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Completeness Identity in Field Theory (*).

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We discuss briefly three notions which are related to the completeness identity for Heisenberg fields: first, Gram-Schmidt orthogonalization of the states $|0\rangle$, $A^*(k)|0\rangle$, $A^*(k_1)A^*(k_2)|0\rangle$, etc. to construct an « ideal » or « improper » complete orthogonal basis for the Hilbert space generated by polynomials in the smeared (momentum space) Heisenberg field $A(k)$ acting on the (assumed cyclic) unique vacuum state $|0\rangle$; secondly, the use of this orthogonal basis to construct an expression for the unit operator $\mathbf{1}$, which we call the completeness identity; and finally, the use of alternative completeness identities to test the mutual consistency of different completeness assumptions; for example, the compatibility of a given kind of space-time completeness with asymptotic completeness. For simplicity, we consider neutral scalar fields throughout.

1. Gram-Schmidt orthogonalization: The sequence of ideal vectors

$$|0\rangle,\ |k\rangle \equiv A^*(k)|0\rangle,\ |k_1, k_2\rangle \equiv A^*(k_1)A^*(k_2)|0\rangle\ ,\ \text{etc.}\ ,$$

is neither normalizable nor orthogonal. We orthogonalize this sequence using the classical Gram-Schmidt procedure. By considering, if necessary,

$$A(k) = B(k) - \langle 0|B(k)|0\rangle\ ,$$

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we assume, without loss of generality, that $\langle 0|A(k)|0\rangle = 0$. Then $|0\rangle$ and $|k\rangle$ are orthogonal at the start. To orthogonalize the next sector, we define

(1) $$|\{k_1, k_2\}\rangle = |k_1, k_2\rangle + \int d^4q\, C_{2,1}(k_1, k_2; q)\,|q\rangle + C_{2,0}(k_1, k_2)\,|0\rangle ;$$

and determine the c-no. functions $C_{2,0}$ and $C_{2,1}$ by requiring orthogonality to the lower sectors. This condition implies that

(2) $$C_{2,0}(k_1, k_2) = -\langle 0|k_1, k_2\rangle \equiv -G^{(2)}(-k_1)\,\delta(k_1 + k_2) ,$$

and

(3) $$G^{(2)}(q)\, C_{2,1}(k_1, k_2; q) = -\langle q|k_1, k_2\rangle \equiv -G^{(3)}(q, q-k_1)\,\delta(q - k_1 - k_2) .$$

Note that $|\{k_1, k_2\}\rangle$ is not symmetric in k_1 and k_2; this lack of symmetry also occurs for a general orthogonalized basis vector. Now we continue by induction, and orthogonalize the n-th sector assuming that all lower sectors are already mutually orthogonal. For this purpose, we write

(4) $$|\{k_1, \dots k_n\}\rangle = |k_1, \dots k_n\rangle + \sum_{r=0}^{n-1} \int d^4q_1 \dots d^4q_r\, C_{n,r}(k_1, \dots k_n; q_1, \dots q_r) \cdot$$
$$\cdot\, |\{q_1, \dots q_r\}\rangle \equiv [\Omega(k_1, \dots k_n)]^*\,|0\rangle .$$

The conditions of orthogonality to lower sectors are

(5) $$C_{n,0}(k_1, \dots k_n) = -\langle 0|k_1, \dots k_n\rangle \equiv -G^{(n)}(-k_1, -k_1-k_2, \dots k_n)\,\delta\Big(\sum_1^n k_i\Big) ,$$

and

(6) $$\int d^4q_1 \dots d^4q_s\, C_{n,s}(k_1, \dots k_n; q_1, \dots q_s)\,\langle\{p_s, \dots p_1\}|\{q_1, \dots q_s\}\rangle =$$
$$= -\langle\{p_s, \dots p_1\}|k_1, \dots k_n\rangle , \qquad s = 1, 2, \dots n-1 ,$$

where the adjoint vectors are defined by

$$\langle\{p_s, \dots p_1\}| \equiv \langle 0|\,\Omega(p_1, \dots p_s) .$$

Having completed the orthogonalization of the different « n-quantum » (not « n-particle ») sectors, we « diagonalize » a given n-quantum sector by choosing functions $\varphi_n(k_1, \dots k_n; q_1, \dots q_n)$ such that the states

$$|[k_1, \dots k_n]\rangle = \int d^4q_1 \dots d^4q_n\, \varphi_n(k_1, \dots k_n; q_1, \dots q_n)\,|\{q_1, \dots q_n\}\rangle ,$$

satisfy

(7) $$\int d^4k_1 \dots d^4k_n\, \langle[k_n, \dots k_1]|[q_1, \dots q_n]\rangle \cdot |[k_1, \dots k_n]\rangle = |[q_1, \dots q_n]\rangle .$$

At discrete values of the masses, this diagonalization must be modified by replacing four dimensional states, functions, volume elements and δ-functions by invariant three dimensional ones; for example if $k_j^2 = k_j'^2 = m^2$ is a discrete mass, eq. (7) should be replaced by

$$(8) \quad \int \frac{d^3k_1}{2(k_1^2+m^2)^{\frac{1}{2}}} \cdots \frac{d^3k_n}{2(k_n^2+m^2)^{\frac{1}{2}}} \langle [k_1, \ldots k_n] | [q_1, \ldots q_n] \rangle \cdot | [k_1, \ldots k_n] \rangle = | [q_1, \ldots q_n] \rangle .$$

These remarks about the diagonalization procedure at discrete masses complete our discussion of the Gram-Schmidt process.

