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Theory of photoelectric detection of light fluctuations†

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Abstract. The basic formulae governing the fluctuations of counts registered by photoelectric detectors in an optical field are derived. The treatment, which has its origin in Purcell's explanation of the Hanbury Brown–Twiss effect, is shown to apply to any quasi-monochromatic light, whether stationary or not, and whether of thermal origin or not. The representation of the classical wave amplitude of the light by Gabor's complex analytic signal appears naturally in this treatment.

It is shown that the correlation of counts registered by N separate photodetectors at N points in space is determined by a $2N$th order correlation function of the complex classical field. The variance of the individual counts is shown to be expressible as the sum of terms representing the effects of classical particles and classical waves, in analogy to a well-known result of Einstein relating to black-body radiation. Since the theory applies to correlation effects obtained with any type of light it applies, in particular, to the output of an optical maser, although, for a maser operating on one mode, correlation effects are likely to be very small.

1. Introduction

In an important note Purcell (1956) has given a very clear explanation of an effect first observed by Brown and Twiss (1956), namely the appearance of correlation in the fluctuations of two photoelectric currents evoked by coherent beams of light. The method employed by Purcell is semi-classical, but it brings out the essence of the phenomenon much more clearly than most other approaches. Purcell's method has been developed further by Mandel (1958, 1959, 1963 a, b) and applied to the analysis of related problems by Alkemade (1959) and by Wolf (1960) (see also Kahn 1958).

In all the publications just referred to, the light incident on the photodetector was implicitly or explicitly assumed to be of thermal origin. However, in view of the development of optical masers the question has been raised in recent months as to the existence of the Hanbury Brown–Twiss effect with light from other sources. In this connection Glauber (1963 a, b) has expressed doubts about the applicability of these stochastic semi-classical methods to the analysis of fluctuation and correlation experiments obtained with light from non-thermal sources. Although partial answers to Glauber's critical remarks have already been given (Mandel and Wolf 1963, Sudarshan 1963 a, b; see also Jaynes and Cummings 1963), it is, of course, desirable to examine this semi-classical approach more closely. This is done in the present paper where new results on fluctuations and correlations are also obtained. The main conclusions are:

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(i) In the conditions under which light fluctuations are usually measured by photoelectric detectors, the semi-classical treatment applies as readily to light of non-thermal origin as to thermal light, and to non-stationary as well as to stationary fields.

(ii) The representation of the optical field by complex analytic signals (as customary in the classical theory of optical coherence) appears naturally in this treatment.

(iii) There exists a very simple formula for the variance of the number of photoelectrons registered by a photodetector in a given time interval, when it is illuminated by any quasi-monochromatic light beam. This formula expresses the variance as the sum of two terms, one of which can be interpreted as representing the effect of fluctuations in a system of classical particles, and the other as arising from the interference of classical waves. This result, which may also be interpreted as describing the fluctuations of the light itself, is strictly analogous to a well-known result first established by Einstein (1909 a, b) for energy fluctuations in an enclosure containing black-body radiation, under conditions of thermal equilibrium.

(iv) The correlation between the number of photoelectrons registered by N separate photodetectors is expressible in terms of functions of $2N$th order correlations of the classical complex wave amplitudes.

The simplicity of the semi-classical theory and its wide range of validity makes it well suited for the analysis of many problems relating to photoelectric detection of light fluctuations. Our work demonstrates that a full quantum field theoretical treatment is not at all necessary for the analysis of such problems.

2. The probability distribution for photoelectrons ejected from a photo cathode illuminated by a light beam

First, consider an electromagnetic wave interacting with a quantum mechanical system, playing the role of a 'detector', in a bound state $|\psi_b\rangle$. Suppose that $|\psi_b\rangle$, together with a continuum of unbound states $|\psi_\kappa\rangle$, form a complete set of orthonormal eigenstates of the unperturbed time-independent Hamiltonian H_0 of the system, i.e.

$$H_0|\psi_\kappa\rangle = E_\kappa|\psi_\kappa\rangle. \tag{2.1}$$

Here κ stands collectively for all the indices labelling the eigenstates of H_0. If a time-dependent perturbation $H_1(t)$ is applied to the system at time t_0, then the state at time t may be expressed in terms of the set $|\psi_\kappa\rangle$:

$$|\psi(t)\rangle = \int C_\kappa(t)\exp\left(-\frac{iE_\kappa t}{\hbar}\right)|\psi_\kappa\rangle\, d\kappa + C_b(t)\exp\left(-\frac{iE_b t}{\hbar}\right)|\psi_b\rangle. \tag{2.2}$$

