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REPRESENTATIONS OF PARAFERMI RINGS

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Abstract: We consider the operator algebra corresponding to a system consisting of an arbitrary finite number v of parafermi oscillators with a view to obtaining all irreducible representations of the operator algebra defining it. It is shown that this algebra is isomorphic to the Lie algebra of the orthogonal group B_v in $2v+1$ dimensions. The known results of the representation theory of the orthogonal group are then used to find the representations of the ring of parafermi operators. In particular it is seen that the so-called Green ansatz for a parafermi ring arises in a natural way and that, in a certain sense, it furnishes the most general "solution" of the parafermi algebra. The case of a parafermi ring of order 2, which is the simplest non-trivial case, is considered in some detail.

1. Introduction

It has been known for some time that Fermi-Dirac and Bose-Einstein statistics do not exhaust the possible quantal descriptions of identical particle assemblies †††. The classic work of Green [2]) is now recognized as being the decisive mathematical formulation of the problem and the first step toward its solution. Considerable effort has been devoted to various aspects of the quantum theory involving the generalized statistics [3-9]) but some of the basic technical questions still remain unanswered. In this paper we address ourselves to the solution of one of these questions, the representation theory for a parafermi ring.

Sect. 2 is devoted to the mathematical statement of the problem and definition of terms. In sect. 3 we demonstrate the isomorphism of the algebra of v parafermi oscillators to the Lie algebra of the orthogonal group B_v in $2v+1$ dimensions and state some results from the representation theory of this latter algebra which are relevant to our purpose. The implications of these results for an arbitrary parafermi algebra and in particular for the case of a parafermi ring of order 2 are discussed in sect. 4. Sect. 5 contains some concluding remarks.

2. Parafermi Rings

Let a_α, $\alpha = 1, 2, \ldots, v$, be a set of (unbounded linear) operators and a_α^+ their adjoints. These operators are said to constitute a parafermi ring if the following

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††† See ref. [1]) for a critical discussion of the physical implications and comparison with experimental evidence.

commutation relations are satisfied:

$$[\tfrac{1}{2}(a_\alpha a_\beta^+ - a_\beta^+ a_\alpha),\ a_\gamma] = -\delta_{\beta\gamma} a_\alpha,$$
$$[\tfrac{1}{2}(a_\alpha a_\beta - a_\beta a_\alpha),\ a_\gamma] = 0, \tag{2.1}$$

for all α, β, γ. Green [2]) showed that if one had commuting fermi oscillator operators

$$b_\alpha^{(r)} b_\beta^{(s)} + (2\delta_{rs}-1) b_\beta^{(s)} b_\alpha^{(r)} = 0,$$
$$b_\alpha^{(r)} b_\beta^{(s)+} + (2\delta_{rs}-1) b_\beta^{(s)+} b_\alpha^{(r)} = \delta_{\alpha\beta}\delta_{rs}, \tag{2.2}$$

then the construction

$$a_\alpha = \sum_{r=1}^{p} b_\alpha^{(r)} \tag{2.3}$$

yields a parafermi ring as defined in (2.1). We shall refer to the construction (2.3) as the Green ansatz of order p; we note that this definition (2.3) leads to

$$(a_\alpha)^{p+1} = 0, \quad (a_\alpha)^j \neq 0, \quad j < p+1, \tag{2.4}$$

for all α. More generally we define a parafermi ring of order p to be the ring of operators a_α satisfying eqs. (2.1) and (2.4).

Following Green we note that the Green ansatz furnishes for $p \geqq 2$ a reducible representation of the parafermi ring. It is from a study of the reduction theory for the Green ansatz that we were led to the present investigation. We observed that in the reduction of a parafermi ring of two oscillators ($v = 2$) corresponding to the Green ansatz of order 2 the irreducible representations were 1, 5 and 10 dimensional and identical with those of the standard Duffin-Kemmer [10]) algebra.

3. Connection with the Orthogonal Group B_v

The algebra (2.1) is in fact nothing other than the Lie algebra of the orthogonal group B_v in $2v+1$ dimensions. To demonstrate this we utilize a construction used previously by Takahashi and Kamefuchi [5]). We define the $2v$ hermitian operators

$$\beta_{2\alpha-1} = \tfrac{1}{2}(a_\alpha + a_\alpha^+), \quad \beta_{2\alpha} = \frac{i}{2}(a_\alpha - a_\alpha^+), \quad \alpha = 1, 2, \dots v. \tag{3.1}$$

By virtue of (2.1) these operators satisfy the commutation relations

$$[\beta_\lambda \beta_\mu - \beta_\mu \beta_\lambda, \beta_k] = -\delta_{\lambda k}\beta_\mu + \delta_{\mu k}\beta_\lambda. \tag{3.2}$$

It is to be emphasized that the relations (3.2) are neither more nor less general than (2.1); they are simply the same relations in a different guise. For the special case of a ring of order $2 (p = 2)$ we have the relation

$$\beta_\lambda^3 = \beta_\lambda. \tag{3.3}$$

The two sets of relations (3.2) and (3.3) are equivalent to the single set

$$\beta_\lambda\beta_\mu\beta_k+\beta_k\beta_\mu\beta_\lambda = \delta_{\lambda\mu}\beta_k+\delta_{\mu k}\beta_\lambda, \tag{3.4}$$

which of course are the defining relations of the Duffin-Kemmer algebra. Thus the algebra of a ring of v parafermi operators of order 2 is identical with the Duffin-Kemmer algebra of $2v$ β matrices.

