NONCOMPACT GROUPS IN PARTICLE PHYSICS*

E. C. G. Sudarshan

Syracuse University, Syracuse, New York

I. Introduction

In conventional treatments of particle physics one starts with dynamical systems consisting of a collection of particles, or of fields associated with particles, and one attempts to describe the interactions of the particles A somewhat more modern version invokes self-consistency to prefer certain combination of particles over others. In most of these theories, it is difficult to see a natural role for the multiplet grouping of particles; true, in most versions if one or more sets of particles fall into multiplets with common characteristics, it makes it likely that the other particles also do this. However, the remarkable observation is that particles do seem to fall into multiplets which may be identified with irreducible representations of various Lie groups, though the groups are by no means invariance groups of the system.

The problems of particle spectra and particle interactions are however interrelated. One must therefore look for a suitable framework within which groups which are not invariance groups enter in an essential manner into the specification of the dynamical system.[1] In the application of approximate symmetries to particle physics, we see that the essential role played by the supermultiplets is in the organization of several irreducible representations of subgroups which are more exact invariance groups. In special dynamical models approximate symmetries can be related to the dynamics of the system; but in general the model itself is formulated in terms of entities other than groups.

II. Dynamical Structure and Noninvariance Groups

Given a symmetry group which is an invariance group of the Hamiltonian one has the result that the states must furnish representations of the group. But the specification of the irreducible representations which will actually

*Work supported by the U. S. Atomic Energy Commission.

occur goes beyond the specification of the group. However there are some cases when one can specify the spectrum; for the system of two spins of magnitudes S_1 and S_2, interacting via a rotationally invariant coupling, the states fall into $(2S+1)$- component multiplets with S ranging from $S_1 + S_2$ to $|S_1-S_2|$. In such cases the substructure of the dynamical system results in a "spectrum" of representations. Conversely, a spectrum of representations is the signature of a dynamical substructure.

On the other hand, it is possible to associate the spectrum with a higher group of which the symmetry group is a subgroup. In this case there is no question of the higher group being an invariance group[2]. Consider for example the elementary example of an isotropic rigid rotator. The system is invariant under O(3) and the states of the system fall into $(2\ell+1)$-dimensional representations of O(3). There is however the further regularity that for each nonnegative integral value of ℓ there is one and only one multiplet of states furnishing the $(2\ell+1)$-dimensional representation of O(3). Can we incorporate this information into a group-theoretic formulation? The answer is "yes". We could identify the entire spectrum with a single irreducible representation of E(3) or of O(3,1). In the E(3) case the rotation generators are the dynamical variables of angular momentum, while the translation generators are the components of the radial vector on the unit sphere[1]. Conversely, given a representation of E(3) we could construct the dynamical variables of angular momentum and direction vector by suitable choice of elements of the generalized enveloping algebra of the Lie algebra of the group.

The organization of the spectrum in terms of a noninvariance group can be extended to other systems as well[2]. For the simple solvable systems of quantum mechanics like the harmonic oscillator or the hydrogen atom this construction is quite straightforward. In these cases again we have the choice of starting with the dynamical system formulated in terms of canonical variables (the oscillator and the hydrogen atom) or in terms of related entities (the rotator). Then we search for a noninvariance group (which has as a subgroup any invariance group of the Hamiltonian) such that the entire spectrum of states constitutes an irreducible representation. In this case we note that the generalized enveloping algebra of the noninvariance group contains all the dynamical variables.

But instead of this. we could start at the other end.

We identify the system by the complete set of its states <u>defined</u> as furnishing a faithful representation of a group. Since most interesting systems have infinitely many linearly independent states, to be able to organize them into a single irreducible representation, the noninvariance group must be <u>noncompact</u>. The dynamical system is now <u>defined</u> by the noninvariance group and the dynamical variables are identified with the elements of the generalized enveloping algebra. The noncompact noninvariance group thus becomes a dynamical model[3].