The integration with respect to κ is to be interpreted as an integration over ω_k, where

$$\hbar\omega_k = E_\kappa - E_b \geqslant 0, \qquad E_\kappa \equiv E_{k\mu}$$

and a summation over μ, where μ denotes the set of all quantum numbers other than k. The coefficients $C_\kappa(t)$ are given by the familiar formula of first-order perturbation theory

$$C_\kappa(t) = \frac{1}{i\hbar}\int_{t_0} \langle\psi_\kappa|H_1(t')|\psi_b\rangle \exp(i\omega_k t')\, dt \tag{2.3}$$

Let $\rho(\omega_k)\, d\omega_k$ be the number of states $|\psi_\kappa\rangle$ in the energy interval $\hbar\, d\omega_k$. Then the probability that a transition has occurred to any of the unbound states by the time t is

$$|C_\kappa(t)|^2\rho(\omega_k)\, d_{\imath}$$

If $t - t_0 = \Delta t$ we define a transition probability per unit time by

$$\Pi(t) = \frac{1}{\Delta t}\int_0^{\infty} |C_\kappa(t)|^2 \rho(\omega_k)\, d\kappa. \tag{2.4}$$

Both $C_\kappa(t)$ and $\Pi(t)$ depend also on the position of the atomic nucleus (to be specified by position vector $\mathbf{R}$ later on).

Let us now consider the interaction between a *single* atom and an incident electromagnetic wave, represented by the vector potential $\mathbf{A}(\mathbf{r}, t)$. Let the momentum of a typical electron of the atom be represented by $\mathbf{p}$. Then the interaction Hamiltonian, in the usual notation is

$$H_1(t) = \frac{e}{mc}\mathbf{A}(\mathbf{r}, t).\mathbf{p}.$$

We now express $\mathbf{A}(\mathbf{r}, t)$ in the form of a Fourier integral

$$\mathbf{A}(\mathbf{r}, t) = \int_{-\infty}^{\infty} \mathscr{A}(\mathbf{r}, \omega)\, e^{-i\omega t}\, d\omega.$$

Since $\mathbf{A}$ is real,

$$\mathscr{A}(\mathbf{r}, -\omega) = \mathscr{A}^*(\mathbf{r}, \omega)$$

Then, from (2.3), (2.5) and (2.6), we obtain

$$\begin{aligned} C_\kappa(t) &= \frac{e}{i\hbar mc}\int_{t_0}^{t_0+\Delta t} dt' \int_{-\infty}^{\infty} d\omega \exp\{i(\omega_k - \omega)t'\} M_\mu(\mathbf{R}, \omega, \omega_k) \\ &= \frac{e\Delta t}{i\hbar mc}\int_{-\infty}^{\infty} d\omega \exp\{i(\omega_k - \omega)(t_0 + \tfrac{1}{2}\Delta t)\}\frac{\sin\{\frac{1}{2}(\omega_k - \omega)\Delta t\}}{\frac{1}{2}(\omega_k - \omega)\Delta t} M_\mu(\mathbf{R}, \omega, \omega_k) \end{aligned} \tag{2.8}$$

where

$$M_\mu(\mathbf{R}, \omega, \omega_k) = \langle \psi_\kappa | \mathscr{A}(\mathbf{r}, \omega).\mathbf{p} | \psi_b \rangle \tag{2.9}$$

is the matrix element between the ground state and the continuum state and is a function of $\mathbf{R}$, the position of the atomic nucleus. Now provided $\Delta t \gg 1/\omega_k$ for all k for which $|\mathbf{M}_\mu(\mathbf{R}, \omega, \omega_k)|^2$ is appreciable, the integrand considered as a function of ω_k will be very sharply peaked about ω. In practice the smallest value of ω_k likely to matter when we are dealing with photoelectric transitions is of the order of 10^{14} c/s. The condition $\omega_k \Delta t \gg 1$ is therefore likely to hold for all measurable intervals Δt. The contribution to $C_\kappa(t)$ arises from values of ω in the neighbourhood of ω_k, and since $\omega_k > 0$ we may replace the lower limit in (2.8) by zero. From (2.4) and (2.8),

$$\begin{aligned} \Pi(t) = \frac{e^2\Delta t}{\hbar^2 m^2 c^2}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} d\omega\, d\omega'\, d\omega_k \exp\{i(\omega' - \omega)(t_0 + \tfrac{1}{2}\Delta t)\}\frac{\sin\{\frac{1}{2}(\omega_k - \omega)\Delta t\}}{\frac{1}{2}(\omega_k - \omega)\Delta t} \\ \times \frac{\sin\{\frac{1}{2}(\omega_k - \omega')\Delta t\}}{\frac{1}{2}(\omega_k - \omega')\Delta t}\sum_\mu \rho(\omega_k) M_\mu^*(\mathbf{R}, \omega', \omega_k) M_\mu(\mathbf{R}, \omega, \omega_k). \end{aligned} \tag{2.10}$$

