51

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Reduction of Operator Rings and the Irreducibility Axiom in Quantum Field Theory*

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The mathematical theory of the reduction of operator rings is used to investigate some structures which can occur in quantum field theory when the postulate that the field operators generate an irreducible ring is relaxed. In particular, it is shown that if a quantum field theory has a commutator which commutes with all field operators it is a direct integral of theories in each of which the commutator is a scalar. If in addition it satisfies the postulates of Lorentz covariance, existence of an invariant vacuum, and mass and energy spectra, then it is a direct integral of generalized free field theories whenever the unitary representation of the Lorentz group can be constructed in terms of functions of the field operators and every state can be constructed by applying field operators on the vacuum. It is also shown that the latter two assumptions together with the requirement of a unique invariant vacuum state are sufficient to prove that the ring generated by the field operators is irreducible. In other words, under these conditions the irreducibility postulate is redundant.

I. INTRODUCTION

ONE of the postulates which is often included in studies of quantum field theory is that the ring generated by "smeared polynomials in the field operators" be irreducible.[1] But for many purposes it is neither necessary nor desirable to limit the scope of investigations by this assumption. For example, if a field theory is defined in terms of its Wightman functions[2] it is possible for the resulting set of field operators to be reducible. In fact a weighted mean of two sequences of Wightman functions is again a sequence of Wightman functions so that the set of all sequences of Wightman functions forms a convex set.[3] But only the extremal points of this convex set yield theories for which the field operators generate an irreducible ring.[4] It is also interesting to note that the theories which do not correspond to extremal points will not necessarily have a vacuum state which is the unique invariant state.

In general, the operators representing the observables for a quantum mechanical system generate an irreducible ring if and only if the system has no superselection rules (or equivalently admits no supersymmetry transformation).[5] However we do not wish to assume that the ring of observables and the ring generated by the field operators are identical; the latter may contain operators, e.g., a baryon creation operator, connecting different superselection subspaces.

In this paper we explore some of the situations which can occur when the irreducibility postulate

* Supported in part by the Atomic Energy Commission.
[1] This is listed as postulate 6 by R. Haag and B. Schroer, J. Math Phys. 3, 248 (1962), which contains a rigorous survey of axiomatic quantum field theory.
[2] A. S. Wightman, Phys. Rev. 101, 860 (1956).
[3] E. C. G. Sudarshan and K. Bardakci, J. Math Phys. 2, 767 (1961).
[4] As was discussed in detail in reference 1 by Haag and Schroer, this is a general property of the construction of a representation of a ring from a positive linear functional on the ring.
[5] J. M. Jauch, Helv. Phys. Acta 33, 711 (1960).

of quantum field theory is relaxed. It is to be expected that the presently available models of field theories can be extended to yield examples of structure richer than have so far been evident. We consider in particular the extension of the generalized free field theory of Greenberg.[6] Under the assumption of irreducibility, it has been shown that if a field theory has a commutator which either commutes with all field operators or is translation invariant and vanishes for space-like separation of its arguments, then the commutator is a scalar. It has also been shown that if, in addition to the latter property, the field theory satisfies the customary postulates of Lorentz covariance, the existence of the vacuum state, and mass and energy spectra, then the theory is a generalized free field theory.[7] We will see that these results can be generalized in a straightforward way when the irreducibility postulate is relaxed.

Our method of investigation is based on the mathematical theory of rings of operators. In the next section we define some of the main concepts of this theory and outline the results which we will use. In Sec. III we review the defining properties of field operators in terms of this mathematical language. The structure of commutators which commute with all field operators is treated in the following section. The result is that the field theory is a direct integral of field theories in each of which the commutator is a scalar. In the final section, in addition to this property of the commutator, we postulate Lorentz invariance, an invariant vacuum state, and positive energy and square of the mass. If we also assume that the unitary representation of the Lorentz group can be constructed as functions of the field operators, and that all of the states of the theory can be obtained by operating with field operators on the invariant vacuum state, then we can deduce that the theory is a direct integral of generalized free field theories. If with these latter two assumptions we require the vacuum state to be the unique invariant state we can prove that the field operators are irreducible. In other words, if in a field theory the unitary representation of the Lorentz group is constructed in terms of functions of the field operators, if there exists a vacuum state which is the unique invariant state, and if all states can be obtained by applying field operators to the vacuum state, then the ring generated by the field operators is irreducible.

