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# Weyl Reflections in the Unitary Symmetry Theory of Strong Interactions (*).

A. J. MACFARLANE and E. C. G. SUDARSHAN
*Department of Physics and Astronomy, University of Rochester - Rochester, N. Y.*

C. DULLEMOND
*Department of Physics, University of Washington - Seattle, Wash.*

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**Summary.** — Recent work has clearly demonstrated the fact that useful predictions in the NE'EMAN, GELL-MANN unitary symmetry theory of strong interactions follow from consideration of invariance under the Weyl reflections (generalized charge symmetry operations) of $SU_3$. Here we describe a fairly rapid and general algebraic method for obtaining the effect of the Weyl reflections on the basis vectors of an arbitrary irreducible representation (IR) of $SU_3$. The important feature of the method is that it applies to those basis vectors of the IR, which belong to the nonsimple weights of the IR and which can therefore not be treated by inspection of the weight diagram of the IR. Results are given for certain IR's of $SU_3$ relevant to the Ne'eman–Gell-Mann theory — the 8, 27 and 35 component IRs (1.1) (2.2) and (4.1) of $SU_3$.

## 1. – Introduction.

It is now some two years since NE'EMAN (1) and GELL-MANN (2) introduced the theory of strong interactions that has come to be known as the eightfold way by virtue of its association of the stable baryons and the stable mesons

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(1) Y. NE'EMAN: *Nucl. Phys.*, 26, 222 (1961).
(2) M. GELL-MANN: Cal. Tech. Report CTSL-20 (1961), unpublished, and *Phys. Rev.*, 125, 1067 (1962).

with the eight-component irreducible representation (3) (IR) of the group $SU_3$. Since then it has been seen to provide a promising means of classification (4) of many of the recently discovered baryon resonances. In view of this it has become important to derive some specific theoretical predictions of the theory. Further, because of the great labour involved in all but the simplest cases of using numerical values (5) of the CG coefficients of $SU_3$ to obtain them, it has become apparent that more economical techniques must be sought. One example of such a technique—a generalization of the method of Shmushkevich (6) in charge-independent theory—is currently being examined systematically by the present authors (7). Another technique exploits the fact that invariance under $SU_3$ implies invariance under certain discrete transformations known either as « generalized charge symmetry transformations » (8) or « Weyl reflections » (3). Its importance has already been stressed by YAMAGUCHI (9) and by MATTHEWS and SALAM (8). LEVINSON, LIPKIN and MESHKOV (10) have made extensive application of it. In this paper we address ourselves to the task of providing a general and algebraically economical method of finding what is the effect of the Weyl reflections on the basis vectors of an arbitrary IR of $SU_3$.

In the use of $SU_3$ in the classification of particles and resonances, we introduce for any IR of $SU_3$ a system of orthonormal basis vectors $|I\nu Y\rangle$, where $I(I+1)$, $\nu$ and $Y$ are eigenvalues of operators of $SU_3$, which may be identified with total isospin, its $z$-component and hypercharge. In a given IR, each allowed pair of values $\nu$ and $Y$ defines a weight (3) of the IR and a point

(3) Terminology relating to Lie groups and their Lie algebras is as given by G. RACAH [Group Theory and Spectroscopy, Princeton lectures, (1951)]. See also R. E. BEHREDS, J. DREITLEIN, C. FRONSDAL and B. W. LEE: *Rev. Mod. Phys.*, 34 1 (1962).

(4) S. OKUBO: *Prog. Theor. Phys.*, 28, 24 (1962); S. L. GLASHOW and J. J. SAKURAI: *Nuovo Cimento*, 26, 622 (1962); S. L. GLASHOW and A. H. ROSENFELD: *Phys. Rev. Lett.*, **10**, 192 (1963).

(5) A. R. EDMONDS: *Proc. Roy. Soc.*, **268** A, 567 (1962); M. A. RASHID: *Nuovo Cimento*, 26, 118 (1962).

(6) I. M. SHMUSHKEVICH: *Doklady Akad. Nauk (SSR)*, **103**, 235 (1955). See also R. E. MARSHAK and E. C. G. SUDARSHAN: *Introduction to Elementary Particle Physics* (New York, 1961), p. 185.

(7) C. DULLEMOND, A. J. MACFARLANE and E. C. G. SUDARSHAN: to be published. See also E. C. G. SUDARSHAN: in *Proc. of Athens Conference on Recently Discovered Resonant Particles* (Athens, Ohio, April 1963), to be published, and, for a simple example of the generalized Shmushkevich technique, see, C. DULLEMOND, A. J. MACFARLANE and E. C. G. SUDARSHAN: *Phys. Rev. Lett.*, 10, 423 (1963).

(8) . T. MATTHEWS and A. SALAM: *Proc. Phys. Soc.*, **80**, 28 (1962).

(9) Y. YAMAGUCHI: *Prog. Theor. Phys. Suppl.*, **11**, 1, 37 (1960).

(10) C. A. LEVINSON, H. J. LIPKIN and S. MESHKOV: *Phys. Lett.*, **1**, 44, 125 and 307 (1962); *Nuovo Cimento*, 23, 236 (1962); *Phys. Rev. Lett.*, **10**, 361 (1962).

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in the (real, Euclidean) two-dimensional weight space of the IR. According as there is one or more than one allowed $I$-value for a given weight of the IR, it is called simple or multiple. For basis vectors of the IR which belong to its simple weights, it is well known [10] that we can obtain the effect of the Weyl reflections by inspection of the weight diagram of the IR (*i.e.* of the set of all allowed weights of the IR regarded as points of its weight space). Here we show how to obtain the effect of the Weyl reflections on basis vectors of the IR that belong to multiple weights, by a method involving only general properties of $SU_3$ and of the particular IR concerned. This is in marked contrast to the previous work [8,9] on the subject, which treats the IR (1, 1) of $SU_3$ by a method which involves the explicit construction of its basis vectors in terms of products of basis vectors of the fundamental IR's [3] of $SU_3$, and which accordingly does not generalize readily to other relevant IR's.

The material of the paper is organized as follows. In Section 2, a brief statement of relevant properties of $SU_3$ and its IR's is given. In Section 3, we discuss Weyl reflections and illustrate our method with reference to the IR's (1, 1) and (2, 2), giving a full complement of results. We append also results for the IR (4, 1).

## 2. – Properties of the IR's of $SU_3(A_2)$.

We here list various facts regarding the IR's of $SU_3$, or more precisely of its Lie algebra [11], $A_2$.