It is evident that the integrand effectively vanishes unless ω, ω' and ω_k are nearly equal, to within an amount of the order of $1/\Delta t$. It is reasonable to suppose that $\rho(\omega_k)$ does not vary significantly over such a small range of ω_k and may therefore be replaced by $\{\rho(\omega)\rho(\omega')\}^{1/2}$ in (2.10).

Next let us assume that the incident radiation is quasi-monochromatic, i.e. its effective bandwidth is small compared with the mid-frequency. Then we can choose $1/\Delta t$ small compared with any frequency ω that contributes to (2.10), but large compared with the difference $\omega - \omega'$ between any pair of frequencies ω and ω', and (2.10) then reduces to

$$\Pi(t) = \frac{2\pi e^2}{\hbar^2 m^2 c^2} \int_0^\infty \int_0^\infty d\omega d\omega' \exp\{i(\omega' - \omega)t\}\{\rho(\omega)\}^{1/2}\{\rho(\omega')\}^{1/2} \frac{\sin\{\frac{1}{2}(\omega-\omega')\Delta t\}}{\frac{1}{2}(\omega-\omega')\Delta t}$$

$$\times \sum_\mu M_\mu^*(\mathbf{R}, \omega', \omega') M_\mu(\mathbf{R}, \omega, \omega) \tag{2.11}$$

we have the identity

$$\frac{\sin\{\frac{1}{2}(\omega-\omega')\Delta t\}}{\frac{1}{2}(\omega-\omega')\Delta t} = \frac{1}{\Delta t} \int_{-\Delta t/2}^{\Delta t/2} \exp\{i(\omega'-\omega)\tau\}\, d\tau.$$

Hence the probability of a photoelectric transition per unit time may be expressed in the form

$$\Pi(t) = \frac{2\pi e^2}{\hbar^2 m^2 c^2} \sum_\mu \frac{1}{\Delta t} \int_{-\Delta t/2}^{\Delta t/2} W_\mu^*(\mathbf{R}, t+\tau) W_\mu(\mathbf{R}, t+\tau)\, d\tau \tag{2.12a}$$

where

$$W_\mu(\mathbf{R}, t) = \int_0^\infty d\omega\, \mathrm{e}^{-i\omega t}\{\rho(\omega)\}^{1/2} M_\mu(\mathbf{R}, \omega, \omega) \tag{2.12b}$$

Since the Fourier spectrum of $W_\mu(\mathbf{R}, t)$ contains no negative frequencies, W_μ is, according to a well-known theorem (Titchmarsh 1948), analytic and regular in the lower half of the complex t plane.

So far we have been considering the interaction between an incident electromagnetic field and a single atom only. Now suppose that we are dealing with a plane wave incident normally on an extended detector, which is in the form of a thin photoelectric layer containing a large number of electrons in initial states $|\psi_b\rangle$. With the assumption that these atoms may be treated as independent (i.e. that their electron wave functions do not appreciably overlap) and that the states are not appreciably depopulated, we can express the probability $P(t)\Delta t$ of photoelectric detection from any part of the photosurface by

$$P(t) = \frac{2\pi e^2 N}{\hbar^2 m^2 c^2 \Delta t} \sum_\mu \int_{-\Delta t/2}^{\Delta t/2} W_\mu^*(\mathbf{R}, t+\tau) W_\mu(\mathbf{R}, t+\tau)\, d\tau \tag{2.13}$$

where N is the number of effective electrons. Since we are dealing with an incident plane wave, the right-hand side of (2.13) is independent of $\mathbf{R}$.

