135}$$

Γ_μ^e is the vertex operator, and G_+^e and D_+^e the Feynman propagators for the electron-positron and photon in the presence of the external field. The $\mathfrak{A}^e$ term represents the effective external potential, i.e., the external potential A^e to which has been added the induced field due to the phenomenon of vacuum polarization.

The retarded asymptotic operator satisfies the homogeneous form of Eq. (134), i.e., without the δ-function on the right hand side. To lowest order

$$\{\gamma_\mu(-i\partial^\mu + e\mathfrak{A}_{(0)}^{e\mu}) + m + \mu_{(0)}\tfrac{1}{2}\sigma^{\mu\nu}F_{\mu\nu}^e\}\psi_{\text{R}}^e(x) = 0 \tag{136}$$

where

$$\mathfrak{A}_{(0)}^{e\mu} = A^{e\mu} + \frac{e^2}{15\pi m^2}\Box A^{e\mu} \tag{136a}$$

and

$$\mu_{(0)} = \frac{\alpha}{2\pi}\frac{e}{2m}. \tag{136b}$$

The asymptotic operator that we have defined therefore describes a "dressed" particle which interacts with the external field with its full form factor.

We next turn to the definition of the asymptotic operators for two-particle configurations. We consider the fermion case and restrict ourselves to the situation where there is no external field present. We define the asymptotic retarded two-particle operator as

$$\psi_{\mathrm{R}}^{(2)}(x_1, x_2) = \lim_{T'\to-\infty} \int d^3x_1' \int d^3x_2' \langle 0 \mid T(\psi(x_1)\psi(x_2)\bar{\psi}(x_1')\bar{\psi}(x_2')) \mid 0\rangle \cdot \gamma^0\gamma^0 T(\psi(x_1')\psi(x_2')). \tag{137}$$

In Eq. (137) the limiting procedure is defined in terms of the center-of-mass time

$$T' = \tfrac{1}{2}(x_{10}' + x_{20}') \tag{138}$$

which is required to approach $-\infty$ while the relative time

$$x_0' = x_{10}' - x_{20}' \tag{139}$$

is kept finite and fixed. In that limit, writing

$$X_\mu = \tfrac{1}{2}(x_{1\mu} + x_{2\mu}) \tag{140a}$$

$$x_\mu = x_{1\mu} - x_{2\mu} \tag{140b}$$

we obtain

$$\psi_{\mathrm{R}}^{(2)}(x_1, x_2) = \lim_{x_0'\to-\infty} \sum_{\mid n\rangle} \langle 0 \mid T(\psi(x_1)\psi(x_2)) \mid n\rangle \cdot \int d^3x_1' \int d^3x_2' \langle n \mid T(\bar{\psi}(X' + \tfrac{1}{2}x')\bar{\psi}(X' - \tfrac{1}{2}x')) \mid 0\rangle \cdot \gamma^0\gamma^0 T(\psi(X' + \tfrac{1}{2}x')\psi(X' - \tfrac{1}{2}x')) \tag{141}$$

where the summation $\sum_{\mid n\rangle}$ is over a complete set of states $\mid n\rangle$. (Note that the two gamma matrices operate on distinct indices!) If we partially label these states in terms of the eigenvalues of the total energy momentum operator then

$$\begin{aligned}\langle n \mid T(\bar{\psi}(X' + \tfrac{1}{2}x')\bar{\psi}(X' - \tfrac{1}{2}x')) \mid 0\rangle\gamma^0\gamma^0 &= e^{iP_n\cdot X'}\langle n \mid T(\bar{\psi}(\tfrac{1}{2}x')\bar{\psi}(-\tfrac{1}{2}x')) \mid 0\rangle\gamma^0\gamma^0 \\ &= e^{iP_n\cdot X'} X_n^{\dagger}(x')\end{aligned} \tag{142a}$$

and

$$\langle 0 \mid T(\psi(\tfrac{1}{2}x)\psi(-\tfrac{1}{2}x) \mid n\rangle = \mathrm{X}_n(x). \tag{142b}$$