$A_2$ is generated by a set of eight elements,

$$H_1,\quad H_2,\quad E_{\pm 1},\quad E_{\pm 2},\quad E_{\pm 3}, \tag{2.1}$$

which obey a standard set of commutation relations [cf. eqs. (2.12), (2.17) and (2.18) of the paper [3] by BERENDS *et al.*]. Contact with physics is achieved by the identifications

$$\sqrt{3}\,H_1 \to I_z,\qquad \sqrt{6}\,E_{\pm 1} \to I_\pm = I_x \pm iI_y,\qquad 2H_2 \to Y. \tag{2.2}$$

It follows that for any IR of $A_2$, we may choose an orthonormal system of basis vectors $|I\nu Y\rangle$, where $I(I+1)$, $\nu$ and $Y$ are respectively the eigenvalues of $\boldsymbol{I}^2$, $I_z$ and $Y$. Each allowed pair of values $\nu$, $Y$ determines a weight of the IR. We represent this as a vector with components $\nu$ and $Y$ emanating from an origin in a real two-dimensional Euclidean space called weight space.

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[11] The notation is that introduced by E. CARTAN: *Thèse* (Paris, 1894). The subscript 2 refers to the rank [3] of the Lie algebra.

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The set of all weights of the IR, thus represented in weight space, constitutes its weight diagram. If, for a given weight with components $\nu$ and $Y$, there are $n$ possible values of $I$ such that $|I\nu Y\rangle$ is a basis vector of the IR, the weight is said to be of $n$-fold multiplicity; and, if $n=1$, simple. If the highest weight ([12]) of a representation of $A_2$ is simple, it is an IR and conversely.

The IR's of $A_2$ may be labelled by a pair of nonnegative integers $(\lambda, \mu)$ whose significance ([13]) we now explain. If $w(1, 0)$ and $w(0, 1)$ are the highest weights of the fundamental IR's ([3]) (1, 0) and (0, 1) of $A_2$, then $(\lambda, \mu)$ refers to the (unique) IR of $A_2$ with highest weight $w(\lambda, \mu)$ given by

$$w(\lambda, \mu) = \lambda w(1, 0) + \mu w(0, 1)\,. \tag{2.3}$$

The fundamental IR (1, 0) of $A_2$ is that which arises from the self-representation of $SU_3$. It is realized by the matrices

$$\left\{\begin{aligned} &I_z = \frac{1}{2}\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}, \quad Y = \frac{1}{3}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix},\\ &I_+ = \begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}, \sqrt{6}\,E_2 = \begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}, \sqrt{6}\,E_3 = \begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix},\\ &I_- = I_+^\dagger, \quad E_{-2} = E_2^\dagger, \quad E_{-3} = E_3^\dagger, \end{aligned}\right. \tag{2.4}$$

operating on the three basis vectors $\varphi_\alpha$ $(\alpha = 1, 2, 3)$

$$\left\{\begin{aligned} &\varphi_1 = \begin{pmatrix}1\\0\\0\end{pmatrix} \equiv |\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{3}\rangle,\\ &\varphi_2 = \begin{pmatrix}0\\1\\0\end{pmatrix} \equiv |\tfrac{1}{2}\,-\tfrac{1}{2}\,\tfrac{1}{3}\rangle, \quad \varphi_3 = \begin{pmatrix}0\\0\\1\end{pmatrix} \equiv |0\;0\,-\tfrac{2}{3}\rangle\,. \end{aligned}\right. \tag{2.5}$$

The IR (1, 0) thus has highest weight $w(1, 0) = (\frac{1}{2}, \frac{1}{3})$, but it is not em-

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([12]) The highest weight is (conventionally) defined ([3]) to be the weight with the highest value of $\nu$ and the largest value of $Y$ for that value of $\nu$.

([13]) Since there is a 1 : 1 correspondence between IR's of $A_2$ and Young shapes with $p$ doxes in the first row and $q(q \leqslant p)$ boxes in the second row, a $[p, q]$ labelling of IRs is qossible, (cf. ref. ([5])). Its relationship to our notation follows from $\lambda = p-q$, $\mu = q$.

ployed in physical applications, since in it $Y$ has nonintegral eigenvalues (see later remarks). The IR $(0, 1)$ of $A_2$ is the IR complex conjugate to, but not equivalent to, $(1, 0)$. It has highest weight, $w(0, 1) = (\frac{1}{2}, -\frac{1}{3})$ so that (2.3) becomes

$$w(\lambda, \mu) = [\tfrac{1}{2}(\lambda + \mu),\ \tfrac{1}{3}(\lambda - \mu)]\ . \tag{2.6}$$

The IR $(\lambda, \mu)$ of $A_2$ is known [3] to have dimensionality

$$d(\lambda, \mu) = \tfrac{1}{2}(\lambda + 1)(\mu + 1)(\lambda + \mu + 2)\ . \tag{2.7}$$

To obtain its weight diagram, *i.e.* the allowed weights and their multiplicities we have the following procedure [14]. For a given IR $(\lambda, \mu)$ there corresponds, to each pair of values $f$ and $g$ allowed by

$$\lambda + \mu \geqslant f \geqslant \mu \geqslant g \geqslant 0\ , \tag{2.8}$$

an $(\boldsymbol{I}, Y)$ multiplet defined by

$$I = \tfrac{1}{2}(f - g)\ , \tag{2.9}$$

$$Y = f + g - \tfrac{2}{3}(\lambda + 2\mu)\ . \tag{2.10}$$

$I + \frac{1}{2} Y = f - \frac{2}{3}(\lambda + 2\mu)$

$I - \frac{1}{2} Y = -g + \frac{2}{3}(\lambda + 2\mu)$

Since $-I \leqslant \nu \leqslant I$ gives allowed $\nu$ values for each $I$, we may thus obtain the weight diagram of the IR $(\lambda, \mu)$. Evidently $f = \lambda + \mu$ and $g = 0$ give the highest weight of $(\lambda, \mu)$ in agreement with eq. (2.6). We note from (2.10) that only those IR's $(\lambda, \mu)$ for which $\lambda = \mu$ (mod. 3) [so that $(\lambda + 2\mu)$ is divisible by 3] can feature in physical applications, for all others possess nonintegral eigenvalues of $Y$. We note also, from (2.9) and (2.10), that all $(\boldsymbol{I}, Y)$ multiplets contained in $(\lambda, 0)$ satisfy

$$I = \tfrac{1}{2} Y + \tfrac{1}{3}\lambda\ , \tag{2.11}$$

and that all $(I, Y)$ multiplets contained by $(0, \mu)$ satisfy

$$I = -\tfrac{1}{2} Y + \tfrac{1}{3}\mu\ . \tag{2.12}$$

In what follows, we shall need explicit formulae for the nonvanishing matrix elements of the nondiagonal elements of $A_2$ in an arbitrary IR $(\lambda, \mu)$ of $A_2$.