[6] O. W. Greenberg, Ann. Phys. **16**, 158 (1961).

[7] A. L. Licht and J. S. Toll, Nuovo cimento **21**, 346 (1961); G. F. Dell'Antonio, J. Math. Phys. **2**, 759 (1961); see also R. Acharya, Nuovo cimento **23**, 580 (1962).

II. MATHEMATICAL BACKGROUND

In this section we give a brief review of the theory of operator rings and their reduction. This survey is not intended to be complete in any way. Its purpose is only to provide enough information to define the main concepts used in this paper. No proofs are given. For further details we refer the reader to the original papers by Murray and von Neumann,[8] Naimark and Fomin,[9] and to the lucid book by Naimark.[10]

We call either a finite dimensional linear inner-product space or an infinite dimensional separable Hilbert space simply a Hilbert space.[11]

Definition. A set R of bounded linear operators on a Hilbert space $\mathcal{H}$ is a *ring* if it is closed under multiplication by scalars, addition, multiplication, and taking the adjoint. That is, if A and B belong to R and a and b are scalars, then $aA + bB$, AB, and A^+ belong to R. (This is also referred to as a *algebra, symmetric ring, ring with involution, etc.)

We will be mainly interested in rings which contain the identity. An example is the ring of all bounded operators on the space.

Definitions. A ring R is *weakly closed* if it is closed in the weak operator topology, that is if A_n are a sequence of operators in R such that for any vectors f, g in $\mathcal{H}$, $(f, A_n g) \to (f, Ag)$ for some bounded operator A, then A belongs to R. A vector f in $\mathcal{H}$ is a *cyclic vector* for a ring R if the set of vectors Af for all A in R is dense in $\mathcal{H}$, that is given any vector g in $\mathcal{H}$ there exists a sequence of operators A_n in R such that $||A_n f - g|| \to 0$. A ring R is *irreducible* if there is no proper subspace of $\mathcal{H}$ which is invariant under R, that is if there is no subspace $\mathfrak{M}$ of $\mathcal{H}$, different from $\mathcal{H}$ and from the zero element, such that Af belongs to $\mathfrak{M}$ for every A in R and f in $\mathfrak{M}$.

A ring R is irreducible if and only if every bounded operator which commutes with every operator in R is a scalar multiple of the identity operator (scalar operator). An irreducible ring is identical with the ring of all bounded operators. Every non-zero vector is a cyclic vector for a ring if and only if the ring is irreducible.

[8] F. V. Murray and J. von Neumann, Ann. Math. **37**, 116 (1936); J. von Neumann, ibid. **50**, 401 (1949).

[9] M. A. Naimark and S. V. Fomin, Trans. Am. Math. Soc. **5**, 35 (1957).

[10] M. A. Naimark, *Normed Rings*, translated by L. F. Boron (P. Noordhoff, Groningen, 1959).

[11] The assumption of separability of the Hilbert space is probably not needed to obtain most of the results of this paper, but we need it in order to be able to directly apply the considerable work of von Neumann.

Definitions. If F is a set of bounded linear operators on $\mathcal{H}$, the *commutant* F' of F is the set of bounded linear operators on $\mathcal{H}$ which commute with all of the operators in F and with the adjoints of all of the operators in F. The *center* Z_R of a ring R is the intersection $Z_R = R \cap R'$ of the ring and its commutant ring, that is Z_R is the set of all elements of R which commute with all of the elements of R.

F' is a weakly closed ring containing the identity operator. By applying this operation a second time we can form F''. Clearly F is contained in F''. In fact F'' is the smallest weakly closed ring which contains F and the identity operator. If F itself is a weakly closed ring containing the identity operator then $F'' = F$. Clearly the center Z_R of a ring R is an Abelian ring; it is weakly closed if R is. If a ring R is Abelian then R is contained in R' or $Z_R = R$. A ring R is irreducible if and only if R' is the ring containing only scalar operators. This is the most simple structure possible for a ring. The next most simple structure is that the center of the ring contain only scalar operators.

Definition. A weakly closed ring R is a *factor* if the center Z_R of R contains only scalar operators, that is if every operator in the ring which commutes with every operator in the ring is a scalar.

Clearly an irreducible ring is a factor, as is the ring of scalar operators. If R is a factor then R' is also a factor.

