CURRENT ALGEBRAS AND GENERALIZED WARD-TAKAHASHI IDENTITIES*

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The general consequences of current algebras for may particle amplitudes are derived using generalized Ward-Takahashi indentities. Several applications are indicated.

The relations between measurable physical quantities consequent on postulated equal time commutation relations between local (field or source) operators have attracted much attention in recent years. Fubini et al. [1] showed that equal time commutation relations of the kind suggested by Gell-Mann [2] lead to a large class of dispersion sum rules; in recent months several such relations have been studied by various authors and applied to a variety of particle phenomena. In this letter we give a general method of deriving all the exact consequences of a local current algebra structure generated by conserved and partially conserved currents. The essential idea is a generalization of the Ward-Takahashi identities in quantum electrodynamics [3]. We demonstrate the generality of our results by considering the relations following for two-body amplitudes; we obtain as special cases the Adler-Weisberger relation [4] for g_A, the relations between g_A and the meson-baryon scattering lengths [5], and a low energy theorem for soft meson production [6,7] by photons.

Identities from current algebras. We proceed as in quantum electrodynamics, using the differential conservation law of the vector current $v_\alpha^\mu(x)$ to deduce the Ward-Takahashi identity [3]. Define

$$I_V^\mu = \int d^4x d^4y d^4z d^4z' \exp\{i(p_f z - p_i z' + qy - kx)\} \tau_V^\mu , \tag{1}$$

$$\tau_V^\mu = \langle 0 | T(v_\alpha^\mu(x)\, \varphi_\beta(y) \psi_\gamma(z)\, \bar{\psi}_{\gamma'}(z')) | 0 \rangle , \tag{2}$$

where $\varphi_\beta(y)$ is a pseudoscalar meson field and $\psi_\gamma(z)$ a baryon field, and α, β, γ, ... are SU(3) octet indices. Differentiating (2) with respect to x_μ and taking the Fourier transform as in (1), we obtain the generalized Ward-Takahashi identity

$$k_\mu I_V^\mu = f_{\alpha\beta\sigma} J_1(\sigma\gamma\gamma') + f_{\alpha\gamma\sigma} J_2(\beta\sigma\gamma') - f_{\alpha\gamma'\sigma} J_3(\beta\gamma\sigma) , \tag{3}$$

where for example,

$$J_1(\sigma\gamma\gamma') = \int d^4y\, d^4z\, d^4z' \exp\{i(p_f z - p_i z' + (q-k)y\}\langle 0 | T(\varphi_\sigma(y) \psi_\gamma(z)\, \bar{\psi}_{\gamma'}(z')) | 0 \rangle \tag{4}$$

and $if_{\alpha\beta\gamma}$ are the structure constants of SU(3). Replacing the vector current by a partially conserved axial vector current $a_\alpha^\mu(x)$ satisfying $\partial_\mu a_\alpha^\mu(x) = C_\alpha\, \varphi_\alpha(x)$, we obtain the corresponding generalized Ward-Takahashi identity

$$k_\mu I_A^\mu = \frac{-i C_\alpha}{\mu_\alpha^2 - k^2} I_p + d_{\alpha\beta\sigma} \mathscr{H}_1(\sigma\gamma\gamma') + h_{\alpha\gamma\sigma} \gamma_5 G_2(\beta\sigma\gamma') - h_{\alpha\gamma'\sigma} G_3(\beta\gamma\sigma)\gamma_5 , \tag{5}$$

where the coefficients $d_{\alpha\beta\sigma}$, $h_{\alpha\gamma\sigma}$ define the commutation of a_α^μ with φ_β and ψ_γ respectively. The amplitude I_p is obtained by replacing v_α^μ in I_V^μ by φ_α and acting on the τ-function by $(\mu_\alpha^2 + \Box_x^2)$. The

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** We use the U(3) × U(3) algebra generated by the time components $v_\alpha^0(x)$ and $a_\alpha^0(x)$.

relations for $k_\mu I_V^\mu$ and $k_\mu I_A^\mu$ are immediately extended to arbitrary n-point functions; elsewhere we use these for studying production amplitudes. We note that the relations (3) and (5) are exact and follow directly from the current algebra without any use of the analytic properties of the amplitudes.

We may derive another series of relations involving the partially time-ordered Green's functions $\bar{\tau}_V^\mu$, $\bar{\tau}_A^\mu$; e.g.

$$\bar{\tau}_V^\mu = \langle 0 | T(\varphi_\beta(y)\, \psi_\gamma(z)\, \psi_{\gamma'}(z'))\, v_\alpha^\mu(x) | 0 \rangle. \tag{6}$$

Their Fourier transform $\bar{I}_V^\mu$, $\bar{I}_A^\mu$, which may be related to the absorptive parts of the amplitude (on replacing T-products by R-products), satisfy

$$k_\mu \bar{I}_V^\mu = 0\,; \quad k_\mu \bar{I}_A^\mu = \frac{-\mathrm{i}\, C_\alpha}{\mu_\alpha^2 - k^2}\, \bar{I}_p. \tag{7}$$

Eqs. (5) and (7) are similar to the equations used by Fubini [8,9]. The differences are (i) our equations are for Green's functions and hence contain information about general off-mass-shell amplitudes; (ii) our equations contain extra terms.

