NUCLEON MAGNETIC MOMENTS IN COVARIANT VERSION OF WIGNER'S SUPERMULTIPLET THEORY*

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Among the most notable features of the recent covariant $\tilde{U}(12)$ generalization[1] of the nonrelativistic SU(6) theory[2] are (a) the derivation of an experimentally quite reasonable absolute value for the proton magnetic moment; (b) the prediction of the value $-\frac{3}{2}$ for the ratio of the proton and neutron magnetic moments; and (c) the occurrence of a single baryon form factor. Of course, the ratio $-\frac{3}{2}$ is present already in the nonrelativistic SU(6) theory.[3] Indeed, this ratio can be even obtained[4] in an appropriate reformulation[5] of Wigner's (nonrelativistic) supermultiplet theory,[6] which shall be referred to simply as the SU(4) theory.

It is the purpose of the present paper to develop a covariant $\tilde{U}(8)$ generalization of the SU(4) theory and to show how the features (a), (b), and (c), mentioned above, are contained in it. The $\tilde{U}(8)$ theory is obtained from the SU(4) theory by a procedure exactly parallel to that used in the development of $\tilde{U}(12)$ from the SU(6) theory. We show that results (a) and (b) survive, and the theory contains a single nucleon form factor. Interest in the $\tilde{U}(8)$ theory stems from the following considerations. Firstly, since the $\tilde{U}(8)$ theory is a covariant merging of isospin and space-time symmetries in the same sense as the $\tilde{U}(12)$ theory is a covariant merging of SU(3) and space-time symmetries, our work indicates the extent to which the features (a), (b), and (c) of the latter theory survive the breaking of the internal SU(3) symmetry. Secondly, the SU(4) derivation[4] of the magnetic-moment ratio $-\frac{3}{2}$ contains an assumption which can be fully justified only by the extension to $\tilde{U}(8)$. Finally, in view of the simplicity of the isospin group relative to SU(3), the $\tilde{U}(8)$ theory may allow the essential features of the $\tilde{U}(12)$ theory to be more clearly seen.

It is well known that the underlying symmetry group of the Wigner supermultiplet theory[6] is an SU(4) group containing as a subgroup the direct product of the spin and isospin SU(2) groups. In the version[4,5] of the theory to be generalized here, however, the nucleon is assigned not to the defining representation $\underline{4}$, but rather, along with the 3-3 nucleon resonance, to the totally symmetric representation $\underline{20}$ occurring in the triple direct product $\underline{4}\times\underline{4}\times\underline{4}$. This agrees with the decomposition $\underline{20}\rightarrow(\underline{4},\underline{4})\oplus(\underline{2},\underline{2})$ of the representation $\underline{20}$ of SU(4) with respect to its spin-isospin $SU(2)\otimes SU(2)$ subgroup, SU(2) representations being referred to by their dimensionalities, and is in analogy with the assignment of baryons and baryon resonances to the representation $\underline{56}$ of SU(6).

In order to give a covariant generalization of the SU(4) theory, we assume that the fundamental entity is an eight-component Dirac particle with isospin $\frac{1}{2}$. This then belongs to the defining representation of a $\tilde{U}(8)$ group with structure specified by the algebra of 64 matrices $\gamma^R S^i$, where

$$\gamma^R\equiv(1,\gamma^\mu,\sigma^{\mu\nu}=\tfrac{1}{2}[\gamma^\mu,\gamma^\nu],i\gamma^\mu\gamma^5,\gamma^5=\gamma^0\gamma^1\gamma^2\gamma^3);$$

$$S^i\equiv(\tfrac{1}{2}I,\tfrac{1}{2}\vec{\tau}).$$

The nucleon and 3-3 nucleon resonance are to be assigned to the totally symmetric representation $\underline{120}$ of $\tilde{U}(8)$ occurring in the triple direct product of three fundamental representations. This has the decomposition

$$\underline{120}\rightarrow(\underline{4},\underline{20})\oplus(\underline{2},\underline{20}')$$

with respect to the $U(2)\otimes\tilde{U}(4)$ subgroup, where U(2) refers to isospin and $\tilde{U}(4)$ to the space of the Dirac matrices, and where $\underline{20}$ and $\underline{20}'$ are the $\tilde{U}(4)$ representations of symmetry types [3] (total symmetry) and [2, 1], respectively. We may give a basis for the $\underline{120}$, explicitly exhibiting the above decomposition, according to

$$\begin{aligned}\Psi_{ABC}&=\Psi_{\alpha p,\beta q,\gamma r}\\&=D_{\alpha\beta\gamma,p,qr}+\tfrac{1}{2}\times6^{-1/2}(\epsilon_{pq}N_{\alpha\beta\gamma,r}\\&\qquad+\epsilon_{qr}N_{\beta\gamma\alpha,p}+\epsilon_{rp}N_{\gamma\alpha\beta,q}),\end{aligned}\tag{1}$$