It is far from true that all rings have the simple structure of irreducible rings and factors. However it turns out that any ring can be built up as a generalized direct sum of either of these kinds of building blocks, just as all representations of many groups can be formed as generalized direct sums of irreducible representations. In order to characterize the general structure of operator rings we need then the concept of a direct integral of Hilbert spaces.

Definition. Let $\sigma(t)$ be the weight function for a Lebesgue-Stieltjes measure on the real line [$\sigma(t)$ is a real, nondecreasing, right continuous bounded function of t for all real t]. For each t let $\mathcal{H}(t)$ be a Hilbert space, and let $\mathcal{H}$ be the set of all vector-valued functions f of t, with $f(t)$ a vector in $\mathcal{H}(t)$, which satisfy the conditions:

(i) For any two functions f and g in $\mathcal{H}$, $(f(t), g(t))$ is a σ measurable function of t;

(ii) For any f in $\mathcal{H}$, $||f(t)||^2$ is a σ measurable and also a σ summable function of t, that is

$$\int ||f(t)||^2 \, d\sigma(t) < \infty$$

An inner product is defined in $\mathcal{H}$ by

$$(f, g) = \int (f(t), g(t)) \, d\sigma(t)$$

The space $\mathcal{H}$ with this inner product is called the *direct integral* of the Hilbert spaces $\mathcal{H}(t)$ with respect to the measure σ.

It can be shown that $\mathcal{H}$ is a linear space if addition and multiplication by scalars are defined as for functions of t,

$$(af + bg)(t) = af(t) = bg(t)$$

for f, g in $\mathcal{H}$ and a, b scalars. Two vectors f and g are considered to be identical if $f(t) = g(t)$ for σ almost all t (that is except for a set of σ measure zero). With this identification it can be shown that $\mathcal{H}$ is a Hilbert space.

Two Hilbert spaces are equivalent if there exists an isometric linear mapping of one onto the other. We will freely identify and interchange equivalent spaces. If in forming the Hilbert space $\mathcal{H}$ as the direct integral of the Hilbert spaces $\mathcal{H}(t)$, as in the above definition, we were to change a set of the spaces $\mathcal{H}(t)$ corresponding to a set of t of σ measure zero, or if we exchanged the measure σ for another measure which is completely continuous with σ (has the same sets of zero measure), then we would get a Hilbert space equivalent to $\mathcal{H}$. For all f in $\mathcal{H}$ the vectors $f(t)$ form a linear manifold in $\mathcal{H}(t)$. We can assume that the closure of this manifold is $\mathcal{H}(t)$. We will use the notation $\mathcal{H} = \int \mathcal{H}(t)\, [d\sigma(t)]^{1/2}$ to denote that $\mathcal{H}$ is the direct integral of the $\mathcal{H}(t)$, and we will call $\mathcal{H}(t)$ the *component spaces* of the direct integral *decomposition* of $\mathcal{H}$. For an element f in $\mathcal{H}$ we will write $f = \int f(t)\, [d\sigma(t)]^{1/2}$.

Definitions. Let A be a bounded linear operator on $\mathcal{H}$. A is *reduced* by the direct integral decomposition $\mathcal{H} = \int \mathcal{H}(t)\, [d\sigma(t)]^{1/2}$ if for every f in $\mathcal{H}$, $(Af)(t) = A(t)f(t)$ where $A(t)$ is a bounded linear operator on $\mathcal{H}(t)$ for σ almost all t. A set R of bounded linear operators on $\mathcal{H}$ is reduced if every operator in R is reduced.

We will write $A = \int A(t)\, d\sigma(t)$ and call $A(t)$ the *part* of A in the component space $\mathcal{H}(t)$. A particular class of operators which are reduced are those which are scalar operators in each component space. These have the form $(Af)(t) = a(t)f(t)$ for any f in $\mathcal{H}$, where $a(t)$ is a complex valued, σ measurable, essentially bounded function of t. These operators form a weakly closed Abelian ring containing the identity operator which we will call the *kernel ring* P associated with the given direct integral decomposition. A necessary and sufficient condition for a bounded linear operator A to be reduced is that A be in P'. A necessary and sufficient condition for a ring R to be reduced is then that R be contained in P'.

