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Reduction of Cartesian Tensors and its Application to Stochastic Dynamics.

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Summary. — An explicit reduction scheme to dispaly the irreducible parts of a fourth-rank cartesian tensor is presented. Such a reduction scheme is of interest in connection with crystal physics, hydrodynamic turbulence, etc. As an immediate application, the stochastic dynamics of a spin-one system is briefly discussed.

1. Introduction.

Space has three dimensions; consequently invariance considerations for physical systems include a systematic approach to the properties of the three-dimensional rotation group and its representations. But since most of this work was undertaken within the framework of quantum mechanics the complex spherical representation of the tensors is the one employed in these studies; in connection with some physical problems (mainly in hydrodynamic turbulence and crystal physics) it is more convenient to have a scheme employing Cartesian tensors. In this paper we develop a Cartesian scheme for the reduction of a fourth-rank tensor (1) which is of particular interest in connection with the stochastic dynamics of a spin-one system.

The reduction scheme we employ is the natural generalization (2) of the famil-

(1) The general theory of such reductions has been developed by A. PAIS: *Ann. Phys.*, **9**, 548 (1960); and the explicit reduction carried out here is consistent with this general theory. See also, G. GOLDHABER S. GOLDHABER, W. LEE and A. PAIS: *Phys. Rev.*, **120**, 300 (1960).

(2) The symmetry characterization of the various irreducible parts of a tensor of rank n is treated in a forth coming paper by A. PAIS. We are indebted to Professor PAIS for a communication on this point.

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iar reduction of a second-rank Cartesian tensor into its trace, the antisymmetric part and the symmetric traceless part which transform respectively as the 1-dimensional, 3-dimensional and 5-dimensional irreducible representations of the rotation group. We refer to this last property by saying that these tensors have spin 0, spin 1 and spin 2, respectively. When one goes beyond the second-rank tensors the reduction is not so straightforward and the various invariant parts are not labelled by the « spin » label alone. We have confined our attention to fourth-rank tensors here; for an arbitrary tensor T_{ijkl} of the fourth rank we can define two « intermediate spins » λ, $\mu = 0, 1, 2$ which are the « spins » associated with the first pair of indices and the second pair of indices, respectively. These intermediate spins now add vectorially to give the spin of the tensor with the familiar relation $|\lambda - \mu| \leqslant \nu \leqslant |\lambda + \mu|$. This scheme is illustrated in Fig. 1 and is the one employed throughout the sequel and we shall refer to this as the $(\lambda, \mu; \nu)$ part. Each « part » is invariant and constitutes an irreducible tensor.

Of course one might choose to discuss the reduction in terms of some other scheme. Since the part with intermediate spin λ has the eigenvalue $(-1)^\lambda$ under the interchange $i \rightleftarrows j$ etc., the only nontrivial one is the second scheme illustrated in Fig. 1. The part with spin labels λ', μ', ν' according to the second

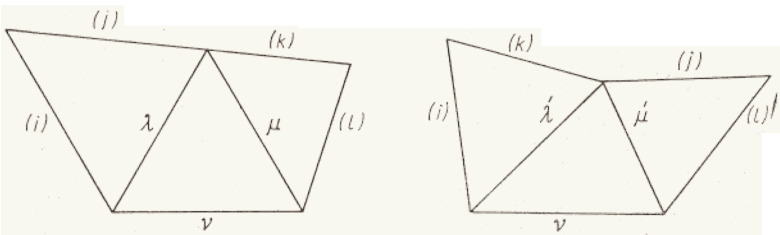

Fig. 1. – Two schemes of reduction of the fourth rank tensor T_{ijkl}

scheme is a linear combination of parts with spin labels λ, μ, ν according to the first scheme with the restriction $\nu = \nu'$ and the coefficients of linear combination (with appropriate normalization!) are Racah coefficients and are thus « known ».

The parts (λ, μ, ν) with the restriction $\mu = 1$ are in one-to-one correspondence with the invariant parts of the third-rank tensor T_{ijk} in a scheme of reduction where λ is the intermediate spin corresponding to the indices i, j. So no separate treatment of the third-rank tensor is necessary.

In Section **2** we carry out the reduction of an arbitrary fourth-rank tensor into the 19 irreducible parts $(\lambda, \mu; \nu)$. The next section deals with the application to the dynamics of spin-one systems. Section **3** deals with the axially symmetric case. In the last section some related points are discussed.

2. – Reduction of an arbitrary fourth-rank tensor.

Let us consider an arbitrary second-rank tensor T_{ij}. Then the irreducible parts of this tensor are as follows:

TABLE I.

Reduction of T_{ij}

Part	Spin
$\frac{1}{3}T_{kk}\delta_{ij}$	0
$\frac{1}{2}(T_{ij}-T_{ji})$	1
$\frac{1}{2}(T_{ij}+T_{ji} \quad \frac{2}{3}T_{kk}\delta$	2

In the sequel we shall often disregard absolute normalizations; with this understanding we can now write down the invariant (but, in general, reducible) parts of the fourth-rank tensor T_{ijkl} with intermediate spins λ, μ as follows:

TABLE II.