For a finite number of degrees of freedom, or more generally a system usually characterized in terms of a finite number of <u>algebraically</u> independent variables, the connected part of the corresponding noninvariance group is a finite parameter group, and vice versa. It is arbitrary to start with a noncompact noninvariance group; but then, it is arbitrary to start with a system defined in terms of, say, a finite number of canonical degrees of freedom!

So far we have made no specific choice of the Hamiltonian. Once the Hamiltonian is chosen[1] as a suitable element of the generalized enveloping algebra, the energy spectrum can be determined. When the choice of the Hamiltonian changes so does the energy spectrum. But the dynamical system itself continues to remain the same!

The identification of noninvariance groups as dynamical system has been studied in connection with the problem of position operators[4]. If one identifies particles with representations of the Poincaré (or extended Galilei) groups, the canonical variables appropriate for described particles can be constructed in terms of the momentum operators and the position operators. It is also well known that when one considers zero mass particles of higher spin, suitable position operators do not exist; hence the representations of the noncompact Poincaré group as a starting point is decidedly more general.

One may also observe that the noninvariance group of a relativistic system described by a relativistic wave equation is larger than the Poincaré group. The local transformation of the covariant tensor or spinor quantities and the space time gradients imply automatically invariance under the complex Lorentz group. An immediate consequence of this is the existence of both positive and negative energy states for the system. If we now consider the complex Poincaré group as the noninvariance group of the system, we have elements of the generalized enveloping algebra connecting positive and negative energy states. The existence of dynamical variables of this kind corresponds physically to the possibility of pair creation by interactions.

III. Noninvariance Groups in Particle Physics

What do all these things do for particle physics? Or what use are noninvariance groups in particle physics? To answer these questions we note that the sequence of multiplet levels is the most striking feature of particle physics. We do know of several multiplets of stable or quasistable particles which constitute representations of isotopic spin, and invariance under isotopic spin transformations seems to be satisfied by strong interactions. But the spectrum of representations is not dictated by isospin invariance; the organization of the isospin multiplets into unitary octets and decuplets does prescribe, to a certain extent, the isospin representations. The problem is now shifted to the question of specifying the spectrum of unitary multiplets.

We note that unitary symmetry, if considered as an invariance principle of strong interactions, is very approximate. It is better to consider it as part of a noninvariance group.

What is involved in specifying the spectrum of representations? The usual identification of multiplets consists of states with the same baryon number, but there are infinitely many such states, most of them unbound and merging into the continuum. Hence, the spectrum of representations of the isospin (or unitary) group must contain infinitely many entries. If these are to be identified with an irreducible representation of a noninvariance group, the noninvariance group must be noncompact. The generators and hence, the entire enveloping algebra of such a noninvariance noncompact group (which spans all particles with the same baryon number) are "meson operators which commute with the baryon number. These generators take us from a state bound or unbound with a fixed baryon number to another such state (bound or unbound) and are therefore associated with actual physical transitions between states of strongly interacting particles. It thus becomes possible to use the noncompact noninvariance group as a dynamical model of particles to describe transition between the quasistationary levels themselves or between them and the continuum.

Before going into further discussion of the relation between generators and physical transitions, it is well to note the lumping together of stationary (or quasistationary) hadron states with the continuum hadron states. This possibility as well as the infinite dimensional nature of the (irreducible) multiplets are features of the noncompact group that are not present in the representation theory of compact groups. The representation theory of arbitrary noncompact groups is only beginning to be explored.[5] In some special cases like SL(2,R), O(3,1), O(4,1), E(3) etc., these problems have been studied, but one often wishes to consider other cases. It is also interesting to study the spectra of various elements of the enveloping algebra of the group as well as the relation between representations of different groups which are related in one manner or other. Following earlier work by Weyl and by Dirac we may consider the relation between the representations of two groups whose Lie algebras have the same complex Lie algebras. The work in this area by Kuriyan, Mukunda and Sudarshan is a systematic exploitation of the method of the Master Analytic Representation of the local Lie group.[6] It would take me too far afield to give any but a brief outline of the method.