Conversely, if we are given a weakly closed Abelian ring P containing the identity operator, there exists a direct integral decomposition of the space $\mathcal{H}$ for which P is the kernel ring. Now our main question is this: Suppose we are given a weakly closed ring R containing the identity operator. Can we find a direct integral decomposition $\mathcal{H} = \int \mathcal{H}(t)\ [d\sigma(t)]^{1/2}$ which reduces R such that the part of R in each component space $\mathcal{H}(t)$ is a factor or is an irreducible ring? The answer found by von Neumann[12] is that any decomposition which has a kernel ring P equal to the center $R \cap R'$ of the ring R will reduce R such that the part of R in the component space $\mathcal{H}(t)$ is a factor for σ almost all t. A necessary and sufficient condition for the part of R in almost every component space to be an irreducible ring as well as a factor is that the center $R \cap R'$ of R be a maximal Abelian subring of R'.[9] The latter condition is true whenever R contains a subring Q which has the property that $Q = Q'$ (Q is called a maximal Abelian ring).[5] It is also sufficient that R' be equal to the center Z of R, or equivalently that R' be Abelian. In any case we can always find a decomposition of $\mathcal{H}$ which reduces R into factors, and every operator in the center $R \cap R'$ of R will be reduced such that its part in each component space is a scalar operator.

If, instead of the Lebesgue-Stieltjes measure σ on the real line, we use any Borel measure ρ on a locally compact Hausdorff space, we can define the direct integral of Hilbert spaces and the reduction of operators and of a ring in a similar manner. Within this more general framework there always exists a direct integral decomposition of the Hilbert space which reduces a given weakly closed ring R containing the identity operator for which there is a cyclic vector, such that the part of R in ρ almost every component space is an irreducible ring.[13] ρ can be taken to be a measure on a compact Hausdorff space X which has the property that open sets have positive measure. This has the consequence that either X consists of a single point or else can be divided into two disjoint measurable sets each having positive measure. (Assume that X has at least two distinct points, x and y. Then there exist disjoint open sets V and W in X with x in V and y in W. Since W has positive measure and is contained in the complement of V, both V and the complement of V have positive measure.)

So far we have considered only bounded operators. But the unbounded self-adjoint operators, which are often of interest in physics, can be handled very easily.

Definition. An unbounded self-adjoint operator A is *associated* with the ring R of bounded operators, if every projection operator E_x in the spectral decomposition $A = \int x\, dE_x$ of A belongs to R.

If A is associated with R then we can say that R contains all bounded functions of A. If R is reduced by a direct integral decomposition of the Hilbert space then each projection operator E_x will be reduced and A will act as a reduced operator. In such a case we will say that A is reduced.

III. FIELD OPERATOR RING AND ITS REDUCTION

For every point x of space-time let $\phi(x)$ be a neutral scalar field operator on the separable Hilbert space $\mathcal{H}$. The rigorous version of this statement is that ϕ is an operator valued distribution over space-time which is defined as follows. Let S be some suitable class of complex testing functions of one or several space-time variables, for example those which are infinitely many times differentiable and vanish at infinity faster than any power of a space-time variable. Then ϕ is a linear mapping of S into linear operators on $\mathcal{H}$ which we denote symbolically by

$$\int f(x)\phi(x)\ d^4x, \tag{3.1}$$

if the element f of S is a function of a single space-time variable, and by

$$\int f^{(n)}(x_1 \cdots x_n)\phi(x_1) \cdots \phi(x_n)\ d^4x_1 \cdots d^4x_n, \tag{3.2}$$

if $f^{(n)} \in S$ is a function of n variables. It is postulated that all of these operators have a common dense domain so that they can be added to form "smeared polynomials in the field operators." It is also postulated that an operator of the form (3.1) is self-adjoint whenever f is real, and in general that an operator of the form (3.2) is self-adjoint whenever

$$f^{(n)*}(x_1, \cdots x_n) = f^{(n)}(x_n, \cdots x_1). \tag{3.3}$$

We denote the set of all such self-adjoint operators by F, Since any function $f^{(n)}$ belonging to S can be written as

[12] J. von Neumann, reference 8, Theorem VII, p. 460.
[13] M. A. Naimark, reference 10, p. 515.

$$f^{(n)} = g^{(n)} + ih^{(n)},$$

where $g^{(n)}$ and $h^{(n)}$ belong to S and satisfy the reality condition (3.3) [set

$g^{(n)}(x_1 \cdots x_n) = \frac{1}{2}\{f^{(n)}(x_1, \cdots x_n) + f^{(n)*}(x_n, \cdots x_1)\}$

and

$h^{(n)}(x_1, \cdots x_n)$
$= -i/2 \{f^{(n)}(x_1 \cdots x_n) - f^{(n)*}(x_n, \cdots x_1)\}],$

we see that every operator of the form (3.2), or any smeared polynomial in the field operators, has form $A + iB$ where A and B are self-adjoint operators which are members of the set F.