Reduction of T_{ijkl}

Part	Intermediate spins λ, μ
$T_{mmnn}\delta_{ij}\delta_{kl}$	0, 0
$(T_{ijnn}-T_{j\,nn})\,\delta_{kl}$	1, 0
$(T_{mmkl}-T_{mmlk})\,\delta_{ij}$	0, 1
$(T_{ijnn}+T_{j\,nn}-\frac{2}{3}\,\delta_{ij}\,T_{mmnn})\,\delta_{kl}$	2, 0
$(T_{ijkl}-T_{j\,kl}+T_{jilk}-T_{\,jlk})$	1, 1
$(T_{mmkl}+T_{mmlk}-\frac{2}{3}\,\delta_{kl}\,T_{mmnn})\,\delta_{ij}$	0, 2
$(T_{ijkl}+T_{j\,kl}-\frac{2}{3}\,\delta_{ij}\,T_{mmkl}-T_{ijlk}-T_{jilk}\quad\frac{2}{3}\,\delta_{\,j}\,T_{mmlk})$	2, 1
$(T_{ijkl}+T_{ijlk}-\frac{2}{3}\,\delta_{kl}\,T_{ijnn}-T_{j\,kl}-T_{jilk}-\frac{2}{3}\,\delta_{kl}\,T_{j\,nn})$	1, 2
$(T_{ijkl}+T_{ijlk}-\frac{2}{3}\,T_{ijnn}\,\delta_{kl}+T_{j\,kl}+T_{jilk}\quad\frac{2}{3}\,T_{jinn}\,\delta_{kl})-$ $-\frac{2}{3}\,\delta_{\,j}(T_{mmkl}+T_{mmlk}-\frac{2}{3}\,T_{mmnn}\,\delta_{kl})$	2. 2

To complete the reduction it is necessary to determine the final spin ν with $|\lambda-\mu|<\nu<|\lambda+\mu|$. Consequently there are 19 such irreducible tensors; three irreducible parts (0, 0; 0), (1, 1; 0), (2, 2; 0), with spin 0; six parts (1, 0; 1), (0, 1; 1), (1, 1; 1), (2, 1; 1), (1, 2; 1), (2, 2; 1) with spin 1; six parts (2, 0; 2), (1, 1; 2), (0, 2; 2), (2, 1; 2), (1, 2; 2), (2, 2; 2) with spin 2; three parts (2, 1; 3), (1, 2; 3), (2, 2; 3) with spin 3; and one part (2, 2; 4) with spin 4. One easily verifies that the total number of independent components is $3\times1+6\times3+$ $+6\times5+3\times7+1\times9=81=3^4$ as it should be for arbitrary fourth-rank tensor. These tensors, labelled T_{ijkl}, are written down below:

Complete reduction of the tensor T_{ijkl} in $(\lambda, \mu; \nu)$ notation)

Spin zero

(1 $\quad T_{mmnn}\delta_{ij}\delta_{l}$ $\quad$ (0, 0; 0)

2) $\quad (T_{mnmn} \quad T_{mnnm})(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})$ $\quad$ (1, 1; 0)

(3) $\quad (T_{mnmn} + T_{mnnm} - \frac{2}{3}T_{mmnn})(\delta_{ik}\delta_{j} \quad \delta_{il}\delta_{jk} \quad \frac{2}{3}\delta_{ij}\delta_{kl})$ $\quad$ (2, 2; 0)

Spin one

(4) $\quad T_{rr'nn}\varepsilon_{rr'm}\varepsilon_{mij}\delta_{kl}$ $\quad$ (1, 0; 1)

(5) $\quad T_{nnrr'}\varepsilon_{rr'm}\varepsilon_{mkl}\delta_{ij}$ $\quad$ (0, 1; 1)

(6) $\quad [T_{rr'ss'}(\varepsilon_{rr'm}\varepsilon_{ss'n} - \varepsilon_{rr'n}\varepsilon_{ss'm})\varepsilon_{mij}\varepsilon_{nkl}]$ $\quad$ (1, 1; 1)

(7 $\quad [T_{irr's}\varepsilon_{rr's}\varepsilon_{jkl} - T_{nrr's}\varepsilon_{rr's}(\delta_{ij}\varepsilon_{nkl} + \delta_{ik}\varepsilon_{jnl} + \delta_{il}\varepsilon_{jkn})] +$
$+ [T_{jrr's}\varepsilon_{rr's}\varepsilon_{ikl} - T_{nrr's}\varepsilon_{rr's}(\delta_{ij}\varepsilon_{nkl} + \delta_{ik}\varepsilon_{inl} + \delta_{jl}\varepsilon_{ikn})]$ $\quad$ (2, 1; 1)

