Electromagnetic and Decay Properties of $G_2$ Multiplets

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In this paper we write down consequences of exact and approximate invariance under the group $G_2$ for the baryons, mesons, and resonances. We use the method of Weyl reflections and the Shnashkevich algorithm, previously applied to the group $SU_3$, to derive electromagnetic mass and magnetic moment relations, and also relationships between the one- to two-particle decay vertices (coupling constants). The assignment of multiplets is the usual one, using the singlet, the 7-fold, and the 14-fold representations of $G_2$.

1. INTRODUCTION

The purpose of this paper is to write down consequences of exact and approximate $G_2$ invariance, and to emphasize how little numerical work need be done in the process. The methods used are analogs of those already used for $SU_3$.

We wish to give special acknowledgment to Dr. R. E. Behrends for encouraging us to do this work, for discussions regarding it, and for information about his work on $G_2$ done in collaboration with Dr. L. Landovitz.

The group $G_2$ has a 7-dimensional irreducible representation $(IR)^3$ which can accommodate the baryons $\Sigma, \Xi, \Lambda$ so that $\Lambda$ has to be a $G_2$ singlet; a similar IR can be used for the mesons $\pi, K, \bar{K}$ or $\rho, K^*, K^\circ$. We shall deal with only one other IR of $G_2$, the 14-dimensional one into which we can put the $\frac{2}{3}$ baryon resonances $N^*$ (1240 MeV), $Y^*$ (1385 MeV); the $\bar{Y}^*$ (1465 MeV), and the $\Xi^*$ (1530 MeV) if it has $I=\frac{3}{2}, J^P=\frac{3}{2}^+$; and the $\Xi^- (I=0, Y^- = -2)$ and $I^* (I=0, 2)$, if they are discovered and have $J^P=\frac{3}{2}^+$. The group $G_2$ has a Weyl group of order 12, whose elements can be seen from the root diagram of $G_2$ and the weight diagrams of its IR's to imply specific consequences of exact $G_2$ invariance. We work only with the single involution $(W_2)$ which is a reflection of the weight diagrams about the axis of a set of Behrends et al., and takes the hypercharge $Y$ into the negative of the electric charge $(-Q)$. Its effect on the basis of the 7- and 14-dimensional IR's of $G_2$ is evident from inspection of their weight diagrams, except for the double weight in the 14-dimensional IR. But we shall not use the effect of $W_2$ on these $(Y, Y^*, Y^*)$.

2. ELECTROMAGNETIC MASS DIFFERENCES

We follow the general method developed in our previous work on electromagnetic properties of $SU_3$ multiplets. If $W_3$ takes isotopic multiplets into $U$ multiplets of equal charge, all electromagnetic properties of the first kind are equal for the components of any $U$ multiplet. For the 14-fold representation we thus have

$$
\Phi(I^*) = \Phi(N^*) = \Phi(Y^*) = \Phi(\Xi^*),
$$

$$
\Phi(N^*) = \Phi(Y^*) = \Phi(\Xi^*) = \Phi(Y^*),
$$

$$
\Phi(N^*) = \Phi(Y^*) = \Phi(\Xi^*) = \Phi(Y^*).
$$

Considering the special case of electromagnetic mass differences, these lead to the three mass difference equations

$$
m(N^*) - m(N^*) = m(Y^*) - m(Y^*) = m(\Xi^*) - m(\Xi^*),
$$

$$
m(N^*) - m(Y^*) = m(\Xi^*) - m(Y^*),
$$

$$
m(N^*) - m(Y^*) = m(Y^*) - m(Y^*).
$$

These are to be compared with the $SU_3$ predictions

$$
m(N^*) - m(N^*) = m(Y^*) - m(Y^*),
$$

$$
m(N^*) - m(Y^*) = m(Y^*) - m(Y^*),
$$

$$
m(N^*) - m(Y^*) = m(Y^*) - m(Y^*).
$$

Similarly for the 7-fold representation of $G_2$ we have the following electromagnetic relations of the first kind:

$$
\Phi(\rho) = \Phi(\Sigma^*), \quad \Phi(\Sigma^*) = \Phi(\Sigma^*), \quad \Phi(\Sigma^*) = \Phi(\Sigma^*).
$$

These lead to the mass difference equations

$$
m(\Sigma^*) - m(\Sigma^*) = m(\rho) - m(\Sigma^*) - m(\Sigma^*),
$$

$$
m(\Sigma^*) - m(\Sigma^*) = m(\rho) - m(\Sigma^*).
$$

Experimentally both the relations are in qualitative agreement only; there is an approximate equal spacing of the masses $m(\Sigma^*)$, $m(\Sigma^*)$, $m(\Sigma^*)$ according to the latest compilation, and consequently both the above equations predict the correct signs but not the magnitudes. Comparing with the $SU_3$ predictions: the first

equation is equally valid for SU₃, but in place of the second relation we have one involving the transition mass $m_T(\Sigma, \Lambda)$.