In the dipole approximation we may express the matrix element M_μ, given by (2.9), in the form

$$M_\mu(\mathbf{R}, \omega, \omega_k) \sim \mathscr{A}(\mathbf{R}, \omega) . \langle\psi_\kappa|\mathbf{p}|\psi_b\rangle.$$

If $\rho(\omega)$ and the matrix element $\langle\psi_\kappa|\mathbf{p}|\psi_b\rangle$ are effectively independent of ω over the narrow frequency band of the incident light, and if the incident wave is plane as we assumed, then (2.12b) reduces to

$$W_\mu(\mathbf{R}, t) = \{\rho(\omega_0)\}^{1/2}\mathbf{V}(\mathbf{R}, t) . \langle\psi_{\mu,k_0}|\mathbf{p}|\psi_b\rangle \tag{2.14}$$

where $\omega_0 = k_0 c$ is the mid-frequency, and $\mathbf{V}(\mathbf{R}, t)$ is the vector function obtained from $\mathbf{A}(\mathbf{R}, t)$ by suppressing the negative frequency components in the Fourier integral† (2.6):

$$\mathbf{V}(\mathbf{R}, t) = \int_0^\infty \mathscr{A}(\mathbf{R}, \omega)\, \mathrm{e}^{-i\omega t}\, d\omega \tag{2.15}$$

† Such a representation of the field, obtained by suppressing the negative Fourier components is customarily employed in the classical theory of optical coherence, under the name of *analytic signal*, a concept due to D. Gabor (cf. Born and Wolf 1959). The same representation has played an important role already in the early quantum mechanical investigations of radiation and coherence based on the correspondence principle, and is also implicit in some older pioneering researches of von Laue relating to coherence and thermodynamics of light.

From Maxwell's equation $\mathbf{V}(\mathbf{R}, t)$ is transverse, i.e. normal to the direction of propagation of the wave. Let us write

$$\mathbf{V}(\mathbf{R}, t) = V(\mathbf{R}, t)\boldsymbol{\epsilon} \tag{2.16}$$

where $\boldsymbol{\epsilon}$ is a unit (generally complex) vector and $V(\mathbf{R}, t)$ is a complex scalar function. Now one may readily show that for a time interval Δt, which is short compared with the reciprocal of the effective bandwidth of the light (as is here assumed),

$$\frac{1}{\Delta t}\int_{-\Delta t/2}^{\Delta t/2} V^*(\mathbf{R}, t+\tau)V(\mathbf{R}, t+\tau)\,d\tau \simeq V^*(\mathbf{R}, t)V(\mathbf{R}, t)$$

With the aid of this result and (2.14) and (2.16), (2.13) becomes

$$P(t) = \frac{2\pi e^2 N}{\hbar^2 m^2 c^2}\rho(\omega_0)V^*(\mathbf{R}, t)V(\mathbf{R}, t)\sum_{\mu} |\boldsymbol{\epsilon}\,.\,\langle\psi_{\mu,k_0}|\mathbf{p}|\psi_{\mathrm{b}}\rangle|^2. \tag{2.17}$$

Because Σ_μ involves summation over all possible polarizations of the electron the result will be independent of $\boldsymbol{\epsilon}$, so that we may write (2.17) in the form

$$P(t) = \alpha\mathbf{V}^*(\mathbf{R}, t)\,.\,\mathbf{V}(\mathbf{R}, t) \tag{2.18a}$$

where α represents the quantum efficiency of the photoelectric detector. This result does not depend in an essential way on all the simplifying assumptions made. If there is a whole range of initial electron states $|\psi_{\mathrm{b}}\rangle$, the factor N in (2.17) has to be replaced by a sum over these states. On the other hand, if the electron wave functions overlap, as in a metal, the electron system has to be treated appropriately, and the calculation must be modified. Nevertheless, as long as we are dealing with plane waves falling normally on a thin photoelectric layer, a factorization of the kind embodied in equation (2.14) will still be permissible. The general form of (2.18*a*) will therefore remain valid, although the total cross section for the process, and therefore the constant α, will be affected.

We may identify $\mathbf{V}^*(\mathbf{R}, t)\,.\,\mathbf{V}(\mathbf{R}, t)$ with the instantaneous intensity† $I(t)$ of the classical field, and express (2.18*a*) in the form

$$P(t)\Delta t = \alpha I(t)\Delta t. \tag{2.18b}$$

The probability of photoemission of an electron is therefore proportional to the classical measure of the instantaneous light intensity, defined in terms of the complex analytic signal. In the idealized case of strictly monochromatic radiation this result is, of course, well known, but its generalization to a field which exhibits arbitrary fluctuations is essential for the purposes of the present discussion (see also Brown and Twiss 1957 a).