For the general case, when the order of the parafermi ring is p, eq. (3.3) is replaced by

$$(\beta_{2\alpha-1}-i\beta_{2\alpha})^{p+1} = 0, \quad (\beta_{2\alpha-1}-i\beta_{2\alpha})^j \neq 0, \quad j < p+1. \tag{3.5}$$

Let us now choose the indices m and n to run over the values 0, 1, 2, . . ., $2v$, and define

$$\begin{aligned} J_{mn} &= -J_{nm} = \frac{1}{i}(\beta_m\beta_n-\beta_n\beta_m), \quad m, n \neq 0,\\ J_{m0} &= -J_{0m} = \beta_m,\\ J_{00} &= -J_{00} = 0. \end{aligned} \tag{3.6}$$

Then the two-index operators J_{mn} are antisymmetric in their indices and satisfy the commutations relations

$$[J_{mn}, J_{m'n'}] = i(\delta_{mm'}J_{nn'}+\delta_{nn'}J_{mm'}-\delta_{nm'}J_{mn'}-\delta_{mn'}J_{nm'}), \quad m, n = 0, 1, 2, \ldots 2v. \tag{3.7}$$

The algebra of the J_{mn} is thus isomorphic to the Lie algebra of the (proper) orthogonal group B_v in $2v+1$ dimensions [11]). This completes the demonstration of the isomorphism of the algebra (2.1) to the Lie algebra of the proper orthogonal group in $2v+1$ dimensions. We note in passing, that the $v(2v-1)$ quantities $(1/i)(\beta_m\beta_n-\beta_n\beta_m)$ constitute an algebra isomorphic with the Lie algebra of the orthogonal group D_v in $2v$ dimensions.

The known results of the representation theory of the orthogonal group [12]) can now be used to find all representations of the parafermi ring. In particular we note the following points:

i) Every irreducible representation of $B_v = 0_{2v+1}$ is characterized by v non-negative numbers $\{\lambda_r\}$, $\lambda_1 \geqq \lambda_2 \geqq \ldots \geqq \lambda_v$, which are either all integral or all half-integral.

ii) Every irreducible representation is finite dimensional and every (reducible) representation is fully reducible.

iii) The spinor representation Δ corresponding to $\lambda_r = \frac{1}{2}$ for all r is the fundamental (spinor) representation and is 2^v dimensional.

iv) Every irreducible representation of B_v occurs in the reduction of the Kronecker product of Δ with itself a sufficient number of times.

Closed expressions are available for the dimensionality and characters of every representation but they are too lengthy to be reproduced here. The reduction of the Kronecker products of irreducible representations is discussed in detail by Murnaghan [12]).

4. Representations of the Parafermi Ring

Returning now to the parafermi rings we apply to them the results of the last section. We observe, first of all, following Brauer and Weyl [13]), that the fundamental spinor representation Δ corresponds to the Jordan-Wigner representation of a fermi ring. Further it is clear that the Kronecker product representation $\Delta \times \Delta \times \ldots \times \Delta$ corresponds to the Green ansatz for a parafermi ring, the number of factors in the product corresponding to the order of the Green ansatz. The Green ansatz is thus not irreducible but its reduction can be worked out by standard techniques. However, the important observation is the following; in view of the remark iv) concerning the representations of B_v, it follows that every irreducible representation of a parafermi ring is contained in the reduction of some Green ansatz. Loosely speaking, we may say that Green found all representations of an arbitrary parafermi ring.

The parafermi ring of order 2 is the simplest non-trivial case and deserves consideration by itself. We have remarked that this case is identical with the generalized Duffin-Kemmer ring of $2v$ β matrices. All representations of the latter are, however, known from the work of Fujiwara [14]) who showed that there are $v+1$ inequivalent irreducible representations which we may denote by $\Gamma(r)$, $r = 0, 2, 1, \ldots v$. The dimensionality of $\Gamma(r)$ is ${}^{2v+1}C_r = \{(2v+1)!/r!(2v+1-r)!\}$, which coincides with the number of independent components of an r^{th} rank totally antisymmetric tensor in $(2v+1)$ dimensions. It is worthwhile noting that this coincidence is not accidental, for in the reduction of the Kronecker product representation $\Delta \times \Delta$ we get precisely these $v+1$ irreducible representations, each of which occurs once, and which transform as the components a totally antisymmetric tensor of rank $r(0 \leqq r \leqq v)$ in $2v+1$ dimensions. The Green ansatz of order 2 for a parafermi ring thus yields, on reduction, every irreducible representation of the (generalized) Duffin-Kemmer ring once and only once.

5. Conclusion

It appears from the foregoing analysis that the representation theory of a finite ring of parafermi operators can be deduced simply from that of the Lie algebra of the orthogonal group in an odd number of dimensions. This fact explains the origin and significance of Green's ansatz (2.3) and yields the curious result that there is no representation of the parafermi ring which is not contained in a Green ansatz of some order.

In concluding this discussion we wish to make the following three remarks. First, the Green ansatz of order p may be viewed as a multispinor of rank p in $2v+1$ dimensions generalizing the theory of multispinors in three and four dimensions. Second, the parafermi rings of order $p > 2$ are generalizations the Bhabha-Rao [15, 16]) rings (studied by these authors for the five dimensional pseudo-Euclidian group) to arbitrary odd-dimensional (Euclidean) spaces. Finally, for the case of a simple oscillator $v = 1$, these results coincide with those of Jordan, Mukunda and Pepper [14]).

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