IV. Analytic Continuation of Group Representations

Consider for example the deSitter group O(4,1). It is a ten parameter group with six generators J_{ab}, belonging to an O(4) group with respect to which the other four generators B_a behave as a tensor of the type (1/2,1/2). The commutator

$$\left[B_a, B_b\right] = i\theta \quad J_{ab}$$

yield respectively O(4,1), E(4), and O(5) according as θ=+1, 0, or -1. It is known that the representations of the O(5) group are labelled by two parameters, c, d, which may be identified with the highest and the lowest values of an O(3) angular momentum (generated by J_{ab}, $1 \leq a$, $b \leq 3$) occurring in the irreducible representation of O(5) labelled by (c,d). By varying c,d over suitable values we get all the representations of O(5). The representation is completely described by the reduced matrix elements of the hermitian operator B_a in the O(4) basis. By analytic continuation of these matrix elements (multiplied by i) into the domain where the corresponding operator becomes hermitian we get the (local) unitary representations of O(4,1). It is worthwhile to point out that we not only get some classes of representations but all the (known!) unitary representations[8] of O(4,1). By the method of Segal, - Wigner, - and Inonu we can, from these, construct the representations of E(4).

In this example, while the group for which we eventually find the representation is noncompact, the basis of the representation is the compact group O(4). This simplifies the problem considerably. In many cases this may not be the appropriate thing to do; it may be necessary to consider the representation of a noncompact group in a basis which consists of representations of a noncompact group. In this case we can again make use of the method of analytic continuation, but with proper caution. As an example, consider the group O(2,1) with the three generators $H = J_{01}$, $K = J_{02}$, $L = J_{12}$ satisfying

$$[H, K] = i L$$

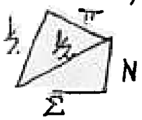
$$[L,H] = i K \qquad [L,K] = -i H$$

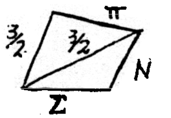
If the compact O(2) basis L is chosen diagonal (it has discrete integral or half integral eigenvalues) and the matrix elements of $H \pm iK$ are the familiar quantities $\sqrt{m(m\pm 1) - j(j+1)}$ for $m^2 > j^2$ with j real, or $+\infty > m \geqslant -\infty$ for j+1/2 pure imaginary. These give the familiar[9] discrete and continuous representations (of the covering group) of O(2,1). With H chosen diagonal, the situation is quite different. The spectrum of H is continuous and there are no proper eigenvectors of H; the (normalizable) vectors of H are to be chosen as integrals over the improper eigenvectors with a weight function which is the square integrable boundary value along the real axis of an analytic function defined in the entire complex plane. For the continuous representations the spectrum $-\infty < m < \infty$ is covered twice, while for the discrete representations it is covered only once. The detailed discussion of these representations is beyond the scope of this paper. The general characteristics are of sufficient interest to warrant mention. The spectra of the noncompact generators and the Casimir operators of noncompact subgroups are in general contunuous; the ladder operators tend to change a vector with an allowed eigenvalue to a vector with a nonallowed eigenvalue. For example, for the O(2,1) group $L \pm K$ tend to change the eigenvalue of H by $\pm i$. But with a vector which is a linear combination of such "ideal" basis vectors with analytic coefficients these changes can be defined as linear operators on these weight functions. It is also possible to have a combination of a discrete and a continuous spectrum for the spectrum of Casimir operators of a subgroup; an example in point is the spectrum of the first Casimir operator of the Lorentz group O(3,1) in the reduction[10] of the space-like representation of the Poincaré group E(3,1). The use of a noncompact noninvariance group thus provides sufficient richness to accommodate all the physically related quasistationary and continuum states.