It should be noted that in equation (2.18) probability enters in two different ways: in the fundamental uncertainties associated with the photoelectric interaction and in the fluctuations of the radiation field itself. This, of course, is a general feature of quantum statistical mechanics (cf. Landau and Lifschitz 1958).

Equation (2.18) shows that the probability of a single photoelectric transition in a small time interval t, $t+\Delta t$ is proportional to Δt. However, we are mainly interested in the probability distribution $p(n, t, T)$ of emission of n photoelectrons in a finite time interval t, $t+T$. If the different photoelectric emissions could be considered as independent statistical events in the sense of classical probability theory, it would follow from (2.18)

† $\mathbf{V}^*\,.\,\mathbf{V}$ is not strictly proportional to the instantaneous energy density; it may be easily shown that $\mathbf{V}^*\,.\,\mathbf{V}$ represents a short-time average of $\mathbf{A}^2$ taken over a time interval of a few mean periods of the light vibrations.

that (see Mandel 1959, 1963 a)

$$p(n, t, T) = \frac{1}{n!}\{\alpha U(t, T)\}^n \exp\{\ \ \alpha U(t, T)\} \tag{2.19}$$

where

$$U(t, T) = \int_t^{t+T} I(t')\, dt'. \tag{2.20}$$

Actually, it is possible to see that it is legitimate to proceed in this way. Consider, for example, a system of two atoms, both of which interact with the incident radiation but do not interact with each other. The product of the unperturbed energy eigenfunctions of the two individual atoms are the energy eigenfunctions of the two-atom system. It is now possible to calculate the probability amplitude for a transition to a final state in which either one or two photoelectrons have been emitted. In particular, the probability amplitude for emission of two photoelectrons in a small time interval $(t, t+\Delta t)$ may be shown to be given by the product of two expressions of the type (2.8). When we take account of the (infinite) degeneracy of the two-atom energy eigenstates, the probability for such a transition in this time interval may then be shown to be given by a product of two factors† of the type (2.10). In any case it seems plausible to look on successive photoelectric emissions from the whole photoelectric surface considered as one system as events which are substantially independent with respect to the electron system, provided the photoelectric layer is not appreciably depopulated. Similar assumptions are implicit in the derivaton of (2.19) from (2.18) by Mandel (1963 a).

Equation (2.19) refers to the photoelectric counting distribution appropriate to a single realization of the incident electromagnetic field. The average of $p(n, t, T)$ over the ensemble of the incident fields is the probability that would normally be derived from counting experiments. If, as is usually the case, $I(t)$ represents a stationary ergodic process, this average will be independent of t. If we denote the ensemble average by $\bar{p}(n, t, T)$, we have from (2.19)

$$\bar{p}(n, t, T) = \overline{\frac{1}{n!}\{\alpha U(t, T)\}^n \exp\{-\alpha U(t, T)\}} \tag{2.21}$$

which, in general, will *not* be a Poisson distribution.

It should be noted that for radiation fields in some states, for example in an eigenstate of the number operator, the probability distribution of U will exhibit somewhat unusual properties, not normally encountered in classical theory. However, in view of a theorem relating to the equivalence of the semi-classical and quantum representations of light beams, established recently by Sudarshan (1963 a, b), such probability distributions can in principle nevertheless always be found.

3. Statistical properties of the counting distribution

Let us next consider the variance of the counts n, recorded in time intervals of duration T. The averages of n and n^2 are given by

$$\bar{n} = \sum_{n=0}^{\infty} n\bar{p}(n, t, T) \tag{3.1}$$

$$\overline{n^2} = \sum_{n=0}^{\infty} n^2\bar{p}(n, t, T). \tag{3.2}$$

† This situation must be contrasted with the situation in which a direct two-electron transition from a single atom takes place.

Using well-known expressions for the first two moments of the Poisson distribution (see, for example, Levy and Roth 1951), we readily find from (3.1), (3.2) and (2.19) that

$$\bar{n} = \alpha \overline{U(t, T)}$$

$$\overline{n^2} = \alpha \overline{U(t, T)} + \alpha^2 \overline{\{U(t, T)\}^2} \tag{3.4}$$

that the variance

$$\overline{(\Delta n)^2} = \overline{(n-\bar{n})^2} = \overline{n^2} - (\bar{n})^2$$

is given by

$$\overline{(\Delta n)^2} = \bar{n} + \alpha^2 \overline{(\Delta U)^2}$$

where

$$\overline{(\Delta U)^2} = \overline{(U-\bar{U})^2} = \overline{U^2} - (\bar{U})^2$$

is the variance of $U(t, T)$.