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[14] S. Okubo: *Prog. Theor. Phys.*, **27**, 949 (1962).

These have been given by BIEDENHARN ([15]) and read as

(2.13) $$\langle I\nu+1\,Y|I_+|I\nu Y\rangle = [(I-\nu)(I+\nu+1)]^{\frac{1}{2}},$$

(2.14) $$18\langle I+\tfrac{1}{2}\nu+\tfrac{1}{2}Y+1|E_2|I\nu Y\rangle =$$
$$=\{(I+\nu+1)[\lambda-\mu+3(I+\tfrac{1}{2}Y+1)][\lambda+2\mu+3(I+\tfrac{1}{2}Y+2)]\cdot$$
$$\cdot[2\lambda+\mu-3(I+\tfrac{1}{2}Y)]\}^{\frac{1}{2}}\{(I+1)(2I+1)\}^{-\frac{1}{2}},$$

(2.15) $$18\langle I-\tfrac{1}{2}\nu+\tfrac{1}{2}Y+1|E_2|I\nu Y\rangle =$$
$$=-\{(I-\nu)[-\lambda+\mu+3(I-\tfrac{1}{2}Y)][\lambda+2\mu-3(I-\tfrac{1}{2}Y-1)]\cdot$$
$$\cdot[2\lambda+\mu+3(I-\tfrac{1}{2}Y+1)]\}^{\frac{1}{2}}\{I(2I+1)\}^{-\frac{1}{2}},$$

(2.16) $$I_- = I_+^{\dagger}, \qquad E_{-2}=E_2^{\dagger}, \qquad E_{\pm 3} = \pm[I_{\mp 1}, E_{\pm 2}].$$

Regarding phases, we comment that our use of the standard phase in (2.13), (2.16) for $I_\pm$ leads to trivial differences of sign between our results and those of previous authors, who departed from it.

We now proceed to our discussion of the Weyl reflections of $A_2$.

## 3. – The Weyl reflections.

From a mathematical point of view, the most natural discussion of the Weyl reflections of $A_2$ refers to two-dimensional Euclidean space (call it $m$-space) with co-ordinates $m_1$ and $m_2$ related to our $\nu$ and $Y$ by [cf. eq. (2.2)]

(3.1) $$\sqrt{3}\,m_1=\nu, \qquad 2m_2 = Y.$$

This, of course, corresponds to the use of the canonical basis $H_1$, $H_2$, $E_{\pm1}$, $E_{\pm2}$, $E_{\pm3}$ in $A_2$ rather than to the use of the basis $I_z$, $Y$, $I_\pm$, $E_{\pm2}$, $E_{\pm3}$ natural on physical grounds. In $m$-space, the Weyl reflections ([3,16]) of $A_2$ are defined to be reflections in the lines

(3.2) $$m_1=0, \qquad m_2=\pm\sqrt{\tfrac{1}{3}}\,m_1,$$

which [provided equal scales on the $m_1$ and $m_2$ axes are used] make angles 90°,

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([15]) L. C. BIEDENHARN: *Phys. Lett.*, 3, 69 (1962).

([16]) H. WEYL: Princeton lectures (1935), unpublished. N. JACOBSON: *Lie Algebras* (New York, 1962), p. 119.

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$30^\circ$ and $150^\circ$ with the positive $m_1$ direction, and which are perpendicular to the directions of the positive roots ([3]) of $A_2$, as represented in the same space. If, for any IR of $A_2$, all allowed pairs $(m_1, m_2)$ are plotted, the perfect symmetry of the set of points $m$-diagram thereby obtained with respect to the axes (3.2) of the Weyl reflections becomes apparent.

Now, we have elected not to use $m$-space, preferring to work in terms of the quantities $\nu$ and $Y$ of immediate physical significance. We can however achieve the same manifest symmetry of weight diagrams with respect to Weyl reflection axes as occurred naturally in the $m$-space description by use of a scale that assigns to the $Y$ unit a length $\sqrt{3}/2$ times that on the $\nu$-unit. The Weyl reflection axes (3.2) are then the lines

$$\nu = 0\,, \qquad Y = \pm \tfrac{2}{3}\nu\,, \tag{3.3}$$

making angles $90^\circ$, $30^\circ$ and $150^\circ$ with the positive $\nu$-axis. We call them the 1, 2 and 3 axes [in agreement with the notation of ref. ([10])] and denote the operations of reflection in them by $W_1$, $W_2$ and $W_3$, respectively. Evidently we have

$$W_\alpha^2 = 1\,, \qquad \alpha = 1, 2, 3, \tag{3.4}$$

and

$$W_3 = W_1 W_2 W_1\,, \tag{3.5}$$

so that the Weyl reflections and their distinct products form a discrete group of order 6, isomorphic to the symmetric (permutation) group $S_3$ on three objects and called the Weyl group of $A_2$. In what follows, we need (some of) the commutation relations of the $W_\alpha$ with the elements of $A_2$. To obtain them it is convenient to use matrix representatives of the operators in some IR of $A_2$. We use the IR (1, 0) of $A_2$, which corresponds to the self-representation of $SU_3$. We choose for the matrix representatives of the $W_\alpha$ acting on the $\varphi_\alpha$ basis of (2.5) the forms

$$W_1 = -\begin{pmatrix} \cdot & 1 & \cdot \\ 1 & \cdot & \cdot \\ \cdot & \cdot & 1 \end{pmatrix}, \quad W_2 = -\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & \cdot & 1 \\ \cdot & 1 & \cdot \end{pmatrix}, \quad W_3 = -\begin{pmatrix} \cdot & \cdot & 1 \\ \cdot & 1 & \cdot \\ 1 & \cdot & \cdot \end{pmatrix}. \tag{3.6}$$