Now all of the operators in F will not necessarily be bounded so we can not form a ring containing F. But if we let F' be the set of all bounded linear operators which commute with every operator in F, that is commute with every projection operator which occurs in the spectral decomposition of an operator in F, and if we let $R = F''$, then R and F' ($= R'$) are weakly closed rings containing the identity operator which have the following properties.[14] Every bounded operator in F is in R, as is every projection operator which occurs in the spectral decomposition of an operator in F. In fact, R is the minimal weakly closed ring containing these projection operators. We may say that R is the smallest weakly closed ring containing bounded functions of the operators in F, or containing bounded functions of "smeared polynomials in the field operators."

Every unbounded operator in F is associated with R, so that if R is reduced by a direct integral decomposition of the Hilbert space $\mathcal{H}$ then F is also reduced according to the terminology introduced in the preceding section. In such a situation we will say that the field operator ϕ is reduced since it gives "smeared polynomials" all of which are reducible.

IV. FIELDS WITH COMMUTATORS WHICH REDUCE TO SCALARS

We will take the statement that the commutator $[\phi(x), \phi(y)]_-$ is a scalar (c number) to mean that for every testing function $f^{(2)}(x, y)$ belonging to S the operator

$$C_{f^{(2)}} = \int f^{(2)}(x, y)[\phi(x)\phi(y) - \phi(y)\phi(x)]\, d^4x\, d^4y$$
$$= \int [f^{(2)}(x, y) - f^{(2)}(y, x)]\phi(x)\phi(y)\, d^4x\, d^4y,$$

is a scalar operator on $\mathcal{H}$. Similarly, the statement that the field has a vanishing double commutator,

$$[[\phi(x), \phi(y)]_-, \phi(z)]_- = 0, \qquad (4.1)$$

implies that when integrated with any testing function $f^{(3)}(x, y, z)$ belonging to S the left-hand side of the above equation gives the zero operator on $\mathcal{H}$. But, Eq. (4.1) is also taken to imply that

$$[[\phi(x), \phi(y)]_-, \phi(z_1)\phi(z_2)]_- = \phi(z_1)[[\phi(x), \phi(y)]_-, \phi(z_2)]_- + [[\phi(x), \phi(y)]_-, \phi(z_1)]_-\ \phi(z_2) = 0,$$

and by induction that

$$[[\phi(x), \phi(y)]_-, \phi(z_1)\phi(z_2) \cdots \phi(z_n)]_- = 0. \qquad (4.2)$$

By integrating the latter equation with testing functions of the form $f^{(n+2)}(x, y, z_1, \cdots z_n) = f^{(2)}(x, y)f^{(n)}(z_1, \cdots z_n)$ where $f^{(2)}$ and $f^{(n)}$ belong to S we can deduce that for any $f^{(2)}$ in S the operator $C_{f^{(2)}}$ commutes with every operator in F, or commutes with every operator in R. This implies that $C_{f^{(2)}}$ is associated with the center $Z_R = R \cap R'$ of R.

Now if R is irreducible we have that $C_{f^{(2)}}$ is a scalar operator. But this conclusion can be drawn from the weaker assumption that R is a factor. In general it is expected that the assumption that R is a factor will be sufficient to prove most of the statements of this type which are of interest in field theory. In any case if R is not a factor we can find a direct integral decomposition of the Hilbert space $\mathcal{H}$ which reduces R into component rings which are factors.[15] Since for each $f^{(2)}$ in S the commutator operator $C_{f^{(2)}}$ is in the center of R, it will be reduced to component operators each of which is a scalar operator on the component space. Hence, the field theory has the structure of a direct integral of field theories, in each of which the commutator is a scalar, whenever Eq. (4.1) is true.

By the proof of Licht and Toll,[7] using the Jacobi identity, it can be shown that Eq. (4.1) is valid if the commutator $[\phi(x), \phi(y)]_-$ is translation invariant and vanishes for space-like $x - y$. We can summarize the results of this section then in the following.