If there exists two fermion bound states $\mid b\rangle$ of mass M_b then the contribution to the Green's function

$$G(x, X; x', X') = \langle 0 \mid T(\psi(x_1)\psi(x_2)\bar{\psi}(x_1')\bar{\psi}(x_2')) \mid 0\rangle \tag{143}$$

from these states is given for $T > T'$ by

$$G_b(x, X; x', X') = \sum_{\mid b\rangle} \int d^4P \mathrm{X}_b(x)\mathrm{X}_b^\dagger(x') \cdot e^{-iP\cdot(X-X')}\theta(T - T')\theta(P_0)\delta(P^2 - M_b^2) \tag{144}$$

where

$$\mathrm{X}_b(x) = \langle 0 \mid T(\psi(\tfrac{1}{2}x)\psi(-\tfrac{1}{2}x)) \mid b\rangle \tag{145}$$

is the two fermion Bethe-Salpeter wave function from which the dependence on the center of mass has been removed. Considerations similar to the ones outlined in the nonrelativistic case can now be carried out. The work of Mandelstam (*6*) (see also Blankenbecler (*13*), Klein and Zemach (*14*), Nishijima (*15*)) on the normalization and orthogonality properties of Bethe-Salpeter amplitudes allows us to define operators which create (or destroy) bound systems of particles specified by their Bethe-Salpeter amplitudes. We here only mention that the so defined operator for the two-particle situation is the natural generalization of the two-particle operator considered by Zimmerman (*4*). These matters are taken up in a sequel to the present paper.

IV. DISCUSSION

In the previous sections we have formulated the asymptotic condition in quantum field theory by a systematic procedure involving the retarded vacuum expectation values of products of field operators. In addition to the elimination of the artificial nature of the mass parameter introduced in the usual formulation of the asymptotic condition this formulation also has the advantage of describing the fully dressed particles with their characteristic interaction with external fields. Bound states appear along with the primitive particles associated with singularities of the retarded Green's functions in this formulation. In this sense the present work may be thought of as completing the program of particle interpretation of interacting quantized fields (*16*).

It is important to note that according to the formulation given here the asymptotic field operators do not span the Hilbert space of the Heisenberg field operators if there are fields without associated stable particles. Thus if the two-point retarded vacuum expectation value has no single-particle singu-

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larities the field $\psi_R(x)$ identically vanishes (*17*). For a theory involving several kinds of Heisenberg fields coupled to each other, if a particular Heisenberg field has no associated one-particle state the asymptotic fields will span a set of scattering states involving asymptotically only the stable particles, primitive or composite. It must be recalled at this point that the use of the so-called "unstable particle states" has always been either in the external source approximation or in an effort to approximate an actual scattering matrix element as a product of two independent matrix elements. In a complete dynamical *theory* of particles there are no states involving unstable "particles." If asymptotic fields are to describe only genuine particle states then the asymptotic fields must span only these states; and the asymptotic limits involving the retarded vacuum expectation value do precisely this. Needless to say, the coupled fields which do not have corresponding asymptotic fields affect the structure of the other asymptotic fields and, consequently, of the scattering amplitudes.

This circumstance appears even more desirable in a field theory in which there are states with undesirable characteristics as candidates for the role of physical states. In quantum field theories involving an indefinite metric (i.e., the field is a linear operator in a generalized Hilbert space with an indefinite fundamental quadratic form) only a subset of the "states" has a positive square; the probability interpretation of such a theory then needs a postulate of identification of the "physical states" (*18*). Such postulates can be stated and some simple solvable models exhibiting these characteristics are known (*19*). But the circumstance discussed in the preceding paragraph provides a natural postulate of identification: asymptotic field operators acting on the vacuum state (an arbitrary number of times) generate the Hilbert space of physical states. This natural postulate would assure that the physical states admit a standard probability interpretation provided the abnormal Heisenberg fields (the "ghost fields") occurring in such a theory do not have any associated stable particles. In any actual theory the ghost fields have sufficiently high bare mass (for reasons which may be thought of as stemming from the kinematics of the Feynman graph formalism) and are thus not associated with any stable particles. Thus "unstable ghost fields" together with the present formulation of the asymptotic condition promise to lead to a simple finite covariant field theory of interacting particles. This program is currently being pursued and we hope to present the results elsewhere.

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