where α,β,γ with values 1, 2, 3, 4 are $\tilde{U}(4)$ indices and p,q,r with values 1, 2 are U(2) indices. D is totally symmetric in the sets α,β,γ and

p, q, r separately and corresponds to isospin $\frac{3}{2}$. N obeys[1]

$$N_{\alpha\beta\gamma}+N_{\beta\alpha\gamma}=0, \quad N_{\alpha\beta\gamma}+N_{\beta\gamma\alpha}+N_{\gamma\alpha\beta}=0, \tag{2}$$

and corresponds to isotopic spin $\frac{1}{2}$. Contact with space-time symmetry is acheived by restriction of $\tilde{U}(4)$ to $\mathcal{L}(4)$, the homogeneous Lorentz group, and by imposing upon D and N the Bargmann-Wigner wave equations.[7] This procedure makes D describe a spin-$\frac{3}{2}$ particle, and N a spin-$\frac{1}{2}$ particle, so that finally the particle content of the representation $\underline{20}$ of SU(4) is reproduced. Explicitly, for N we have[1]

$$mN_{\alpha\beta\gamma,r}=[(m+\not{p})\gamma_5 C]_{\alpha\beta}\psi_{\gamma,r}, \tag{3}$$

where $\psi_{\gamma,r}$, a nucleon Dirac spinor with isospace index r, obeys $(\not{p}-m)\psi_{\gamma,r}=0$.

For mesons, the SU(4) theory[5] assigns ρ, ω, and π to the adjoint representation $\underline{15}$ of SU(4), whose spin-isospin decomposition is

$$\underline{15}\to(\underline{3},\underline{3})\oplus(\underline{3},\underline{1})\oplus(\underline{1},\underline{3}).$$

To obtain a covariant description of the same particles, we take the adjoint representation $\underline{63}$ of $\tilde{U}(8)$, write down its $U(2)\otimes\tilde{U}(4)$ decomposition explicitly, and apply the Bargmann-Wigner equations. This leads to the result $\Phi_A{}^B = P_A{}^B + V_A{}^B$, where P is the pseudoscalar meson part and V the vector-meson part, the latter given explicitly by

$$\mu V_{\alpha r}{}^{\beta s}=(\mu+\not{q})_\alpha{}^\beta (S^i)_r{}^s V^{\nu i}, \tag{4}$$

where $V^{\nu i}$ describes a vector meson of mass μ and isospace index i in the usual way: $(q^2-\mu^2)V^{\nu i}=0$, $q_\nu V^{\nu i}=0$.

We now use Eqs. (1), (3), and (4) to compute the vector-meson nucleon part of the interaction

$$g\bar{\Psi}^{ABC}(p')\Phi_A{}^D(q)\Psi_{DBC}(p), \quad q=p'-p. \tag{5}$$

Writing this as $gJ_{\nu i}V^{\nu i}$, direct calculation gives the results

$$J^{\nu i}=\frac{P^\nu}{2m}\left(1+\frac{q^2}{2m\mu}\right)\bar{\psi}^{\gamma,r}(p')(\tfrac{1}{2}\tau^i)_r{}^s\psi_{\gamma,s}(p)+\frac{5}{12m^2}\left(1+\frac{2m}{\mu}\right)\bar{\psi}^{\gamma,r}(p')(\tfrac{1}{2}\tau^i)_r{}^s(\tau^\nu)_\gamma{}^\beta\psi_{\beta,s}(p), \tag{6}$$

$$J^{\nu 0}=\frac{3}{2m}P^\nu\left(1+\frac{q^2}{2m\mu}\right)\bar{\psi}^{\gamma,r}(p')(\tfrac{1}{2}I)_r{}^s\psi_{\gamma,s}(p)+\frac{1}{4m^2}\left(1+\frac{2m}{\mu}\right)\bar{\psi}^{\gamma,r}(p')(\tfrac{1}{2}I)_r{}^s(\tau^\nu)_\gamma{}^\beta\psi_{\beta,s}(p), \tag{7}$$

where

$$P=p+p',$$

$$\tau^\nu=\epsilon^{\nu\rho\sigma\tau}P_\rho q_\sigma\gamma_\tau\gamma_5=-\tfrac{1}{2}[\not{P}\gamma^\nu\not{q}-\not{q}\gamma^\nu\not{P}]. \tag{8}$$

Note that in Eqs. (6) and (7), in which any two of the four quantities $P^\nu, \gamma^\nu, \sigma^{\mu\nu}q_\mu, \tau^\nu$ could have been used, we have chosen to use P^ν and τ^ν.[8] P^ν is used in order to obtain the electric charge directly in the static limit $q\to 0$. Having chosen P^ν, τ^ν is then chosen because it is precisely that linear combination of the above four quantities which is orthogonal to P^ν, i.e.,

$$\tau^\nu P_\nu=0. \tag{9}$$

This simple property of τ^ν does not appear to have been mentioned in the literature, although a physical consequence of it, namely that the choice P^ν and τ^ν "diagonalizes" the Rosenbluth cross section for electron-proton scattering, has been noted.[8]

Equations (6) and (7) describe the nuclear-vector-meson vertex and imply that for this vertex there is only one over-all form factor.