2. – Completeness identity (1): Using the vectors $|[q_1, \ldots q_n]\rangle$, the completeness identity takes the simple form

$$(9) \qquad \mathbf{1} = |0\rangle\langle 0| + \sum_{n=1}^{\infty} \int d^4q_1 \ldots d^4q_n \, |[q_n, \ldots q_1]\rangle \langle [q_1, \ldots q_n]| \; .$$

At discrete values of the masses, the completeness formula must be modified in the way indicated in **1** above.

3. – Mutual consistency of different completeness assumptions: As far as we know, no systematic study has been made of the compatibility of various types of space-time completeness or irreducibility with completeness of the in fields or with the other postulates of quantum field theory. The extreme types of space-time irreducibility are (*a*) theories in which the field operator together with a finite number of other operators at a single time are irreducible, and (*b*) theories for which no finite time slice suffices for irreducibility, but instead all of space-time is required. Examples of (*a*) are: (i) the free scalar field, in which the field itself suffices (2), (ii) a canonical field theory in which the field and its canonical conjugate (perhaps its time derivative) are required, (iii) a direct sum of n free fields of different masses in which the field together with its first $n-1$ time derivatives are required. Cases in which a single time is replaced by a finite number of discrete times, or of space-like surfaces are included in this category. An example of (*b*) is: a generalized free field (3) whose weight function has some continuous part (4). Presumably there are also intermediate types of space-time irreducibility. The example of convex models (5) shows that for each type of irreducibility there is a theory whose field operators are complete, but which has a degenerate vacuum (6) and requires additional space-time independent operators to form an irreducible set. For each type of space-time completeness a completeness identity, *i.e.* an expression for the operator $\mathbf{1}$, can be constructed in analogy with the work of **1** and **2**.

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(1) A later note by one of us (HJS) will develop the completeness identity for several models including the generalized free field and the convex model.

(2) O. W. GREENBERG: *Phys. Rev.*, **115**, 706 (1959), in particular p. 708.

(3) G. F. DELL'ANTONIO: *Journ. Math. Phys.*, **2**, 759 (1961); O. W. GREENBERG: *Ann. Phys.*, **16**, 158 (1961); A. L. LICHT and J. S. TOLL: *Nuovo Cimento*, **21**, 346 (1961).

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(6) K. HEPP, R. JOST, D. RUELLE and O. STEINMANN: *Helv. Phys. Acta*, **34**, 542 (1961); H.-J. BORCHERS: *Nuovo Cimento*, **24**. 214 (1962); T. F. JORDAN and E. C. G. SUDARSHAN: University of Rochester preprint (1962).

Let

$$(10)\qquad \mathbf{1}_s = \sum_{n=0}^{\infty} |[n]\rangle_{s\,s}\langle[n]| \,,$$

where we have rewritten eq. (9) in an abbreviated form for this case, be the unit operator constructed using some space-time completeness assumption, and

$$(11)\qquad \mathbf{1}_{in} = \sum_{n=0}^{\infty} |[n]\rangle_{in\,in}\langle[n]| \,,$$

be the unit operator constructed using the assumption of asymptotic completeness. Then the equation

$$(12)\qquad \mathbf{1}_s = \mathbf{1}_{in} \,,$$

is a consistency requirement for these different kinds of completeness assumptions. A useful form of this equation follows by considering the unitary uperator U (or U^*) which maps $|[n]\rangle_{in}$ into $|[n]\rangle_s$ (or *vice versa*). Clearly

$$(13)\qquad U = \sum_{n=0}^{\infty} |[n]\rangle_{s\,in}\langle[n]| \,.$$

Since U is unitary

$$(14)\qquad \| U|[n]\rangle_s\|^2 = \Big\| \sum_{n'=0}^{\infty} |[n']\rangle_{s\,in}\langle[n']|[n]\rangle_s\Big\|^2 = \sum_{n'=0}^{\infty} |{}_{in}\langle[n']|[n]\rangle_s|^2 = 1 \,.$$

Since eq. (14) is a sum of positive terms, an inequality results if the series is terminated at a finite value of n'. We hope that eq. (14), or inequalities following from it, will be useful in studying the consistency of different completeness assumptions.