V. Application to Hadron Phenomena

We now return to the description of hadron phenomena in terms of noninvariance groups. To proceed beyond the classification scheme to the description of interactions we have to make a hypothesis. The simplest such hypothesis is to identify the tensor character of the sources of weak and electromagnetic interactions. By a direct application of the Wigner - Eckart theorem, one could then deduce consequences of such an assumption.[11] For the linear electromagnetic properties and for weak interactions these considerations in relation to the isospin and unitary groups have already been studied elsewhere. We could however make the stronger hypothesis that the sources of the electromagnetic and of the weak interaction and, more generally, of any mesons, are in fact some of the generators of the noninvariance group. In this case quite immediate and useful relations can be obtained provided we neglect the dependence of the transition matrix elements on the momentum. (This latter neglect is necessitated by the fact that we are working with finite-parameter groups but the inclusion of the momentum dependence would convert it into an infinite parameter group--see below.) As an example of the method we may consider the hypothesis[12] that the source of leptonic weak interactions is the positive chiral part of the current to which the pion or the kaon is coupled. If we now identify this current with the generator[13] of a chiral isospin (or unitary) group, it follows that in any process involving an arbitrary number of pions the weak interaction matrix element (in the limit of vanishing pion four-momentum) must be the same as given by an $I = 1/2$ (or $I = 1$) current rule[12] for the strangeness violating (strangeness conserving) weak interactions, between the other particles. (Compare the discussion of nonleptonic weak interactions below). Similar considerations would suggest that the photoproduction of any number of pions be also subject to the same limitation. It is a linear combination of an isoscalar and an isovector in the coupling of the remaining hadrons (as well as any number of pions). Within the framework of unitary symmetry the predictions are stronger, but because of the approximate nature of the symmetry one does not expect these predictions to be borne out.

Within the framework of a current-current weak interaction theory we could make predictions about nonleptonic decays also. Consider for example the nonleptonic decay of a hyperfragment of isotopic spin I into a nucleus with isotopic spin I with the emission of an arbitrary number of pions. Then, by hypothesis the weak interaction Hamiltonian transforms[12] as the product of two currents with positive chiral isospins $\frac{1}{2}$ and $\mathcal{L}$ negative chiral isospins 0. The pion sources do not change these tensor characters[14] (in the limit of zero, four-momentum transfer), Consequently the transition can be characterized in terms of two amplitudes for $|I - I'| = \frac{1}{2}$ (which reduce to one if I or I' is zero) and only these transitions are allowed, or if $|I - I'| = 3/2$. The transitions therefore contain only $\Delta I = 1/2$ and $\Delta I = 3/2$ but no other; there is the added restriction that the inclusion of any number of pions does not alter this character. In particular, for the transitions $\Xi \to \Lambda\pi$ and $\Lambda \to N\pi$, these imply the $\Delta I = \frac{1}{2}$ rule[15], since $\Xi\ \Lambda$ or ΛN cannot couple to $I = 3/2$). For the $\Sigma \to N\pi$ transition there are precisely two amplitudes corresponding to the $\Delta I = 1/2$ and $\Delta I = 3/2$ contribution. Each of these contribute only one (rather than two) matrix elements according to the schemes





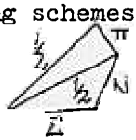
and these amplitudes imply the triangular relation

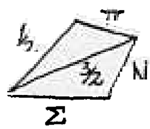
$$\sqrt{2}\, M(\Sigma^+ \to p\pi^0) = M(\Sigma^- \to N\pi^-) + M(\Sigma^+ \to N\pi^+)$$

which is experimentally indistinguishable from the familiar $\Delta I = \frac{1}{2}$ triangular relation[16]

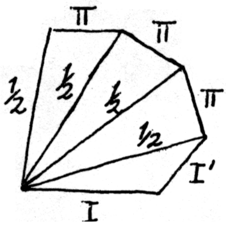
$$\sqrt{2}\, M(\Sigma^+ \to p\pi^0) = M(\Sigma^- \to n\pi^-) - M(\Sigma^+ \to n\pi^+)$$

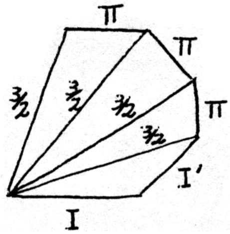
at the present time. (In principle, the decay of hyperfragments would lead to phase information distinguishing the two triangular relations but this seems to be beyond present experimental techniques.) The difference in the structure could be understood in terms of the $\Delta I = \frac{1}{2}$ rule coupling schemes





More generally, with an arbitrary number of pions the coupling suggested earlier is according to the scheme





Needless to say, one could equally well consider any other symmetry scheme with respect to which the source of zero momentum mesons act as generators.