(8) $\quad [T_{rr'sl}\varepsilon_{rr's}\varepsilon_{ijk} - T_{rr'sn}\varepsilon_{rr's}(\delta_{lk}\varepsilon_{ijn} + \delta_{lj}\varepsilon_{ink} + \delta_{li}\varepsilon_{njk})] +$
$+ T_{rr'sk}\varepsilon_{rr's}\varepsilon_{ijl} - T_{rr'sn}\varepsilon_{rr's}(\delta_{lk}\varepsilon_{ijn} + \delta_{kj}\varepsilon_{inl} + \delta_{ki}\varepsilon_{njl})]$ $\quad$ (1, 2; 1)

(9) $\quad T_{rr'nn}\varepsilon_{rr'm}[\varepsilon_{mik}\delta_{jl} \quad \varepsilon_{mjk}\delta_{il} \quad \varepsilon_{mil}\delta_{jk} \quad \varepsilon_{mjl}\delta_{ik})]$ $\quad$ (2, 2; 1)

Spin two

$(T_{ijnn} + T_{jinn} - \frac{2}{3}\delta_{ij}T_{mmnn})\delta_{kl}$ $\quad$ (2, 0; 2)

11) $\quad (T_{nnkl} + T_{nnlk} - \frac{2}{3}\delta_{kl}T_{mmnn})\delta_{ij}$ $\quad$ (0, 2; 2)

$[(T_{inkn} + T_{knin} - \frac{2}{3}\delta_{ik}T_{mnmn})\delta_{jl} - (T_{jnkn} \quad T_{knjn} \quad \frac{2}{3}\delta_{ik}T_{mnmn})\delta_{i}$
$- [(T_{inln} + T_{lnin} - \frac{2}{3}\delta_{il}T_{mnmn})\delta_{jk} -$
$- (T_{jnln} + T_{lnjn} - \frac{2}{3}\delta_{jl}T_{mnmn})\delta_{ik}]$ $\quad$ (1, 1; 2)

$[(T_{inkn} + T_{knin} - \frac{2}{3}\delta_{ik}T_{mnmn})\delta_{jl} + (T_{jnkn} - T_{kjn} - \frac{2}{3}\delta_{jk}T_{mnmn})\delta_{il}]$
$[(T_{inln} + T_{jnnin} - \frac{2}{3}\delta_{il}T_{mnmn})\delta_{jk} + \quad T_{inln} + T_{lnjn} -$
$\frac{2}{3}\delta_{jl}T_{mnmn})\delta_{ik}]$ $\quad$ (2, 1; 2)

(14) $\quad [(T_{inkn} + T_{knin} - \frac{2}{3}\delta_{ik}T_{mnmn})\delta_{jl} + (T_{inln} + T_{lnin} -$
$- \frac{2}{3}\delta_{il}T_{mnmn})\delta_{ik}] - [(T_{jnkn} + T_{knjn} - \frac{2}{3}\delta_{jk}T_{mnmn})\delta_{il}$
$+ (T_{jnln} + T_{lnjn} - \frac{2}{3}\delta_{jl}T_{mnmn})\delta_{ik}]$ $\quad$ (1, 2; 2)

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$$\{[(T_{inkn}+T_{knin}-\tfrac{2}{3}\delta_{ik}T_{mnmn})\delta_{jl}+(T_{jnkn}+T_{kknjn}-$$
$$-\tfrac{2}{3}\delta_{jk}T_{mnmn})\delta_{il}-\tfrac{2}{3}\delta_{ij}(T_{lnkn}+T_{knln}-\tfrac{2}{3}\delta_{lk}T_{mnmn})]+$$
$$+[(T_{inln}+T_{lnin}-\tfrac{2}{3}\delta_{il}T_{mnmn})\delta_{ik}+(T_{jnln}+T_{lnjn}-$$
$$-\tfrac{2}{3}\delta_{jl}T_{mnmn})\delta_{ik}-\tfrac{2}{3}\delta_{ij}(T_{lnkn}+T_{knln}-\tfrac{2}{3}\delta_{lk}T_{mnmn})]-$$
$$-[\tfrac{4}{3}\delta_{lk}(T_{injn}+T_{jnin}-\tfrac{2}{3}\delta_{ij}T_{mnmn})]\} \qquad (2,2;2)$$

Spin three

(16)
$$[\tfrac{5}{3}(T_{ijrr'}+T_{irr'j}+T_{jirr'}=T_{jrr'i}+T_{rr'ij})\varepsilon_{rr'm}\varepsilon_{mkl}-$$
$$-\tfrac{2}{3}\delta_{ij}(T_{nnrr'}+T_{nrr'n}+T_{rr'nn})\varepsilon_{rr'm}\varepsilon_{mkl}-$$
$$-\tfrac{1}{3}(T_{njrr'}+T_{nrr'j}+T_{jnrr'}+T_{jrr'n}+T_{rr'jn}+T_{rr'nj})\varepsilon_{rr'n}\varepsilon_{ikl}-$$
$$-\tfrac{1}{3}(T_{nirr'}+T_{nrr'i}+T_{inrr'}+T_{irr'n}+T_{rr'ni}+T_{rr'in})\varepsilon_{rr'n}\varepsilon_{jkl}] \qquad (2,1;3)$$