The formula (3.5) has evidently a very simple interpretation. It shows that the variance of the fluctuations in the number of ejected photoelectrons may be regarded as having two separate contributions: (i) from the fluctuations in the number of particles obeying the classical Poisson distribution (term $\bar{n}$), and (ii) from the fluctuations in a classical wave field (wave interference term $\alpha^2 \overline{(\Delta U)^2}$). This result, which holds for any radiation field, is strictly analogous to a celebrated result of Einstein (1909 a, b)† relating to energy fluctuations in an enclosure containing black-body radiation, under conditions of thermal equilibrium. Fürth (1928) has later shown that the same result holds for energy fluctuations of thermal radiation of spectral compositions other than that appropriate to black-body radiation, but like Einstein's analysis, Fürth's considerations apply to closed systems only. We have now shown that a fluctuation formula of this type is also valid for counting fluctuations in *time* intervals, for any light beam (i.e. thermal or non-thermal and stationary and non-stationary), at points that may be situated far away from the sources of the light. Although the result refers to the fluctuations of the photoelectric counts, it can be regarded as reflecting the fluctuation properties of the light itself, in so far as they are accessible to measurement.

Since equation (3.5) applies to any light beam, whether of thermal origin or not, it is likely to be useful in connection with photoelectric experiments relating to fluctuations of light generated by optical masers. In this connection we note that, if an optical maser operates on a single mode and is well stabilized, then the variance $\overline{(\Delta U)^2}$ will be negligible (absence of classical wave intensity fluctuations). In this case (3.5) reduces to‡

$$\overline{(\Delta n)^2} = \bar{n} \tag{3.6}$$

i.e. the variance is the same as for a system of classical particles.

We may draw some further conclusions from the results of § 2. First, let us again consider the case when the light intensity of the classical wave field does not fluctuate significantly. In this case (2.21) becomes

$$\bar{p}(n, t, T) = \frac{1}{n!} \bar{n}^n \exp(-\bar{n})$$

† For a lucid account of the significance of Einstein's result, see Born (1949).

‡ It should be born in mind that the formula was derived on the basis of the first-order perturbation theory. When very intense maser beams are employed the effect of multiple photon interactions might have to be included.

where

$$\bar{n} = \alpha \bar{U} = \alpha \bar{I} T. \tag{3.8}$$

Thus we see that the photoelectrons now obey the Poisson distribution. This situation may be expected to arise in the case already referred to, namely when the photoemission is triggered off by a well-stabilized, single-mode laser beam. That in this case the distribution $\bar{p}(n, t, T)$ will be Poisson's was already noted elsewhere (Mandel 1964, Glauber 1964). This result clearly shows that departure from Poisson's statistics is not a universal consequence of the Bose–Einstein statistics of light quanta as is often erroneously believed to be the case†.

If, on the other hand, the light is of thermal origin, the probability distribution $\bar{p}(n, t, T)$ may be expected to be quite different. For in this case the incident field will fluctuate appreciably and its distribution will as a rule be Gaussian (van Cittert 1934, Blanc-Lapierre and Dumontet 1955, Janossy 1957, 1959). This implies (Mandel 1963 a, p. 191) that for polarized thermal light the probability distribution of the intensity is exponential:

$$p(I) = \frac{1}{\bar{I}} \exp\left(\frac{-I}{\bar{I}}\right)$$

It may then be shown from (2.21), (3.9) and (3.8) that, if T is much smaller than the coherence time of the light, the probability distribution of the photoelectrons becomes (Mandel 1958, 1959, 1963 a)

$$\bar{p}(n, t, T) = \frac{\bar{n}^n}{(\bar{n}+1)^{n+1}} \tag{3.10}$$

This will be recognized as the Bose–Einstein distribution (Morse 1962). It was derived by Bothe (1927) by a somewhat similar argument long ago.

4. Multiple correlations

The one-dimensional counting distributions do not exhaust the range of application of the semi-classical theory, for the relation (2.19) can be applied to any number N of photodetectors, each situated at a different point of the radiation field. Let n_j be the number of counts registered at the jth detector in a time interval t_j, $t_j + T$. Then

$$\overline{n_1 n_2 \ldots n_N} = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_N=0}^{\infty} \left\{ n_1 n_2 \ldots n_N \overline{\prod^{N} p_j(n_j, t, T)} \right\}$$

where the $p_j(n_j, t, T)$ $(j = 1, 2 \ldots N)$ are given by expressions such as (2.19). Equation (4.1) may be rewritten in the form

$$\overline{n_1 n_2 \ldots n_N} = \overline{\sum^{\infty} n_1 p_1(n_1, t_1, T) \sum^{\infty} n_2 p_2(n_2, t_2, T) \ldots \sum^{\infty} n_N p_N(n_N, t_N, T)}$$