As for $SU₃$ electromagnetic relations of the first kind are equally applicable to arbitrary order electromagnetic properties like form factors, double Compton amplitudes, etc.

3. MAGNETIC MOMENTS

Specializing now to linear electromagnetic properties we have additional relations of the second kind. Since the first-order mass formula for $G_2$ is an equal spacing rule for the 7-fold and 14-fold representations, it turns out that such electromagnetic properties of $U$ multiplets are proportional to their electric charge for these two representations. We get in this 14-fold representation:

$$
\mu(1^+) = \frac{1}{2} \mu(N^{++}) = \mu(N^+) = \mu(Y^+) = \mu(\Xi^+) = 0,
$$

We compare with the corresponding predictions of $SU₃$ for the 10-fold representation:

$$
\frac{1}{2} \mu(N^{++}) = \mu(Y^+) = \mu(\Xi^+) = 0,
$$

For the baryons we get, according to $G_2$, the results of $SU₃$ predictions:

$$
\mu(p) = \mu(\Sigma^+) = -\mu(\Xi^+) = 0,
$$

$$
\mu(n) = \mu(\Sigma^0) = \mu(\Lambda) = 0,
$$

to be compared with the $SU₃$ predictions:

$$
\mu(p) = \mu(\Sigma^+); \quad \mu(\Sigma^-) = \mu(\Xi^+);
$$

$$
\mu(n) = -2 \mu(\Sigma^0) = \mu(\Lambda) = 2 \mu(\Lambda) = 0.
$$

4. THE 7 → 7+7 DECAY WEIGHTS

Decay vertices of the type $7 → 7+7$ (baryon resonance into baryon plus meson) allowed by considerations other than energy conservation are listed in Table I. Assuming exact $G_2$ invariance, Weyl reflections give:

$$
a = b = c = d = e = f = g,
$$
and the Shmushkevich equations are automatically satisfied.

In the first order of symmetry breaking, using the first-order mass formula, we get the modified Shmushkevich equations:

$$
3(a+b+c+g) = 4(c+d+e),
$$

$$
3(a+e+f) = 4(b+d+g),
$$

$$
3(b+c+e+g) = 4(a+d+f),
$$

which imply

$$
b+g = c + e = a + f; \quad 6d = a + b + c + e + f + g.
$$

Unfortunately, only the $(N,N\pi)$, $(\Xi,\Xi\pi)$, $(\Sigma,\Sigma\pi)$ transitions are energetically accessible as decays. The only tests for the lowest lying 7-fold resonance are thus the inequalities

$$
c > a + f; \quad 6d > a + c + f.
$$

To translate these results in terms of decay weights (i.e., squared effective matrix elements) into relations between actual widths, one may use the transition rate formula of Behrends and Landovitz.

5. THE 14 → 7+7 DECAY WEIGHTS

Decay vertices of the type $14 → 7+7$ allowed by considerations other than energy conservation are given in Table II. The Weyl reflection equations give

$$
a = 3b = 3c = 3d = 3e = f = g = \frac{1}{2} h = 6j,
$$

The Shmushkevich equations are

$$
2a = 3(b+c) = 2(9h+j) = 2(5h+m+3l) = 3(3d+e) = 2f,
$$

$$
a + b + 3g + k = 4c + 2h + l + 4d = 3j + m + 6e + f,
$$

$$
a + 6c + 3f + m = 4b + 2h + l + 4d = 3g + k + 6d + f.
$$

Thus, in the approximation of exact $G_2$ invariance we get the complete solution:

$$
2a = 6b = 6c = 6d = 6e = 2f = 12g = 3h = 12j = 4k = 4m,
$$

$l = 0$.

Only $(N,N\pi)$, $(\Xi,\Xi\pi)$ are energetically accessible as decays.

In the first order of the symmetry breaking term, we

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3 The symbols in parentheses are squared effective matrix elements for the corresponding transitions, due account of charge independence having been taken.