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$$[\tfrac{5}{3}(T_{klrr'}+T_{krr'l}+T_{lkrr'}+T_{lrr'k}+T_{rr'lk}+T_{rr'kl})\varepsilon_{rr'm}\varepsilon_{mij}-$$
$$-\tfrac{2}{3}\delta_{kl}(T_{nnrr'}+T_{nrr'n}=T_{rr'nn})\varepsilon_{rr'm}\varepsilon_{mij}-$$
$$-\tfrac{1}{3}(T_{nlrr'}+T_{nrr'l}+T_{lnrr'}+T_{lrr'n}+T_{rr'nl}+T_{rr'ln})\varepsilon_{rrn}\varepsilon_{ijk}$$
$$-\tfrac{1}{3}(T_{nkrr}+T_{nrrk}+T_{knrr}+T_{krrn}++T_{rrkn}+T_{rnk})\varepsilon_{rrn}\varepsilon_{ijl}] \qquad (1,2;3)$$

(18)
$$[\tfrac{5}{3}(T_{ikrr'}+T_{irr'k}+T_{kirr'}+T_{krr'i}+T_{rr'ki}+T_{rr'ik})\varepsilon_{rr'm}\varepsilon_{mjl}+$$
$$+\tfrac{5}{3}(T_{jkrr'}+T_{jrr'k}+T_{kjrr'}+T_{krr'j}+T_{rr'jk}+T_{rr'kj})\varepsilon_{rr'm}\varepsilon_{mil}+$$
$$+\tfrac{5}{3}(T_{ilrr'}+T_{irr'l}+T_{lirr'}+T_{lrr'i}+T_{rr'li}+T_{rr'il})\varepsilon_{rr'm}\varepsilon_{mjk}+$$
$$+\tfrac{5}{3}(T_{jlrr'}+T_{jrr'l}+T_{ljrr'}+T_{lrr'j}+T_{rr'lj}+T_{rr'jl})\varepsilon_{rr'm}\varepsilon_{mik}-$$
$$-\tfrac{2}{3}(T_{rr'nn}+T_{nrr'n}+T_{nnrr'})(\varepsilon_{mjl}\delta_{ik}+\varepsilon_{mil}\delta_{jk}+$$
$$+\varepsilon_{mjk}\delta_{il}+\varepsilon_{mik}\delta_{jl})\varepsilon_{mrr'}] \qquad (2,2;3)$$

Spin four

$$\{\tfrac{7}{3}[(T_{ijkl}+T_{ijlk}+T_{ikjl}+T_{iklj}+T_{iljk}+T_{iklkj})+$$
$$+(T_{jikl}+T_{jilk}+T_{jkil}+T_{jkli}+T_{jlik}+T_{jlki})+$$
$$+(T_{kjli}+T_{kjil}+T_{kilj}+T_{kijl}+T_{klij}+T_{klji})+$$
$$+(T_{lijk}+T_{likj}+T_{lkij}+T_{lkji}+T_{ljki}+T_{ljik})]-$$
$$-\tfrac{2}{3}\delta_{ij}[(T_{nnkl}+T_{nnlk}+T_{nknl}+T_{nkln}+T_{nlnk}+T_{nlkn})+$$
$$+T_{knnl}+T_{klnn}+T_{knln}+T_{lknn}+T_{lnkn}+T_{lnnk})]-$$
$$-\tfrac{2}{3}\delta_{ik}[(T_{nnjl}+T_{nnlj}+T_{njnl}+T_{njln}+T_{nlnj}+T_{nljn})+$$
$$+T_{jnnl}+T_{jnln}+T_{jlnn}+T_{lnnj}+T_{lnjn}+T_{ljnn})]-$$
$$-\tfrac{2}{3}\delta_{il}[(T_{nnjk}+T_{nnkj}+T_{njnk}+T_{njkn}+T_{nkjn}+T_{nknj})+$$

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$$+ T_{jnnk} + T_{jnkn} + T_{jknn} + T_{knnj} + T_{knjn} + T_{kjnn}] -$$
$$- \tfrac{2}{3}\delta_{jk}[T_{nnil} + \text{symmetrize in the four indices to get 12 terms}] -$$
$$- \tfrac{2}{3}\delta_{jl}[T_{nnjk} + \text{symmetrize in the four indices to get 12 terms}] -$$
$$- \tfrac{2}{3}\delta_{kl}[T_{nnij} + \text{symmetrize in the four indices to get 12 terms}] -$$
$$+ \tfrac{24}{15}[T_{mmnn}\delta_{ij}\delta_{kl} + T_{nmnm}\delta_{ik}\delta_{jl} + T_{mnnm}\delta_{il}\delta_{jk}]\} .$$