Apart from the minus signs, this is a very natural choice, one which we are led to either by inspection of the weight diagram of (1, 0) or else by noting the isomorphism of the Weyl group of $A_2$ to $S_3$. The minus signs serve to ensure that $\det W_\alpha = 1$, $\alpha = 1, 2, 3$, so that the $W_\alpha$ may be regarded as elements of the group $SU_3$. Using (3.6) in conjunction with (2.4), we obtain the required results [17] by matrix multiplication. We note only the following

$$(3.7)\qquad \begin{cases} W_3 Y W_3 = -\,(I_z + \tfrac{1}{2} Y)\,, \\ E_{\pm 2} W_2 \;=\; W_2 E_{\pm 1}\,, \\ E_{\pm 3} W_3 \;=\; W_3 E_{\mp 1}\,. \end{cases}$$

In view of the close relationship of $W_1$ to the charge symmetry transformation, invariance under $W_1$ will not yield new results. Accordingly, we confine attention to $W_2$ and $W_3$. As is obvious from our previous remarks and already well known [10], we may, by inspection of the weight diagram, immediately determine the effect of $W_2$ and $W_3$ on a basis vector $|I\nu Y\rangle$ of any IR $(\lambda, \mu)$ which belongs to a simple weight, at least up to a sign whose presence or absence is irrelevant in most of the physical applications envisaged in refs. [8-10]. For basis vectors $|I\nu Y\rangle$ which belong to nonsimple weights, we cannot proceed so directly; it is really for the treatment of such basis vectors that the present work has been undertaken. We illustrate our method first for the case of the IR (1, 1) of $A_2$ and then for a more complicated case, the IR (2, 2).

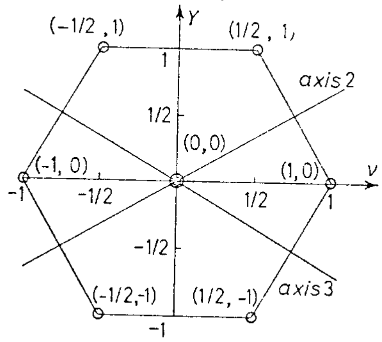


Fig. 1. – The weight diagram for the IR (1. 1) of $A_2$. We use a scale wherein the $Y$ unit bears the ratio $\sqrt{3}/2$ to the $\nu$ unit. A single circle at a point $(\nu, Y)$ signifies that the weight $(\nu, Y)$ is simple; a double circle that it has multiplicity two, etc.

We consider first the case of the eight-component IR (1, 1) of $A_2$, whose weight diagram is displayed in Fig. 1. For the effect of $W$ on the basis vectors of (1, 1) which belong to simple weights, we obtain the results

$$(3.8)\qquad W_2 |1 \;\; -1 \;\;\; 0\rangle = \;\; |\tfrac{1}{2} \;\; -\tfrac{1}{2} \;\; -1\rangle\,,$$

$$(3.9)\qquad W_2 |\tfrac{1}{2} \;\;\; \tfrac{1}{2} \;\; -1\rangle = -|\tfrac{1}{2} \;\; -\tfrac{1}{2} \;\;\; 1\rangle\,,$$

$$(3.10)\qquad W_2 |1 \;\;\; 1 \;\;\; 0\rangle = -|\tfrac{1}{2} \;\;\; \tfrac{1}{2} \;\;\; 1\rangle\,,$$

by inspection of Fig. 1 (except for the signs). The sign of (3.8) is arbitrarily

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[17] We could also have obtained the commutation relations of the $W_1$, $W_2$, $W_3$ with the elements of $A_2$, without reference to the fundamental IR (1, 0) of $A_2$ or any other, by an argument involving the behaviour of the roots [3] of $A_2$ under the Weyl reflections.

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selected and then those in (3.9) and (3.10) follow uniquely, *e.g.* according to

$$\begin{aligned} W_2|\tfrac{1}{2}\ \tfrac{1}{2}\ -1\rangle &\sim W_2 I_+|\tfrac{1}{2}\ -\tfrac{1}{2}\ -1\rangle && \text{by (2.13)},\\ &\sim E_2 W_2|\tfrac{1}{2}\ -\tfrac{1}{2}\ -1\rangle && \text{by (3.7)},\\ &\sim E_2|1\ -1\ 0\rangle && \text{by (3.8)},\\ &= -|\tfrac{1}{2}\ -\tfrac{1}{2}\ 1\rangle && \text{by (2.15)},\end{aligned}$$

the symbol $\sim$ indicating the ommission of some positive numerical factor. The next stage involves the weight (0, 0) to which belong the basis vectors $|1\ 0\ 0\rangle$ and $|0\ 0\ 0\rangle$. Here as in all such situations, we begin with the basis vector with the highest $I$-value and develop

$$(3.11)\qquad \left\{\begin{aligned} W_2|1\ 0\ 0\rangle &= \sqrt{\tfrac{1}{2}}\, W_2 I_+|1\ -1\ 0\rangle,\\ &= \sqrt{3}\, E_2 W_2|1\ -1\ 0\rangle,\\ &= \sqrt{3}\, E_2|\tfrac{1}{2}\ -\tfrac{1}{2}\ -1\rangle,\\ &= \tfrac{1}{2}|1\ 0\ 0\rangle - \frac{\sqrt{3}}{2}|0\ 0\ 0\rangle,\end{aligned}\right.$$

using eq.s (2.13), (3.7), (3.8), (2.14) and (2.15). Since $W_2^2=1$, we get from (3.11) the remaining result

$$(3.12)\qquad W_2|0\ 0\ 0\rangle = -\frac{\sqrt{3}}{2}|1\ 0\ 0\rangle - \tfrac{1}{2}|0\ 0\ 0\rangle.$$

As a check on consistency we may derive (3.11) via an alternative route

$$W_2|1\ 0\ 0\rangle = \sqrt{\tfrac{1}{2}}\, W_2 I_-|1\ 1\ 0\rangle = -\sqrt{3}\, E_{-2}|\tfrac{1}{2}\ \tfrac{1}{2}\ 1\rangle.$$

The treatment of $W_3$ proceeds similarly using the last line of (3.7) and the matrix elements of $E_{\pm 3}$. The results are

$$(3.13)\qquad \left\{\begin{aligned} W_3|1\ \ 1\ 0\rangle &= |\tfrac{1}{2}\ \ \tfrac{1}{2}\ -1\rangle,\\ W_3|\tfrac{1}{2}\ \ \tfrac{1}{2}\ 1\rangle &= |\tfrac{1}{2}\ -\tfrac{1}{2}\ -1\rangle,\\ W_3|1\ -1\ 0\rangle &= |\tfrac{1}{2}\ -\tfrac{1}{2}\ \ 1\rangle,\\ W_3\begin{bmatrix}|1\ 0\ 0\rangle\\ |0\ 0\ 0\rangle\end{bmatrix} &= \begin{bmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & -\frac{1}{2}\end{bmatrix}\begin{bmatrix}|1\ 0\ 0\rangle\\ |0\ 0\ 0\rangle\end{bmatrix}.\end{aligned}\right.$$