Relations for S-matrix elements and applications. For the T-matrix element T_V^μ for the process $V_\alpha + B_\gamma \rightarrow P_\beta + B_{\gamma'}$ described by the Green's function (2), our equations give, when all the particles are on the mass shell,

$$k_\mu T_V^\mu = 0\,; \quad k_\mu \bar{T}_V^\mu = 0, \tag{8}$$

with expresses current conservation; and for the analogous process with v_α^μ replaced by a_α^μ,

$$k_\mu T_A^\mu = \frac{-\mathrm{i}\, C_\alpha}{\mu_\alpha^2 - k^2}\, T_p\,; \quad k_\mu \bar{T}_A^\mu = \frac{-\mathrm{i}\, C_\alpha}{\mu_\alpha^2 - k^2}\, \bar{T}_p, \tag{9}$$

which are generalized Goldberger-Treiman relations.

For application to meson-baryon scattering consider

$$T_{AA}^{\mu\nu} = \int \mathrm{d}^4x\, \mathrm{d}^4y\, \mathrm{d}^4z\, \mathrm{d}^4z' \exp\{\mathrm{i}(p_f z - p_i z' + qy - kx)\} \times$$
$$\times\, \bar{u}(p_f)\, (\not{p}_f - m_f) \langle 0 | T[a_\alpha^\mu(x)\, a_\beta^\nu(y)\, \psi_\gamma(z)\, \bar{\psi}_{\gamma'}(z')] | 0 \rangle\, (\not{p}_i - m_i)\, u(p_i). \tag{10}$$

With p_i, p_f on the mass shell and k, q off-shell, the generalized Ward-Takahashi identity

$$k_\mu q_\nu T_{AA}^{\mu\nu} = \frac{C_\alpha C_\beta}{(\mu_\alpha^2 - k^2)(\mu_\beta^2 - q^2)}\, T_{pp} + f_{\alpha\beta\sigma}\, q_\nu F_V^\nu(\sigma\gamma\gamma') \tag{11}$$

where F_V^ν is the baryon vector form factor:

$$\langle p_f | v_\alpha^\nu(0) | p_i \rangle = \mathrm{i}^2 \left[\frac{m_i\, m_f}{p_i^o\, p_f^o}\right]^{\frac{1}{2}} F_V^\nu\,; \quad F_V^\nu = \bar{u}(p_f) f_V^\nu\, u(p_i), \tag{12}$$

and T_{pp} is the amplitude for meson-baryon scattering. Multiplying (11) by suitable factors, isolating the pole terms and letting $q = k \rightarrow 0$, we get **the following** relations [5] between g_A and the πN scattering lengths a_3, a_1:

$$\lim_{\mu_\pi \rightarrow 0} \frac{4\pi (1 + \mu_\pi / m_N)\, a_3}{\mu_\pi} = -\mu_\pi^4 C_\pi^{-2}\,; \quad a_1 = -2a_3, \tag{13}$$

where $C_\pi = \sqrt{2}\, g_A\, m_N\, \mu_\pi^2 / G_{NN\pi} K_{NN\pi}(0)$. (11) also gives

$$\lim_{q-k\to 0} \frac{\partial}{\partial q^{\mathrm{o}}} F_{\mathrm{pp}} = \frac{\mu_\pi^4}{C_\pi^2} \left\{1 - g_{\mathrm{A}}^2 \left(1 - \frac{m_{\mathrm{N}}^2}{(p^{\mathrm{o}})^2}\right)\right\} , \tag{14}$$

which has the esential content of the Adler-Weisberger relation [4]; here F_{pp} is the proper part of the πN forward scattering amplitude $(T_{\mathrm{pp}})_{k=q}$. Similarly by considering the Green's function

$$\tau_{\mathrm{VA}}^{\mu\nu} = \langle 0 | T[v_\alpha^\mu (x)\, a_\beta^\nu (y) \psi_\gamma(z) \overline{\psi}_{\gamma'}(z')] | 0 \rangle, \tag{15}$$

and the corresponding amplitudes, we obtain a low energy theorem for the electro-production of π^+ on protons [7,10]:

$$G\frac{\sqrt{2}\, m_{\mathrm{N}}}{_{\mathrm{NN}\pi} K_{\mathrm{NN}\pi}}(0) \lim_{q\to 0} T_{\mathrm{VP}}^{\mu} = \bar{u}(p_{\mathrm{f}}) \left\{ f_{\mathrm{A}}^{\mu}(\sigma\gamma\gamma') + \mathrm{i} f_{\mathrm{V}}^{\mu(n)} \left[1 + \frac{m}{p_{\mathrm{i}}^{\mathrm{o}}}\gamma_{\mathrm{o}}\right]\gamma_5\, \tau^{(-)} - \tau^{(-)}\gamma_5 \left[1 + \frac{m}{p_{\mathrm{f}}^{\mathrm{o}}}\gamma_{\mathrm{o}}\right] f_{\mathrm{V}}^{\mu(\mathrm{p})} \right\} u(p_{\mathrm{i}}). \tag{16}$$

Assuming the proton and neutron e.m. form factors $f_{\mathrm{V}}^{\mu(n)}$ and $f_{\mathrm{V}}^{\mu(n)}$, this may be used for determing the baryon axial vector form vector f_{A}^{μ}.

We shall give a detailed exposition of our method and its further consequences and generalizations in a subsequent paper.

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