Now each of the sums on the right-hand side of (4.2) represents, according to a well-known property of the Poisson distribution (2.19), the parameter of that distribution

† In this connection see the interesting discussion by Rosenfeld (1955, especially pp. 7–78).

$$\alpha_j U_j(t_j, T) = \sum_{n_j=0}^{\infty} n_j p_j(n_j, t_j, T). \tag{4.3}$$

Hence, if we substitute from (4.3) into (4.2), we obtain the formula

$$\overline{n_1 n_2 \ldots n_N} = A\overline{U_1(t_1, T)U_2(t_2, T)\ldots U_N(t_N, T)} \tag{4.4}$$

where

$$A = \alpha_1 \alpha_2 \ldots \alpha_N \tag{4.5}$$

represents the product of the quantum efficiencies of the N detectors. Equation (4.4) shows that *the correlation of the counts registered by the N photodetectors is proportional to the correlation in the integrated intensities* (cf. (2.20))

$$U_j(t_j, T) = \int_{t_j}^{t_j+T} I_j(t')\, dt' \qquad (j = 1, 2, \ldots N) \tag{4.6}$$

of the classical field at the location of the N detectors.

If we substitute from (4.6) into (4.4) and recall that $I_j(t) = \mathbf{V}_j^*(t).\mathbf{V}_j(t)$ we readily find that

$$\overline{n_1 n_2 \ldots n_N} = A \int_{t_1}^{t_1+T} \int_{t_2}^{t_2+T} \ldots \int_{t_N}^{t_N+T} \Gamma^{(N,N)}(t_1', t_2', \ldots t_N')\, dt_1' dt_2' \ldots dt_N' \tag{4.7}$$

where

$$\Gamma^{(N,N)}(t_1, t_2, \ldots t_N) = \overline{\mathbf{V}_1^*(t_1).\mathbf{V}_1(t_1)\ldots\mathbf{V}_N^*(t_N).\mathbf{V}_N(t_N)}. \tag{4.8}$$

Thus the correlation of the counts is completely expressible in terms of the $2N$th order cross-correlation function of the classical field (cf. Mandel 1964, Wolf 1963, 1964). For a stationary field this correlation function will, of course, be independent of the origin of time and if, in addition, ergodicity is assumed, it can also be expressed in the form

$$\tilde{\Gamma}^{(N,N)}(\tau_2, \tau_3, \ldots \tau_N)$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \mathbf{V}_1^*(t).\mathbf{V}_1(t)\mathbf{V}_2^*(t+\tau_2).\mathbf{V}_2(t+\tau_2).\ .\mathbf{V}_N^*(t+\tau_N).\mathbf{V}_N(t+\tau_N)\, dt \tag{4.9}$$

where

$$\tau_j = t_j - t_1 \qquad (j = 2, 3, \ldots N).$$

In a quantized field-theoretical treatment the correlation $\overline{n_1 n_2 \ldots n_N}$ would be expressed in terms of the expectation value of the ordered product of the corresponding creation and annihilation operators (Glauber 1963 b, 1964). This expectation value has already been shown to be equivalent to a cross correlation of the complex classical fields (Sudarshan 1963 a, b, Mandel and Wolf 1965) and the formula (4.7) emphasizes this fact once again.

We can also convert (4.4) into a correlation formula for the fluctuations $\Delta n_j = n_j - \bar{n}_j$. By making a multinomial expansion of the product $\Delta n_1 \Delta n_2 \ldots \Delta n_N$ and applying (4.4) repeatedly, we obtain the formula

$$\overline{\Delta n_1 \Delta n_2 \ldots \Delta n_N} = A\overline{\Delta U_1 \Delta U_2 \ldots \Delta U_N} \tag{4.10}$$

where

$$\Delta U_j = U_j - \bar{U}_j.$$

The value of the correlation depends, of course, on the type of light illuminating the detectors. For light from the usual thermal sources the random process $\mathbf{V}(t)$ will, to a good approximation, be stationary, ergodic and Gaussian, and the correlations appearing on the right-hand side of equations (4.4), (4.7) and (4.10) can then be expressed in terms of second-order cross-correlation functions (Reed 1962). In particular, for

$N = 2$, (4.10) then represents the correlation effect discovered by Brown and Twiss (1956, 1957 a, b). However, from (4.10) it follows that the effect will be small when the fluctuations ΔU in the integrated classical intensity are small, as may be the case for light generated by an optical maser oscillating in a single mode.