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The method previously applied ([8,9]) to the IR (1, 1) proceeds in two stages. In the first stage the basis vectors of (1, 1) are explictily constructed in terms of products of basis vectors $\varphi_\alpha$ of the IR (1, 0) and basis vectors ([18]) $\bar{\varphi}_\alpha$ of the IR (0, 1). In the second, explicit performance of the substitutions indicated by (3.6) is carried out for the basis vectors so constructed. In view of many different sign conventions used in the literature in the construction of a basis for the IR (1, 1) in terms of $\varphi_\alpha$, $\bar{\varphi}_\alpha$ we append a construction consistent with eqs. (3.8)–(3.13):

$$(3.14)\qquad \left\{\begin{aligned}
|1\;\;1\;\;0\rangle &= -\,\varphi_1\bar{\varphi}_2\,,\\
|1\;\;0\;\;0\rangle &= \sqrt{\tfrac{1}{2}}\,(\varphi_1\bar{\varphi}_1-\varphi_2\bar{\varphi}_2)\,,\\
|1\;-1\;\;0\rangle &= \varphi_2\bar{\varphi}_1\,,\\
|\tfrac{1}{2}\;\;\tfrac{1}{2}\;\;1\rangle &= \varphi_1\bar{\varphi}_3\,,\\
|\tfrac{1}{2}\;-\tfrac{1}{2}\;\;1\rangle &= \varphi_2\bar{\varphi}_3\,,\\
|\tfrac{1}{2}\;\;\tfrac{1}{2}\;-1\rangle &= -\,\varphi_3\bar{\varphi}_2\,,\\
|\tfrac{1}{2}\;-\tfrac{1}{2}\;-1\rangle &= \varphi_3\bar{\varphi}_1\,,\\
|0\;\;0\;\;0\rangle &= \sqrt{\tfrac{1}{6}}\,(-\,\varphi_1\bar{\varphi}_1-\varphi_2\bar{\varphi}_2+2\varphi_3\bar{\varphi}_3)\,.
\end{aligned}\right.$$

The 27-component IR (2, 2) of $A_2$ has the weight diagram shown in Fig. 2. For those of the basis vectors of (2.2) which belong to simple weights, we obtain

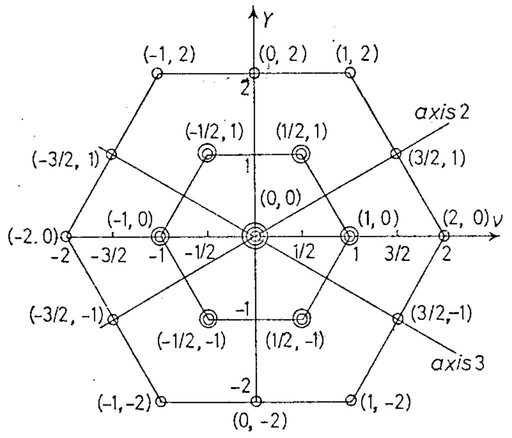


Fig. 2. – The weight diagram for the IR (2, 2) of $A_2$.

([18]) We use a bar to denote complex conjugation, since the more usual star has come to denote a resonance.

the effect of $W_2$, by inspection of Fig. 1, in the form

(3.15) $$W_2|1\;\;1\;\;2\rangle = |2\;\;2\;\;0\rangle,$$

(3.16) $$W_2|1\;\;0\;\;2\rangle = |\tfrac{3}{2}\;\;\tfrac{3}{2}\;-1\rangle,$$

(3.17) $$W_2|1\;-1\;\;2\rangle = |1\;\;1\;-2\rangle,$$

(3.18) $$W_2|\tfrac{3}{2}\;\;\tfrac{3}{2}\;\;1\rangle = |\tfrac{3}{2}\;\;\tfrac{3}{2}\;\;1\rangle,$$

(3.19) $$W_2|\tfrac{3}{2}\;-\tfrac{3}{2}\;\;1\rangle = -|1\;\;0\;-2\rangle,$$

(3.20) $$W_2|2\;-2\;\;0\rangle = |1\;-1\;-2\rangle,$$

(3.21) $$W_2|\tfrac{3}{2}\;-\tfrac{3}{2}\;-1\rangle = |\tfrac{3}{2}\;-\tfrac{3}{2}\;-1\rangle,$$

wherein the first sign has been chosen positive arbitrarily and all that follow are fixed, as in (3.9). We next consider the double weights $(-1, 0)$ and $(-\tfrac{1}{2}, -1)$, whose associated basis vectors are seen from Fig. 2 to mix under $W_2$. As before, we start out from the basic vectors with higher $I$ and, as before, we obtain

$$W_2|2\;-1\;\;0\rangle = \sqrt{\tfrac{1}{6}}\,|\tfrac{3}{2}\;-\tfrac{1}{2}\;-1\rangle - \sqrt{\tfrac{5}{6}}\,|\tfrac{1}{2}\;-\tfrac{1}{2}\;-1\rangle,$$

$$W_2|\tfrac{3}{2}\;-\tfrac{1}{2}\;-1\rangle = \sqrt{\tfrac{1}{6}}\,|2\;-1\;\;0\rangle - \sqrt{\tfrac{5}{6}}\,|1\;-1\;\;0\rangle.$$

Since $W_2^2=1$, the effect of $W_2$ on $|1\;-1\;0\rangle$ and $|\tfrac{1}{2}\;-\tfrac{1}{2}\;-1\rangle$ follows. We summarize the four results in the involutory matrix equation

(3.22) $$W_2\begin{vmatrix}|2\;-1\;0\rangle\\|1\;-1\;0\rangle\end{vmatrix} = \begin{vmatrix}\sqrt{\tfrac{1}{6}} & -\sqrt{\tfrac{5}{6}}\\ -\sqrt{\tfrac{5}{6}} & -\sqrt{\tfrac{1}{6}}\end{vmatrix}\begin{vmatrix}|\tfrac{3}{2}\;-\tfrac{1}{2}\;-1\rangle\\|\tfrac{1}{2}\;-\tfrac{1}{2}\;-1\rangle\end{vmatrix}.$$

Likewise for other pairs of double weights, we have the involutory matrix equations