In the process of reduction we have systematically employed the well-known properties of the Levi-Civita symbol and in particular the identity

$$\varepsilon_{\alpha\beta\gamma}\varepsilon_{\alpha'\beta'\gamma'} \quad \delta_{\alpha\alpha'}\delta_{\beta\beta'}\delta_{\gamma} \quad \delta_{\alpha\beta'}\delta_{\beta\gamma'}\delta_{\gamma\alpha'} + $$
$$+ \delta_{\alpha\gamma'}\delta_{\beta\alpha'}\delta_{\gamma\beta'} \quad \delta_{\alpha\alpha'}\delta_{\beta\gamma'}\delta_{\gamma\beta'} \quad \delta_{\alpha\gamma'}\delta_{\beta\beta'}\delta_{\gamma\alpha'} \quad \delta_{\alpha\beta'}\delta_{\beta\alpha'}\delta_{\gamma\gamma'}$$

One particular point deserves mention! The scheme of reduction employed here exhibits the various parts with spin ν, $0 \leqslant \nu \leqslant 4$ as fourth-rank tensors; of course we could make a one-to-one correspondence of one of these parts $(\lambda, \mu; \nu)$ with a tensor of rank ν obtained from T_{ijkl} by linear operations. But for the applications we have in mind it is more convenient to leave them in their present form. It is also worth noting that in this terminology there are no isotropic tensors with spin different from zero; of course there are isotropic tensors of all ranks (except 1) since there are spin 0 parts in a general tensor of any rank except one.

To illustrate the method, consider the part (1, 1; 1). Since (ij) and (kl) indices are antisymmetrized, they can be contracted using the Levi-Civita symbols $\varepsilon_{\alpha i'j'}$, $\varepsilon_{\beta k'l'}$ leading to a second-rank tensor $t_{\alpha\beta}$ which is now to be antisymmetrized giving $t_{\alpha\beta} - t_{\beta\alpha}$ with spin $\nu = 1$. We can now restore the indices using the Levi-Civita symbols $\varepsilon_{\alpha ij}$, $\varepsilon_{\beta kl}$ thus finally obtaining

$$T^{(1,1;1)}_{ijkl} = T_{i'j'k'l'}\varepsilon_{\alpha i'j'}\varepsilon_{\beta k'l'}(\varepsilon_{\alpha ij}\varepsilon_{\beta kl} \quad \varepsilon_{\beta ij}\varepsilon_{\alpha kl})$$

which is the result given above. By playing with the Levi-Civita and Kronecker symbols the irreducible tensors $(\lambda, \mu; \nu)$ can also be written in apparently different looking forms, which are same within a multiplying constant. For example,

$$T_{i'j'k'l'}\varepsilon_{\beta i'j'}\varepsilon_{\beta k'l'}(\varepsilon_{\alpha ij}\varepsilon_{\beta kl} - \varepsilon_{\beta ij}\varepsilon_{\alpha kl})$$

can be written in the form

$$T_{ijkl} \quad T_{jikl} \quad T_{ijlk} + T_{jilk} \quad T_{klij} + T_{klji} + T_{lkij} + T_{lkij} \quad T_{lkji}$$

by making use of the identity

$$\varepsilon_{\alpha i'j'}\varepsilon_{\alpha ij} \quad (\delta_{ii'}\delta_{jj'} \quad \delta_{ij'}\delta_{ji'}$$

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3. – Application to stochastic dynamics.

A spin-one system in quantum mechanics is a three-level system whose states transform as the components of a vector under rotations. Being a three-level system the most general admissible state is a nonnegative Hermitian 3×3 matrix with unit trace; hence a general state is specified by 8 independent real parameters lying in appropriate domains. The most general dynamical law then consists in stating how these 8 parameters vary as a function of time, with the restriction that the temporal changes do not lead these parameters outside the allowed domains.

Let us consider a general 3×3 Hermitian matrix M and its parametrization in terms of 3 real parameters r, p_0, q_0 and three complex parameters p_1, q_1, q_2:

$$M \quad \begin{pmatrix} r + p_1 \quad q_0 & \dfrac{p_1 + q_1}{\sqrt{2}} & q_2 \\ \dfrac{p_1^* + q_1^*}{\sqrt{2}} & r \quad 2q_0 & \dfrac{p_1 \quad q_1}{\sqrt{2}} \\ q_2^* & \dfrac{p_1^* \quad q_1^*}{\sqrt{2}} & r \quad p_0 + q_0 \end{pmatrix}$$