4 C. Dullin and A. J. Macfarlane, and E. C. G. Sudarshan, Phys. Rev. Letters 10, 423 (1963). The mass formula for a meson multiplet in a charge conjugation invariant theory cannot contain a term linear in the hypercharge. However, the modified Shmushkevich formula used here is not restrictive in this form and does contain a term linear in the hypercharge.
Table II. $14 \rightarrow 7 + 7$ decays.

<table>
<thead>
<tr>
<th>Decay</th>
<th>Hamiltonian</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T^+ \rightarrow pK^+(a)$</td>
<td>$nK^+(a)$</td>
</tr>
<tr>
<td>$N^+ \rightarrow p\pi^+(38)$</td>
<td>$\Sigma^+K^+(3c)$</td>
</tr>
<tr>
<td>$N^+ \rightarrow p\pi^+(20)$</td>
<td>$\Sigma^+K^+(3c)$</td>
</tr>
<tr>
<td>$N^+ \rightarrow n\pi^+(22)$</td>
<td>$\Sigma^+K^+(3c)$</td>
</tr>
<tr>
<td>$N^- \rightarrow n\pi^-(38)$</td>
<td>$\Sigma^-K^+(3c)$</td>
</tr>
<tr>
<td>$Y^+ \rightarrow pK^+(20)$</td>
<td>$\Sigma^+\pi^+(a)$</td>
</tr>
<tr>
<td>$Y^+ \rightarrow pK^-(20)$</td>
<td>$\Sigma^+\pi^-(b)$</td>
</tr>
<tr>
<td>$Y^- \rightarrow nK^+(20)$</td>
<td>$\Sigma^-\pi^+(a)$</td>
</tr>
<tr>
<td>$Y^- \rightarrow nK^-(20)$</td>
<td>$\Sigma^-\pi^-(b)$</td>
</tr>
<tr>
<td>$Z^+ \rightarrow p\pi^+(36)$</td>
<td>$\Sigma^+K^+(3d)$</td>
</tr>
<tr>
<td>$Z^- \rightarrow p\pi^-(36)$</td>
<td>$\Sigma^-K^+(3d)$</td>
</tr>
<tr>
<td>$Y^+ \rightarrow pK^+(36)$</td>
<td>$\Sigma^+K^+(3d)$</td>
</tr>
<tr>
<td>$Y^- \rightarrow pK^-(36)$</td>
<td>$\Sigma^-K^+(3d)$</td>
</tr>
<tr>
<td>$Z^+ \rightarrow p\pi^+(36)$</td>
<td>$\Sigma^+\pi^-(b)$</td>
</tr>
<tr>
<td>$Z^- \rightarrow p\pi^-(36)$</td>
<td>$\Sigma^-\pi^-(b)$</td>
</tr>
</tbody>
</table>

get six modified Shmushkevich equations:

$$2(a + f) = 3(b + c + d + e) = 4(g + h + j)$$

$$= 4(k + m) + 6l,$$

From these equations we can deduce

$$3(b + e) = 2(g + h + j).$$

So that we have the inequality

$$b + e > 2g.$$ 

which may be experimentally tested.

**A Field Theory of Weak Interactions.**

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The study of the Bethe-Salpeter equation for lepton-lepton interaction mediated by charged vector mesons is continued. Starting from the exact integral equation, an improved approximation procedure is developed. This reproduces the low-energy results for "allowed" processes given in a previous paper. Beyond that, it is now found that to leading order the BS equation gives the value $3g^2/4\pi$ for the zero-energy ratio between "forbidden" and "allowed" amplitudes, where $g$ is the bare lepton-lepton coupling constant. Some information on the momentum dependence of the forbidden amplitude is also obtained. The mathematical methods developed in an earlier paper are then applied to the corresponding Bethe-Salpeter equation of the Fermi field theory. It is shown that the calculated amplitudes for both allowed and forbidden processes are equal to zero. This illustrates the fact that if higher-order effects are taken seriously, there is no reason to consider the Fermi field theory as the limiting case of a vector meson theory with a boson mass which tends to infinity.

### 1. Introduction

In the first paper in this series, we have shown that in the vector meson theory of weak interactions (W theory), graphs involving more than one virtual vector meson may give sizable contributions to the matrix element for processes like $\mu$ decay. This is true in spite of the smallness of the meson-lepton coupling constant $g$, and occurs because of the divergences which make the perturbation expansion meaningless. In I we to this paper as I. We denote Eq. (4.19) of that paper by Eq. (14.19) in this work. For the terminology "leptonic" and "semileptonic" see I, Ref. 2.