(3.23) $$W_2\begin{bmatrix}|\tfrac{3}{2}\;-\tfrac{1}{2}\;1\rangle\\ \\|\tfrac{1}{2}\;-\tfrac{1}{2}\;1\rangle\end{bmatrix} = \begin{bmatrix}-\dfrac{2}{3} & \dfrac{\sqrt{5}}{3}\\ \dfrac{\sqrt{5}}{3} & \dfrac{2}{3}\end{bmatrix}\begin{bmatrix}|\tfrac{3}{2}\;\tfrac{1}{2}\;-1\rangle\\ \\|\tfrac{1}{2}\;\tfrac{1}{2}\;-1\rangle\end{bmatrix},$$

(3.24) $$W_2\begin{bmatrix}|2\;1\;0\rangle\\|1\;1\;0\rangle\end{bmatrix} = \begin{bmatrix}-\sqrt{\tfrac{1}{6}} & \sqrt{\tfrac{5}{6}}\\ \sqrt{\tfrac{5}{6}} & \sqrt{\tfrac{1}{6}}\end{bmatrix}\begin{bmatrix}|\tfrac{3}{2}\;\tfrac{1}{2}\;1\rangle\\|\tfrac{1}{2}\;\tfrac{1}{2}\;1\rangle\end{bmatrix}.$$

Now there remains only to find out how the three basis vectors that belong to the weight $(0, 0)$ mix under $W_2$. We develop

$$\begin{aligned}W_2|2\;0\;0\rangle &= \sqrt{\tfrac{1}{6}}\,W_2I_+|2\;-1\;0\rangle,\\ &= E_2\{\sqrt{\tfrac{1}{6}}\,|\tfrac{3}{2}\;-\tfrac{1}{2}\;-1\rangle - \sqrt{\tfrac{5}{6}}\,|\tfrac{1}{2}\;-\tfrac{1}{2}\;-1\rangle\},\\ &= \frac{1}{6}|2\;0\;0\rangle - \frac{\sqrt{15}}{6}|1\;0\;0\rangle + \frac{\sqrt{5}}{3}|0\;0\;0\rangle,\end{aligned}$$

and

$$\begin{aligned} W_2|1\ 0\ 0\rangle &= \sqrt{\tfrac{1}{2}}\,W_2 I_+|1\ -1\ 0\rangle\,,\\ &= \sqrt{3}\,E_2\{-\sqrt{\tfrac{5}{6}}\,|\tfrac{3}{2}\ -\tfrac{1}{2}\ -1\rangle - \sqrt{\tfrac{1}{6}}\,|\tfrac{1}{2}\ -\tfrac{1}{2}\ -1\rangle\}\,,\\ &= -\frac{\sqrt{15}}{6}|2\ 0\ 0\rangle + \frac{1}{2}|1\ 0\ 0\rangle + \frac{\sqrt{3}}{3}|0\ 0\ 0\rangle\,, \end{aligned}$$

with the aid of (3.22) and other equations. Using $W_2^2 = 1$, we can evaluate $W\,|0\ 0\ 0\rangle$ and hence complete the involutory matrix equation

$$W_2\begin{bmatrix}|2\ 0\ 0\rangle\\ |1\ 0\ 0\rangle\\ |0\ 0\ 0\rangle\end{bmatrix} = \begin{bmatrix}\frac{1}{6} & -\frac{\sqrt{15}}{6} & \frac{\sqrt{5}}{3}\\ -\frac{\sqrt{15}}{6} & \frac{1}{2} & \frac{\sqrt{3}}{3}\\ \frac{\sqrt{5}}{3} & \frac{\sqrt{3}}{3} & \frac{1}{3}\end{bmatrix}\begin{bmatrix}|2\ 0\ 0\rangle\\ |1\ 0\ 0\rangle\\ |0\ 0\ 0\rangle\end{bmatrix}. \tag{3.25}$$

The treatment of $W_3$ follows the same pattern and yields the following results

$$(3.26)\quad \begin{cases} W_3|1\ -1\ 2\rangle = |2\ -2\ \ 0\rangle\,, \qquad W_3|1\ \ 0\ 2\rangle = |\tfrac{3}{2}\ -\tfrac{3}{2}\ -1\rangle\,,\\ W_3|1\ \ 1\ 2\rangle = |1\ -1\ -2\rangle\,, \qquad W_3|\tfrac{3}{2}\ \ \tfrac{3}{2}\ 1\rangle = |1\ \ 0\ -2\rangle\,,\\ W_3|2\ \ 2\ 0\rangle = |1\ \ 1\ -2\rangle\,, \qquad W_3|\tfrac{3}{2}\ -\tfrac{3}{2}\ 1\rangle = |\tfrac{3}{2}\ -\tfrac{3}{2}\ \ 1\rangle\,,\\ W_3|\tfrac{3}{2}\ \tfrac{3}{2}\ -1\rangle = |\tfrac{3}{2}\ \ \tfrac{3}{2}\ -1\rangle\,,\\ W_3\begin{bmatrix}|\tfrac{3}{2}\ \tfrac{1}{2}\ 1\rangle\\ |\tfrac{1}{2}\ \tfrac{1}{2}\ 1\rangle\end{bmatrix} = \begin{bmatrix}\frac{2}{3} & \frac{\sqrt{5}}{3}\\ \frac{\sqrt{5}}{3} & -\frac{2}{3}\end{bmatrix}\begin{bmatrix}|\tfrac{3}{2}\ -\tfrac{1}{2}\ -1\rangle\\ |\tfrac{1}{2}\ -\tfrac{1}{2}\ -1\rangle\end{bmatrix},\\ W_3\begin{bmatrix}|2\ -1\ 0\rangle\\ |1\ -1\ 0\rangle\end{bmatrix} = \begin{bmatrix}\sqrt{\tfrac{1}{6}} & \sqrt{\tfrac{5}{6}}\\ \sqrt{\tfrac{5}{6}} & -\sqrt{\tfrac{1}{6}}\end{bmatrix}\begin{bmatrix}|\tfrac{3}{2}\ -\tfrac{1}{2}\ 1\rangle\\ |\tfrac{1}{2}\ -\tfrac{1}{2}\ 1\rangle\end{bmatrix},\\ W_3\begin{bmatrix}|2\ 1\ 0\rangle\\ |1\ 1\ 0\rangle\end{bmatrix} = \begin{bmatrix}\sqrt{\tfrac{1}{6}} & \sqrt{\tfrac{5}{6}}\\ \sqrt{\tfrac{5}{6}} & -\sqrt{\tfrac{1}{6}}\end{bmatrix}\begin{bmatrix}|\tfrac{3}{2}\ \tfrac{1}{2}\ -1\rangle\\ |\tfrac{1}{2}\ \tfrac{1}{2}\ -1\rangle\end{bmatrix},\\ W_3\begin{bmatrix}|2\ 0\ 0\rangle\\ |1\ 0\ 0\rangle\\ |0\ 0\ 0\rangle\end{bmatrix} = \begin{bmatrix}\frac{1}{6} & \frac{\sqrt{15}}{6} & \frac{\sqrt{5}}{3}\\ \frac{\sqrt{15}}{6} & \frac{1}{2} & -\frac{\sqrt{3}}{3}\\ \frac{\sqrt{5}}{3} & -\frac{\sqrt{3}}{3} & \frac{1}{3}\end{bmatrix}\begin{bmatrix}|2\ 0\ 0\rangle\\ |1\ 0\ 0\rangle\\ |0\ 0\ 0\rangle\end{bmatrix}. \end{cases}$$