Under an arbitrary unitary transformation $M \to UMU^+ = M$ we get a new set of six parameters which are linear combinations of the old parameters. But there are three invariants which may be written

$$\begin{aligned} A &= 3r \\ B &= r^2 \quad 4rq_0 - (q_0^2 + q_1 q_1^* + q_2 q_2^* + p_0^2 + p_1 p_1^*)\,, \\ C &= r^3 \quad r(q_0^2 + q_1 q_1^* + q_2 q_2^* + p_0^2 + p_1 p_1^*) + \\ &\quad + q_0(p_0^2 - p_1 p_1^* - q_0^2 - q_1 q_1^* + q_2 q_2^*) + p_0(p_0 q_0 + p_1 p_1^* + p_1^* q_1) + \\ &\quad + \tfrac{1}{2}(q_2^* p_1^2 + q_3 p_1^{*2} - q_0^3 + q_0 q_2 q_2^*) - \tfrac{1}{2}(q_2 q_1^{*2} + q_2^* q_1^2 + q_0^3 - q_0 q_2 q_2^*)\,, \end{aligned}$$

which are the coefficients of the characteristic equation of the Hermitian matrix M:

$$\det|M \quad \lambda I \qquad \lambda^3 + A\lambda^2 \quad B\lambda + C \quad 0$$

If we now restrict M to be a density matrix ϱ then M must have unit trace and be nonnegative. Consequently $A = 1$ and the other two invariants must

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lie within the domain bounded by the C-axis and the arcs OP, PQ defined by the equations

$$27C = 9B - 2 \mp 2(1 \quad 3B)^{\frac{3}{2}}$$

This domain is shown shaded in Fig. 2; note that while the density matrices form a convex set since these unitary invariants are not linear in the density matrix parameters, the allowed domain is not convex.

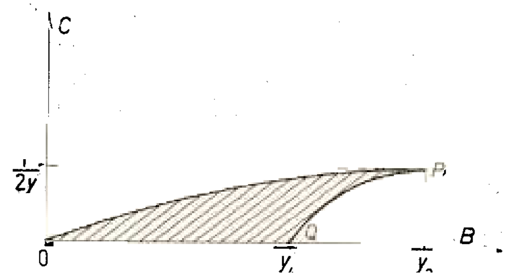

Fig. 2. – Allowed domains for the characteristic parameters A, B of the density matrix of a 3-level system.

Following CARTAN, we observe that the parametrization of the matrix M is such that the matrices depending on p, q, r transform amongst themselves under rotations as irreducible tensors with 3, 5 and 1 independent components respectively, provided the angular momentum matrices are chosen as the following standard set:

$$J_1 = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \qquad J_2 \begin{pmatrix} 0 & \frac{-i}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & 0 & \frac{-i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & 0 \end{pmatrix};$$

$$J_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The stochastic dynamics of this system is completely specified by the $3^2 \times 3^2$ stochastic matrix $A_{rs,r's'}(t)$ given by the defining relation

$$\varrho_{rs}(t) = A_{rsr's'}(t)\,\varrho_{r's'}(0)$$

with summation over the repeated indices $r's'$ being understood. The sto chastic matrix satisfies the following properties:

$$A_{sr\,s'r'} = (A_{rs\,r's})^*$$

$$A_{rsr's'}\,y_r^* y_s x_{r'} x_{s'}^* \geqslant 0 \qquad \text{for all } x \text{ and } y$$

$$A_{rr,r's'} = \delta_{r's'}$$

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corresponding to hermiticity, positive definiteness and normalization of the density matrices.

The operation of summation over repeated indices corresponds to contraction of tensors in view of the transformation properties of the quantities p, q, r under rotations. Hence it follows that the irreducible parts $(\lambda, \mu; \nu)$ in the reduction of the stochastic matrix considered as a fourth-rank tensor A_{ijkl} has the following interpretation: it furnishes the specific coefficient in the contribution of the irreducible spin μ part of the density matrix $\varrho_{kl}(0)$ in the expression for the irreducible spin λ part of the density matrix $\varrho_{ij}(t)$.

In particular the normalization condition $A_{rr,r's'} = \delta_{r's'}$ yields immediately

$$A_{ijkl}[t;\ (0, \mu; \mu)] \qquad 0$$

If we know further that the dynamics possesses any symmetry property, it would be reflected in the dependence of $A_{ijkl}[t, (\lambda, \mu; \nu)]$ on the index ν. For the isotropic case, for example we have

$$A_{ijkl}[t;\ (\lambda, \mu; \nu)] \qquad 0\,,$$

Even if the system is not isotropic, say for the relaxation of a spin system in a cubic crystal lattice, the crystal symmetry prevents the existence of any spin 1 or spin 2 tensors invariantly associated with the lattices; consequently the parts of the stochastic matrix with $\nu = 1, 2$ must vanish for such a relaxing system provided no external magnetic field is present. For more complex crystal classes the discussion is not so straightforward.

4. – Systems with axial symmetry.