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References

ALKEMADE, C. T. J., 1959, *Physica*, **25**, 1145.
BLANC-LAPIERRE, A., and DUMONTET, P., 1955, *Rev. Opt.* (*Théor. Instrum.*), **34**, 1.
BORN, M., 1949, *Natural Philosophy of Cause and Chance* (Oxford: Clarendon Press), p. 80.
BORN, M., and WOLF, E., 1959, *Principles of Optics* (London, New York: Pergamon Press), chap. X.
BOTHE, W., 1927, *Z. Phys.*, **41**, 345.
BROWN, R. HANBURY, and TWISS, R. Q., 1956, *Nature, Lond.*, **177**, 27.
—— 1957 a, *Proc. Roy. Soc.* A, **242**, 300.
—— 1957 b, *Proc. Roy. Soc.* A, **243**, 291.
VAN CITTERT, P. H., 1934, *Physica*, **1**, 201.
—— 1939, *Physica*, **6**, 1129.
EINSTEIN, A., 1909 a, *Phys. Z.*, **10**, 185.
—— 1909 b, *Phys. Z.*, **10**, 817.
FÜRTH, R., 1928, *Z. Phys.*, **50**, 310.
GLAUBER, R. J., 1963 a, *Phys. Rev. Letters*, **10**, 84.
—— 1963 b, *Phys. Rev. Letters*, **130**, 2529.
—— 1964, *Proc. 3rd Int. Congress on Quantum Electronics*, Eds N. Bloembergen and P. Grivet (Paris: Dunod; New York: Columbia University Press), p. 111.
JANOSSY, L., 1957, *Nuovo Cim.*, **6**, 14.
—— 1959, *Nuovo Cim.*, **12**, 369.
JAYNES, E. T., and CUMMINGS, F. W., 1963, *Proc. Inst. Elect. Electron. Engrs*, **51**, 89.
KAHN, F. D., 1958, *Optica Acta*, **5**, 93.
LANDAU, L. D., and LIFSHITZ, E. M., 1958, *Statistical Physics* (London: Pergamon Press; Reading, Mass.: Addison Wesley), p. 18.
LEVY, H., and ROTH, L., 1951, *Elements of Probability* (Oxford: Clarendon Press), p. 143.
MANDEL, L., 1958, *Proc. Phys. Soc.*, **72**, 1037.
—— 1959, *Proc. Phys. Soc.*, **74**, 233.
—— 1963 a, *Progress in Optics*, **2**, 181, Ed. E. Wolf (Amsterdam: North-Holland).
—— 1963 b, *Proc. Phys. Soc.*, **81**, 1104.
—— 1964, *Proc. 3rd Int. Congress on Quantum Electronics*, Eds N. Bloembergen and P. Grivet (Paris: Dunod; New York: Columbia University Press), p. 101.
MANDEL, L., and WOLF, E., 1963, *Phys. Rev. Letters*, **10**, 276
MANDEL, L., and WOLF, E., 1965, *Rev. Mod. Phys.*, in the press.
MORSE, P. M., 1962, *Thermal Physics* (New York: Benjamin), p. 218.
PURCELL, E. M., 1956, *Nature, Lond.*, **178**, 1449.
REED, I. S., 1962, *Trans. Inst. Radio Engrs*, **IT-8**, 194.
ROSENFELD, L., 1955, *Niels Bohr and the Development of Physics*, Ed. W. Pauli (London: Pergamon Press).
SUDARSHAN, E. C. G., 1963 a, *Phys. Rev. Letters*, **10**, 277.
—— 1963 b, *Proc. Symp. on Optical Masers* (New York: Brooklyn Polytechnic Press and John Wiley), p. 45.
TITCHMARSH, E. C., 1948, *Introduction to the Theory of Fourier Integrals*, 2nd edn (Oxford: Clarendon Press), p. 128.
WOLF, E., 1960, *Proc. Phys. Soc.*, **76**, 424.
- 1963, *Proc. Symp. on Optical Masers* (New York: Brooklyn Polytechnic Press and John Wiley), p. 29.
- 1964, *Proc. 3rd Int. Congress on Quantum Electronics*, Eds N. Bloembergen and P. Grivet (Paris: Dunod; New York: Columbia University Press), p. 13.