Theorem If for a set of field operators the double commutator vanishes, that is if equation (4.1) is true, then there exists a direct integral decomposition of the Hilbert space $\mathcal{H}$ which reduces R into factors, such that for every $f^{(2)}$ belonging to S the commutator operator $C_{f^{(2)}}$ is reduced such that its part in each component space is a scalar operator. A particular case in which this is true is when the commutator $[\phi(x), \phi(y)]_-$ is translation invariant and vanishes for space-like $(x - y)$.

[14] M. A. Naimark, reference 10, pp. 444–450.

[15] These factors will be irreducible if we assume that R contains a complete set of commuting observables, that is a maximal Abelian ring. See Sec. II and J. M. Jauch, reference 5.

V. DIRECT INTEGRALS OF GENERALIZED FREE FIELDS

In this section we are interested in some particular cases where the reduction of the field operator ring R into factors yields a generalized free field theory in each component space of the associated direct integral decomposition of the Hilbert space $\mathcal{H}$. Hence, we postulate that the following conditions, having to do with Lorentz invariance, the vacuum state, and the mass and energy spectrum, are satisfied:

(a) There exists a set of unitary operators $U(a, \Lambda)$ on $\mathcal{H}$ which form a true representation of the proper inhomogeneous Lorentz group, that is

$$U(a_1, \Lambda_1)U(a_2, \Lambda_2) = U(a_1 + \Lambda_1 a_2, \Lambda_1\Lambda_2).$$

(b) The field operators transform according to

$$U(a, \Lambda)\phi(x)U^{+}(a, \Lambda) = \phi(\Lambda x + a).$$

Let U be the ring generated by the $U(a, \Lambda)$. That is $U = \{U(a, \Lambda)\}''$ is the smallest weakly closed ring containing all of the $U(a, \Lambda)$, Clearly U contains the identity operator. We assume that the operators $U(a, \Lambda)$ can be formed as functions of the field operators, or more precisely we postulate that

(c) U is contained in R.[16]

We also assume that there exists at least one invariant state which we call the vacuum state, that is we postulate:

(d) There exists a vector w in $\mathcal{H}$, with $(w, w) = ||w||^2 = 1$, which is invariant under the inhomogeneous Lorentz transformation operators $U(a, \Lambda)$, that is $U(a, \Lambda)w = w$ for every $U(a, \Lambda)$.[17]

While we have chosen not to postulate that the field operators form an irreducible ring, we need to limit the size of the Hilbert space $\mathcal{H}$ with respect to the ring R by assuming that all of the state vectors can be obtained by operating with "smeared polynomials in the field operators" on the vacuum state. Thus we postulate the "completeness" property of the field operators that

(e) The vector w of postulate (d) is cyclic for R, that is the vectors of the form Aw, with A in R, are dense in $\mathcal{H}$.

The energy-momentum operators P_μ are defined as the generators of the translation operators $U(a, 1) = e^{iP\cdot a}$, where $P\cdot a = P_0 a_0 - \mathbf{P}\cdot\mathbf{a}$ and the mass operator M is defined by $M^2 = P^2 = P_0^2 - \mathbf{P}^2$. We postulate that the energy-momentum four-vector lies in the forward light cone.

(f) The self-adjoint operators P_0 and M^2 are positive.

Now suppose we decompose the Hilbert space $\mathcal{H}$ into a direct integral of Hilbert spaces such that the ring R is reduced into factors. We would like to know that the conditions (a) through (f) are satisfied by the parts of the field operators in each component space, so that the theory can be thought of as a direct integral of field theories in each of which these conditions, plus the condition that the ring generated by the field operator is a factor, are satisfied. Since U is contained in R each operator $U(a, \Lambda)$ will be reduced, and the parts of these operators in any one component space will form a representation of the proper inhomogeneous Lorentz group and will transform the parts of the field operators covariantly. It is clear that the part of the ring U in any component space will be contained in the part of the ring R in that component space. Also the properties of the generators of the translation operators, in particular the positiveness of the energy and square of the mass, will hold for the parts of these operators in each component space. Hence we have that conditions (a), (b), (c), and (f) are satisfied for the part of the field theory in each component space. The question which remains then is whether each component space has a vacuum state. If we can find a nonzero, cyclic, invariant state we can always normalize it; so a positive answer to this question is given by the following.