Excepting for the familiar case of isotropy, the simplest special case is that of axial symmetry. It is of particular importance in the discussion of the relaxation in a strong external magnetic field in an isotropic medium (or one in which an axis of axial symmetry coincides with the direction of the magnetic field). We denote the unit vector in the distinguished direction by ξ. Given such an axially symmetric system, any tensor of spin ν can be expressed as linear combinations with numerical coefficients of the $(2\nu+1)$ spherical harmonics $Y_\nu^m(\xi)$. Consequently apart from a multiplicative constant $a(\lambda, \mu; \nu)$ the irreducible parts $T_{ijkl}(\lambda, \mu; \nu)$ are uniquely determined and may be written down immediately. They are given below, where we have omitted the possible multiplicative constant; the most general axially symmetric fourth-rank tensor is an arbitrary linear combination of these tensors.

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Reduction of $T_{ijkl}(\xi)$

Spin zero

(1 $\delta_{ij}\delta_{kl}$ (0, 0; 0)

(2) $(\delta_{ik}\delta_{jl} \quad \delta_{il}\delta_{jk})$ (1, 1; 0)

(3) $(\delta_{ik}\delta_{jl} \quad \delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl})$ (2, 2; 0)

Spin one

(4) $\xi_p\varepsilon_{pij}\delta_{kl}$ (1, 0; 1)

(5) $\xi_p\varepsilon_{pkl}\delta_{ij}$ (0, 1; 1)

(6) $\xi_i\varepsilon_{jkl} - \xi_j\varepsilon_{ikl})$ (1, 1; 1)

(7) $(\xi_i\varepsilon_{jkl} + \xi_j\varepsilon_{ikl} - \frac{2}{3}\delta_{ij}\,\xi_m\varepsilon_{mkl})$ (2, 1; 1)

(8) $(\xi_k\varepsilon_{lij} + \xi_l\varepsilon_{kij} - \frac{2}{3}\delta_{kl}\xi_m\varepsilon_{mij})$ (1, 2; 1)

(9) $[\xi_m\varepsilon_{mik}\delta_{jl} + \xi_m\varepsilon_{mjk}\delta_{il} + \xi_m\varepsilon_{mil}\delta_{jk} + \xi_m\varepsilon_{mjl}\delta_{ik}]$ (2, 2; 1)

Spin two

$(\xi_i\xi_j - y_3\delta_{ij})\delta_{kl}$ (2, 0; 2)

$(\xi_k\xi_l - y_3\delta_{kl})\delta_{ij}$ (0, 2; 2)

$[(\xi_i\xi_k - y_3\delta_{ik})\delta_{jl} - (\xi_j\xi_k - y_3\delta_{jk}\delta_{il}] -$

$- [(\xi_i\xi_l - y_3\delta_{il})\delta_{jk}) - (\xi_j\xi_l - ye\,\delta_{jl})\delta_{ik}]$ (1, 1; 2)

(13) $[\xi_i\xi_k\delta_{jl} + \xi_j\xi_k\delta_{il}] - [\xi_i\xi_l\delta_{jk} + \xi_i\xi_l\delta i_k]$ (2, 1; 2)

(14) $[\xi_i\xi_k\delta_{jl} + \xi_i\xi_l\delta_{jk}] - [\xi_j\xi_k\delta_{il} + \xi_j\xi_l\delta_{ik}]$ (1, 2; 2)

(15) $\{[(\xi_i\xi_k - \frac{1}{3}\delta_{ik})\delta_{jl} + (\xi_j\xi_k - \frac{1}{3}\delta_{jk})\delta_{il} - \frac{2}{3}\delta_{ij}(\xi_k\xi_l - y_3\delta_{k2})]_2 +$

$- [(\xi_i\xi_l - \frac{1}{3}\delta_{il})\delta_{jk} + (\xi_j\xi_l - \frac{1}{3}\delta_{jl})\delta_{ik} - \frac{2}{3}\delta_{ij}(\xi_k\xi_l - y_3\delta_{kl})]$

$- [\frac{5}{3}\delta_{kl}(\xi_i\xi_j - \frac{1}{3}\delta_{ij})]\}$ (2, 2; 2)

Spin three

(16) $[\frac{5}{3}\xi_i\xi_j\xi_p\varepsilon_{pkl} - \frac{1}{3}\delta_{ij}\xi_p\varepsilon_{pkl} - \frac{1}{3}\xi_i\varepsilon_{jkl} - \frac{1}{3}\xi_j\varepsilon_{ikl}]$ (2, 1; 3)

(17) $[\frac{5}{3}\xi_k\xi_l\xi_p\varepsilon_{pij} - \frac{1}{3}\delta_{kl}\xi_p\varepsilon_{pij} - \frac{1}{3}\xi_k\varepsilon_{lij} - \frac{1}{3}\xi_l\varepsilon_{kij}]$ (1, 2; 3)