It is surely clear that we can handle any IR of $A_2$ in this way, treating weights in increasing order of multiplicity and basis vectors belonging to any weight in decreasing order of $I$-value. Also it can be seen that treatment of the IR (2, 2) of $A_2$ by a method involving a suitable generalization of (3.14) involves a great amount of arithmetical effort.

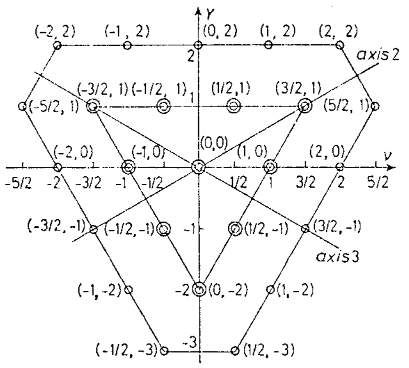


Fig. 3. – The weight diagram for the IR (4, 1) of $A_2$.

The thirty-five-component IR (4, 1) of $A_2$ has the weight diagram shown in Fig. 3. The effect of $W_2$ and $W_3$ on its basis vectors is given by the following set of equations.

$$
(3.27)\quad\left\{
\begin{aligned}
&W_2|2\;\;2\;\;2\rangle = |\tfrac{5}{2}\;\;\tfrac{5}{2}\;\;1\rangle, \qquad W_2|2\;\;1\;\;2\rangle = |2\;\;2\;\;0\rangle,\\
&W_2|2\;\;0\;\;2\rangle = |\tfrac{3}{2}\;\;\tfrac{3}{2}\;-1\rangle, \qquad W_2|2\;-1\;\;2\rangle = |1\;\;1\;-2\rangle,\\
&W_2|2\;-2\;\;2\rangle = |\tfrac{1}{2}\;\;\tfrac{1}{2}\;-3\rangle, \qquad W_2|\tfrac{5}{2}\;-\tfrac{5}{2}\;\;1\rangle = -|\tfrac{1}{2}\;-\tfrac{1}{2}\;-3\rangle,\\
&W_2|2\;-2\;\;0\rangle = -|1\;-1\;-2\rangle, \qquad W_2|\tfrac{3}{2}\;-\tfrac{3}{2}\;-1\rangle = -|\tfrac{3}{2}\;-\tfrac{3}{2}\;-1\rangle,\\
&W_2\begin{bmatrix}|\tfrac{5}{2}\;\tfrac{3}{2}\;1\rangle\\ |\tfrac{3}{2}\;\tfrac{3}{2}\;1\rangle\end{bmatrix} = \begin{bmatrix}-\dfrac{1}{5} & \dfrac{\sqrt{24}}{5}\\ \dfrac{\sqrt{24}}{5} & \dfrac{1}{5}\end{bmatrix}\begin{bmatrix}|\tfrac{5}{2}\;\tfrac{3}{2}\;1\rangle\\ |\tfrac{3}{2}\;\tfrac{3}{2}\;1\rangle\end{bmatrix},\\
&W_2\begin{bmatrix}|\tfrac{5}{2}\;\tfrac{1}{2}\;1\rangle\\ |\tfrac{3}{2}\;\tfrac{1}{2}\;1\rangle\end{bmatrix} = \begin{bmatrix}-\sqrt{\tfrac{1}{10}} & \sqrt{\tfrac{9}{10}}\\ \sqrt{\tfrac{9}{10}} & \sqrt{\tfrac{1}{10}}\end{bmatrix}\begin{bmatrix}|2\;1\;0\rangle\\ |1\;1\;0\rangle\end{bmatrix},\\
&W_2\begin{bmatrix}|\tfrac{5}{2}\;-\tfrac{1}{2}\;1\rangle\\ |\tfrac{3}{2}\;-\tfrac{1}{2}\;1\rangle\end{bmatrix} = \begin{bmatrix}-\sqrt{\tfrac{1}{5}} & \sqrt{\tfrac{4}{5}}\\ \sqrt{\tfrac{4}{5}} & \sqrt{\tfrac{1}{5}}\end{bmatrix}\begin{bmatrix}|\tfrac{3}{2}\;\tfrac{1}{2}\;-1\rangle\\ |\tfrac{1}{2}\;\tfrac{1}{2}\;-1\rangle\end{bmatrix},\\
&W_2\begin{bmatrix}|\tfrac{5}{2}\;-\tfrac{3}{2}\;1\rangle\\ |\tfrac{3}{2}\;-\tfrac{3}{2}\;1\rangle\end{bmatrix} = \begin{bmatrix}-\sqrt{\tfrac{2}{5}} & \sqrt{\tfrac{3}{5}}\\ \sqrt{\tfrac{3}{5}} & \sqrt{\tfrac{2}{5}}\end{bmatrix}\begin{bmatrix}|1\;0\;-2\rangle\\ |0\;0\;-2\rangle\end{bmatrix},\\
&W_2\begin{bmatrix}|2\;0\;0\rangle\\ |1\;0\;0\rangle\end{bmatrix} = \begin{bmatrix}-\dfrac{1}{2} & \dfrac{\sqrt{3}}{2}\\ \dfrac{\sqrt{3}}{2} & \dfrac{1}{2}\end{bmatrix}\begin{bmatrix}|2\;0\;0\rangle\\ |1\;0\;0\rangle\end{bmatrix},\\
&W_2\begin{bmatrix}|2\;-1\;0\rangle\\ |1\;-1\;0\rangle\end{bmatrix} = \begin{bmatrix}-\sqrt{\tfrac{1}{2}} & \sqrt{\tfrac{1}{2}}\\ \sqrt{\tfrac{1}{2}} & \sqrt{\tfrac{1}{2}}\end{bmatrix}\begin{bmatrix}|\tfrac{3}{2}\;-\tfrac{1}{2}\;-1\rangle\\ |\tfrac{1}{2}\;-\tfrac{1}{2}\;-1\rangle\end{bmatrix}.
\end{aligned}\right.
$$