Theorem. Let U be contained in R and let w be a vector which is cyclic for R and invariant under each $U(a, \Lambda)$. Then if R is reduced by a direct integral decomposition of the Hilbert space $\mathcal{H} = \int \mathcal{H}(t)\,[d\sigma(t)]^{1/2}$, the component $w(t)$ of the vector $w = \int w(t)\,[d\sigma(t)]^{1/2}$ is a nonzero (normalizable) vector in $\mathcal{H}(t)$ which is invariant under the part in $\mathcal{H}(t)$ of each $U(a, \Lambda)$ and is cyclic for the part of R in $\mathcal{H}(t)$ for σ almost all t.

[16] If we assume that R can be reduced into irreducible rings by a decomposition of the Hilbert space into a *discrete* direct sum then we can prove that U is contained in R. For then, each unitary operator $U(a, \Lambda)$ must map each component space onto a single component space, that is, it must at worst introduce a permutation of the component spaces. But, since each of these operators is continuously connected to the identity operator, it must leave every component space invariant, which implies that U must be reduced and hence be contained in R. See R. Hagedorn, Nuovo cimento Suppl. 12, 73 (1959). This assumption has also been proved, for the case of a Wightman theory with local commutativity, by H. J. Borchers, "On Structure of the Algebra of Field Operators," Institute for Advanced Study, Princeton, New Jersey, (preprint). Borchers has also proved, for a Wightman field, that the Hilbert space is separable. In this regard see also D. Ruelle, "On the Asymptotic Condition in Quantum Field Theory," (to be published, Helv. Phys. Acta), Appendix.

[17] Note that we do not assume that w is the *unique* vector invariant under every $U(a, \Lambda)$.

Proof. Let K be the set of all t for which $||w(t)||^2 > 0$. Since $||w(t)||^2$ is a measurable function, K is a measurable set, and w belongs to $\int_K \mathcal{H}(t)\ [d\sigma(t)]^{1/2}$ which is a subspace invariant under R. But then we have that

$$Rw \subset R \int_K \mathcal{H}(t)[d\sigma(t)]^{1/2} = \int_K \mathcal{H}(t)[d\sigma(t)]^{1/2}.$$

Since w is cyclic this implies that

$$\mathcal{H} = \int_K \mathcal{H}(t)[d\sigma(t)]^{1/2},$$

which means that K differs from the space of all t only by a set of σ measure zero. Hence we have proved that $w(t)$ is nonzero for σ almost all t. Let $V = \int V(t)\ d\sigma(t)$ be any operator for which $Vw = w$, which we can write as $||Vw - w||^2 = 0$, or $\int ||V(t)w(t) - w(t)||^2\ d\sigma(t) = 0$, which implies that $V(t)w(t) = w(t)$ for all t except possibly a set of σ measure zero. By letting V be in turn each of a finite number of generating elements of the representation of the proper inhomogeneous Lorentz group by the unitary operators $U(a, \Lambda)$, we can deduce that $w(t)$ is invariant under the part in $\mathcal{H}(t)$ of each operator $U(a, \Lambda)$ for all t not in the union of the finite number of corresponding sets of zero σ measure, that is for σ almost all t. Finally, for all operators A belonging to R, the vectors

$$\psi_A = Aw = \int A(t)w(t)[d\sigma(t)]^{1/2} = \int \psi_A(t)[d\sigma(t)]^{1/2}$$

form a dense set in $\mathcal{H}$. For any t the vectors $\psi_A(t) = A(t)w(t)$ form a linear manifold $\mathfrak{M}(t)$ in $\mathcal{H}(t)$. The closure Cl $(\mathfrak{M}(t))$ of this linear manifold is a subspace of $\mathcal{H}(t)$ and $\int$ Cl $(\mathfrak{M}(t))\ [d\sigma(t)]^{1/2}$ is a subspace of $\mathcal{H}$ which contains the dense set of vectors ψ_A. Hence this subspace is equal to the whole of $\mathcal{H}$ and Cl $(\mathfrak{M}(t)) = \mathcal{H}(t)$ for σ almost all t, or the vectors $A(t)w(t)$ are dense in $\mathcal{H}(t)$ for σ almost all t. Hence we have shown that $w(t)$ is cyclic for the part of R in $\mathcal{H}(t)$ for σ almost all t, which completes the proof of the theorem.