(18) $[(\frac{5}{3}\xi_i\xi_k - \frac{1}{3}\delta_{ik})\xi_m\varepsilon_{mjl} + (\frac{5}{3}\xi_i\xi_l - \frac{1}{3}\delta_{il})\xi_m\varepsilon_{mjk} +$

$+ (\frac{5}{3}\xi_j\xi_k - \frac{1}{3}\delta_{jk})\xi_m\varepsilon_{mil} + (\frac{5}{3}\xi_j\xi_l - \frac{1}{3}\delta_{jl})\xi_m\varepsilon_{mik}]$ (2, 2; 3)

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Spin four

$$(19)\qquad [\tfrac{7}{3}\xi_i\xi_j\xi_k\xi_l - \tfrac{1}{3}(\delta_{ij}\xi_k\xi_l + \delta_{ik}\xi_j\xi_l + \delta_{il}\xi_j\xi_k + \delta_{jk}\xi_i\xi_l + \delta_{jl}\xi_i\xi_k + \delta_{kl}\xi_i\xi_j) + \\ + \tfrac{1}{15}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})] \qquad (2,2;4)$$

In its application to stochastic dynamics of axially symmetric spin-1 systems these universal forms imply certain geometrical connections with respect to the changes in multipole polarizations p, q as functions of time. These connections are most early expressed in the following manner: Let $\varrho_{ij}^{(0)}$, $\varrho_{ij}^{(1)}$, $\varrho_{ij}^{(2)}$ be the spin 0, 1, 2 parts of the density matrix ϱ_{ij}. These parts in turn may be decomposed, in relation to the vector ξ as follows:

$$\varrho_{ij}^{(0)} \to \varrho(0,0) = \varrho_{ij}^{(0)}\delta_{ij} \quad 1\,,$$

$$\varrho_{ij}^{(1)} \to \begin{cases} \varrho(1,0) = \varepsilon_{ijk}\varrho_{ij}^{(1)}\xi_k \\ \varrho_i(1,1) = \varrho_{ij}^{(1)}\xi_j \end{cases}$$

$$\varrho_{ij}^{(2)} \to \begin{cases} \varrho(2,0) = \varrho_{ij}^{(2)}\xi_i\xi_j \\ \varrho_i(2,1) = \varrho_{ij}^{(2)}\xi_j - \varrho_{kl}^{(2)}\xi_k\xi_l\xi_i \\ \varrho_{ij}(2,2) = \varrho_{ij}^{(2)} - \varrho_{kl}^{(2)}\xi_k\xi_l\xi_i\xi_j\,. \end{cases}$$

Then the axially symmetric dynamics mixes only $\varrho(0,0)=1$ with itself, $\varrho(1,0)$, $\varrho(2,0)$ amongst themselves, $\varrho_i(1,1)$ and $\varrho_i(2,1)$ amongst themselves and $\varrho_{ij}(2,2)$ with itself.

5. – Concluding remarks.

The previous sections outline a method of reduction of arbitrary tensors in their cartesian form. The same general method used here could be used for higher-rank tensors also ([2]) but the task becomes rapidly complicated. Needless to say, in a physical problem like that of hydrodynamic turbulence where the Cartesian form is very desirable, the decomposition is only the kinematics of the problem and the interesting question is the transcription of the dynamics, contained in the Navier-Stokes equations, in terms of these invariants. The complexity of this aspect of the problem may be seen from the fact that so far only the second-rank axisymmetric case has been completely studied; for a comprehensive discussion of the decay of turbulence of at least a partial

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study of higher-order correlations is necessary, The methods developed here provide the initial steps in such a study; a more detailed discussion is beyond the scope of this work, but will be presented elsewhere.

The restrictions imposed by crystal symmetry on the various irreducible parts of the fourth-rank tensor have only been touched upon here, but this aspect has to be examined in the detailed study of the relaxation mechanism of dipole and quadrupole polarizations of spin-one ions at lattice sites. This connection is however best studied with reference to the particular crystal under investigation.

The decomposition outlined here also suggests a new representation of crystal properties; one splits a tensor into its irreducible parts and each irreducible part is given a geometric representation. This last one is facilitated by the fact that an irreducible part with spin ν can be associated with a homogeneous form of degree ν in three variables; interpretation of these variables as homogeneous co-ordinates in a plane and a canonical choice of the homogeneous fom would give a *plane* curve associated with each irreducible part. This program is currently being carried out by S. MORIN in relation to various tabulated crystalline properties.

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RIASSUNTO (*)

Si presenta uno schema di riduzione per rappresentare le parti irriducibili di un tensore di quarto ordine. Tale schema di riduzione interessa in rapporto alla fisica dei cristalli, turbolenza idrodinamica, ecc. Come applicazione immediata si discute brevemente la dinamica stocastica di un sistema di spin uno.

(*) *Traduzione a cura della Redazione.*

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