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$$
(3.28)\quad
\begin{cases}
W_3|2\;-2\;2\rangle = |\tfrac{5}{2}\;-\tfrac{5}{2}\;\;1\rangle\,, & W_3|2\;-1\;\;2\rangle = |2\;-2\;\;0\rangle\,,\\
W_3|2\;\;0\;2\rangle = |\tfrac{3}{2}\;-\tfrac{3}{2}\;-1\rangle\,, & W_3|2\;\;1\;\;2\rangle = |1\;-1\;-2\rangle\,,\\
W_3|2\;\;2\;2\rangle = |\tfrac{1}{2}\;-\tfrac{1}{2}\;-3\rangle\,, & W_3|\tfrac{5}{2}\;\;\tfrac{5}{2}\;\;1\rangle = |\tfrac{1}{2}\;\;\tfrac{1}{2}\;-3\rangle\,,\\
W_3|2\;\;2\;0\rangle = |1\;\;1\;-2\rangle\,, & W_3|\tfrac{3}{2}\;\;\tfrac{3}{2}\;-1\rangle = |\tfrac{3}{2}\;\;\tfrac{3}{2}\;-1\rangle\,,\\
W_3\begin{bmatrix}|\tfrac{5}{2}\;\tfrac{3}{2}\;1\rangle\\ |\tfrac{3}{2}\;\tfrac{3}{2}\;1\rangle\end{bmatrix} = \begin{bmatrix}\sqrt{\tfrac{2}{5}} & \sqrt{\tfrac{3}{5}}\\ \sqrt{\tfrac{3}{5}} & -\sqrt{\tfrac{2}{5}}\end{bmatrix}\begin{bmatrix}|1\;0\;-2\rangle\\ |0\;0\;-2\rangle\end{bmatrix}\,, &\\
W_3\begin{bmatrix}|\tfrac{5}{2}\;\tfrac{1}{2}\;1\rangle\\ |\tfrac{3}{2}\;\tfrac{1}{2}\;1\rangle\end{bmatrix} = \begin{bmatrix}\sqrt{\tfrac{1}{5}} & \sqrt{\tfrac{4}{5}}\\ \sqrt{\tfrac{4}{5}} & -\sqrt{\tfrac{1}{5}}\end{bmatrix}\begin{bmatrix}\tfrac{3}{2}\;-\tfrac{1}{2}\;-1\rangle\\ \tfrac{1}{2}\;-\tfrac{1}{2}\;-1\rangle\end{bmatrix}\,, &\\
W_3\begin{bmatrix}|\tfrac{5}{2}\;-\tfrac{1}{2}\;1\rangle\\ |\tfrac{3}{2}\;-\tfrac{1}{2}\;1\rangle\end{bmatrix} = \begin{bmatrix}\sqrt{\tfrac{1}{10}} & \sqrt{\tfrac{9}{10}}\\ \sqrt{\tfrac{9}{10}} & -\sqrt{\tfrac{1}{10}}\end{bmatrix}\begin{bmatrix}|2\;-10\rangle\\ |2\;-10\rangle\end{bmatrix}\,, &\\
W_3\begin{bmatrix}|\tfrac{5}{2}\;-\tfrac{3}{2}\;1\rangle\\ |\tfrac{3}{2}\;-\tfrac{3}{2}\;1\rangle\end{bmatrix} = \begin{bmatrix}\dfrac{1}{5} & \dfrac{\sqrt{24}}{5}\\ \dfrac{\sqrt{24}}{5} & -\sqrt{\dfrac{1}{5}}\end{bmatrix}\begin{bmatrix}\tfrac{5}{2}\;-\tfrac{3}{2}\;1\\ \tfrac{3}{2}\;-\tfrac{3}{2}\;1\end{bmatrix}, &\\
W_3\begin{bmatrix}|2\;1\;\;0\rangle\\ |1\;1\;\;0\rangle\end{bmatrix} = \begin{bmatrix}\sqrt{\tfrac{1}{2}} & \sqrt{\tfrac{1}{2}}\\ \sqrt{\tfrac{1}{2}} & -\sqrt{\tfrac{1}{2}}\end{bmatrix}\begin{bmatrix}|\tfrac{3}{2}\;\tfrac{1}{2}\;-1\rangle\\ |\tfrac{3}{2}\;\tfrac{1}{2}\;-1\rangle\end{bmatrix}\,, &\\
W_3\begin{bmatrix}|2\;0\;0\rangle\\ |1\;0\;0\rangle\end{bmatrix} = \begin{bmatrix}\dfrac{1}{2} & \dfrac{\sqrt{3}}{2}\\ \dfrac{\sqrt{3}}{2} & -\dfrac{1}{2}\end{bmatrix}\begin{bmatrix}|2\;0\;0\rangle\\ |1\;0\;0\rangle\end{bmatrix}. &
\end{cases}
$$

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## RIASSUNTO (*)

Lavori recenti hanno chiaramente dimostrato che dallo studio dell'invarianza rispetto alle riflessioni di Weyl (operazioni di simmetria di carica generalizzata) di $SU_3$ seguono utili predizioni nella teoria della simmetria unitaria di Ne'eman, Gell-Mann delle interazioni forti. Qui si descrive un metodo algebrico generale abbastanza rapido per ottenere l'effetto delle riflessioni di Weyl sui vettori basilari di una rappresentazione irriducibile (IR) arbitraria di $SU_3$. Una caratteristica importante del metodo è che esso si applica a quei vettori basilari dell'IR, che appartengono ai pesi non semplici dell'IR e che quindi non possono essere trattati con l'ispezione del diagramma dei pesi dell'IR. Si dànno i risultati per alcune IR di $SU_3$ importanti nella teoria di Ne'eman e Gell-Mann — gli IR a 8, 27 e 35 componenti (1.1), (2.2) e (4.1) di $SU_3$.


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(*) *Traduzione a cura della Redazione.*