To find out what is implied by these equations for the electromagnetic interactions, we make the additional assumption that the coupling of the electromagnetic field to the nucleon is dominated by the ρ and ω intermediate states. From the assignments of the ρ and ω made above, it then follows that the nucleon form factor is

$$\{\alpha J^{\nu 0}+\beta J^{\nu 3}\}f(q^2)/(q^2-\mu^2), \tag{10}$$

where α and β are constants. The ratio β/α is now determined by the condition that the static charge of the nuetron be zero. This condition yields

$$\beta/\alpha = 3. \tag{11}$$

From (6), (7), and (10) we then obtain, using an obvious notation, the following electromagnetic form factors:

$$J_{\mu n}{}^{C} = 0,$$
$$J_{\mu p}{}^{C} = \frac{3}{2m}\left(1 + \frac{q^2}{2m\mu}\right)\frac{\alpha f(q^2)}{q^2-\mu^2}\bar{p}p,$$
$$J_{\mu n}{}^{M} = -\frac{2}{4m^2}\left(1 + \frac{2m}{\mu}\right)\frac{\alpha f(q^2)}{q^2-\mu^2}\bar{n}r_{\mu}n,$$
$$J_{\mu p}{}^{M} = \frac{3}{4m^2}\left(1 + \frac{2m}{\mu}\right)\frac{\alpha f(q^2)}{q^2-\mu^2}\bar{p}r_{\mu}p. \tag{12}$$

One may normalize $\alpha f(0)$ so that $J_{\mu p}{}^{C}$ is equal to e for $q=0$. From (12) it follows in particular that

$$\mu_p/\mu_n = -\tfrac{3}{2},$$

and

$$\mu_p = (1+2m/\mu). \tag{13}$$

Results (12) and (13) are essentially those of reference 1. However, the μ in our formulas is the (common) mass of ρ and ω. Note that $J_{\mu n}{}^{C}$ is zero, not only for $q=0$ but for all q.

It might appear that our input information is more than that of reference 1 since we have demanded $Q_n = 0$, whereas this followed in reference 1 from the use of $M_3 + (\frac{1}{3})^{1/2}M_8$. However, we could have elected to use $\frac{1}{2}(\frac{1}{3}+\tau_3)$ in our work, and this is no more ad hoc than $M_3 + (\frac{1}{3})^{1/2}M_8$, since it may be motivated by the Gell-Mann–Nishijima relation for the fundamental ($B=\frac{1}{3}$) fields. However, we prefer to make a direct appeal to the experimental result $Q_n = 0$, since, in fact, this underlies both $\frac{1}{2}(\frac{1}{3}+\tau_3)$ and $M_3 + (\frac{1}{3})^{1/2}M_8$. Note that the choice of $\frac{1}{2}(\frac{1}{3}+\tau_3)$ can be motivated by the Gell-Mann–Nishijima relation even within the framework of the SU(4) theory. However, in this case there may be some doubt[9] as to whether this quantity or $\frac{1}{2}(1+\tau_3)$ should be preferred. The doubt is clearly resolved by $\tilde{U}(8)$, in which the choice $\frac{1}{2}(\frac{1}{3}+\tau_3)$ comes from the fact that the covariant nature of the theory provides a link between the magnetic moment and the static charge, which in the case of the neutron we know to be zero.

In conclusion, we note that although specialization of $\tilde{U}(12)$ to the nucleons in $\tilde{U}(12)$ results formally in the use of the expression $\frac{1}{2}(\frac{1}{3}+\tau_3)$, this is not equivalent to the same quantity for $\tilde{U}(8)$. The reason is that for $\tilde{U}(8)$ this operator corresponds to a coupling of the physical ρ and ω, whereas for $\tilde{U}(12)$, it corresponds to a coupling of ρ and ω_u, where

$$\omega_u = (\tfrac{1}{3})^{1/2}\omega + (\tfrac{2}{3})^{1/2}\varphi.$$

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[9]Although $\frac{1}{2}(1+\tau_3)$ is the quantity which appears in Eq. (8) of reference 4, it is clear that $\frac{1}{2}(\frac{1}{3}+\tau_3)$ is intended, since simple algebra shows that it is the latter quantity which yields the result $-\frac{3}{2}$ for the proton-neutron magnetic moment ratio.