We next note that, if we make the additional assumption that the vacuum vector w represents the unique state which is invariant under the representation of the proper inhomogeneous Lorentz group, we can prove that the ring R generated by the field operators is a factor or even that it is irreducible. In other words, under the assumptions we have made, the postulate of the irreducibility of the field operators is implied by the postulate of the uniqueness of the vacuum.

Theorem. Let the ring U generated by the representation of the proper inhomogeneous Lorentz group belong to the ring R generated by the field operators. If there exists a vector w which is cyclic for R and which is the unique (up to a scalar factor) vector invariant under each $U(a, \Lambda)$, then R is a factor which is in fact irreducible.[18]

Proof. Let $\mathcal{H} = \int \mathcal{H}(t)\ [d\sigma(t)]^{1/2}$ be a decomposition of the Hilbert space which reduces R into factors. Then in the space of all t there is either only one set, consisting of a single point, which has positive measure, in which case the reduction of R is trivial and R is itself a factor, or there is a set K of t such that both K and its complement K^c have positive measure. In the latter case let E_K be the projection operator defined by

$$(E_K f)(t) = I_K(t)f(t)$$

for any $f = \int f(t)\ [d\sigma(t)]^{1/2}$ belonging to $\mathcal{H}$, where $I_K(t)$ is the real function of t which is equal to one when t is in K and is equal to zero otherwise. Then E_K belongs to R'. For any inhomogeneous Lorentz transformation operator $U(a, \Lambda)$ we then have that

$$U(a, \Lambda)E_K w = E_K U(a, \Lambda)w = E_K w,$$

or $E_K w$ is invariant under each $U(a, \Lambda)$. Now by the preceding theorem we have that $||E_K w||^2 = \int_K ||w(t)||^2\ d\sigma(t) \neq 0$, for otherwise $||w(t)||^2$ would vanish on the set K of positive measure. Similarly, since K^c has positive measure, we can deduce that $w - E_K w = (1 - E_K)w = E_{K^c} w \neq 0$. But this contradicts the uniqueness of w. Hence we must conclude that R is a factor.

By using a direct integral decomposition of the Hilbert space that reduces R into irreducible rings instead of just into factors we can construct a completely analogous argument to show that R must be irreducible. This completes the proof of the theorem. [Note that the above proofs do not depend on the specific transformation properties of the field operators or the mass and energy conditions of our postulates (b) and (f). Neither do they depend on any properties of the Lorentz group. The proof remains valid for any symmetry group, e.g., the Galilei group.]

We have chosen not to postulate that the ring R generated by the field operators is a factor (or an irreducible ring), but to investigate the consequences of the weaker postulates that the ring U generated

[18] This result is also contained in the work of H. J. Borchers, and of D. Ruelle,[16] within the framework of Wightman field theory. In addition, the converse of this theorem, that irreducibility implies uniqueness of the vacuum, has been proved for Wightman theories by Borchers. The work reported in this paper was done independently of that of Borchers and that of Ruelle.

by the unitary representation of the proper inhomogeneous Lorentz group is contained in R and that there exists a vector w which is invariant under each $U(a, \Lambda)$ and cyclic for R. We have seen that we can have two kinds of situations. If we postulate that the vacuum vector w is the unique vector which is invariant under each $U(a, \Lambda)$, we can prove that R is irreducible. If we do not postulate the uniqueness of the vacuum, R may not be a factor, but we can always reduce it into factors and the properties associated with Lorentz covariance and the vacuum will hold in the part of the field theory in each component space of the direct integral decomposition which effects this reduction.

In particular, suppose that a field theory satisfies postulates (a) through (f) and in addition has the property that the commutator commutes with all field operators, or more precisely $C_{f^{(2)}}$ belongs to $R \cap R'$ for all $f^{(2)}$ in S. Then, if we decompose the Hilbert space so that R is reduced into factors, the commutator operator $C_{f^{(2)}}$ will be a scalar in each component space. But the postulates (a) through (f) will also be valid for the part of the theory in each component space. By the argument of Licht and Toll[7] the commutator then has the generalized free field form in each component space and the part of the theory in each component space is equivalent to a generalized free field theory. Thus we summarize and conclude with the following statement: If a field theory satisfies postulates (a) through (f) it is equivalent to a direct integral of generalized free field theories if it has the property that $[[\phi(x), \phi(y)]_-, \phi(z)]_- = 0$. In particular, it will have the latter property whenever the commutator is translation invariant and vanishes for space-like separations of the arguments.