We note that in the above discussions an implicit assumption has been made that the pion sources determine the interaction. This is a sequel to the assumption that the pionic weak interactions are all induced effects and no direct coupling is to be invoked. If a direct coupling is postulated, that term has to be added to the above contributions and any sum rules that still remain would depend upon the nature of the postulated direct coupling.

It is interesting to note two things. First, it was not necessary to restrict attention to weak or electromagnetic processes only, since in the limit of zero meson momenta we could apply similar considerations to the strong interaction transition amplitudes also. It may then happen that some strong interaction transition amplitudes vanish; for example, in the absence of a direct pion - pion coupling, the pion - pion scattering amplitude vanishes in the limit of zero four momenta of all the pions.

The second remark concerns the use of only currents with baryon number zero. It is not clear that strong interactions are to be described solely in terms of current with zero baryon number; it is conceivable that the full characterization of the strong interaction involve currents with nonzero baryonic number. The multiplets associated with these currents would involve resonant states with varying baryon numbers. On the other hand, if the electromagnetic and weak interactions are any guide the hadronic charge or strangeness can change by interactions (charged lepton currents!) but the hadronic baryon number is absolutely conserved and only "equibaryonic" multiplets and the associated currents with zero baryon number are to be considered.

VI. Outlook

We may now ask for a further generalization of these ideas to apply to noninvariance groups whose generators involve nonzero momentum transfer. Since there are no finite invariant four momenta this implies generators with all possible momentum transfers. The associated Lie algebra structure no longer corresponds to a locally compact group. But the full description of a local relativistic theory of particles must imply such infinite parameter Lie groups. It now becomes essential to study their Fourier transforms as "Lie fields" which have standard local relativistic properties. Admittedly these studies would involve a mathematical framework considerable more advanced than the ones referred to above. The study of these local Lie fields can be considered as a "strengthened" form of the algebra of local, von Neumann rings; the strengthening involves the additional Lie algebra structure over and beyond the von Neumann algebra structure. The fields associated with the observed particles are likely to appear as "integration constants" in such a theory.

In recapitulation then, it appears that we have much to gain by formulating the physics of particles in terms of its characteristic noninvariance groups. This takes into account as a primary fact the observed multiplet structure and allows us to characterize particle interactions in a more useful manner. The techniques of analytic continuation of group representations provides us with a convenient technical tool in this study.

Acknowledgement

I wish to thank J. Kuriyan, N. Mukunda, L. O'Raifeartaigh and S. Pakvasa for collaboration and counsel; with their permission several results of joint work are quoted without explicit acknowledgement.

References and Footnotes

1. For earlier work on this aspect see E. C. G. Sudarshan "Principles of Classical Dynamics" (edited by N. Mukunda), University of Rochester Report NYO-10250 (1961)
2. N. Mukunda, L. S. O'Raifeartaigh and E. C. G. Sudarshan, Phys. Rev. Letters 15, 1041 (1965); see also the contribution by A. O. Barut to the International Seminar on Elementary Particles, Trieste (1965), and earlier references cited in these papers.
3. E. C. G. Sudarshan, Proceedings of the Eastern Theoretical Conference at Stony Brook (November 1965).
4. T. D. Newton and E. P. Wigner, Rev. Mod. Phys. 21, 400 (1949).
5. Compare the presentation by E. M. Stein, Proceedings of the International Seminar on Elementary Particles, Trieste (1965).
6. See also E. M. Stein and S. Wainger, Arkiv fur Matematik, 5, 553 (1965).
7. I. M. Gelfand and M. L. Tseitlein, Doklady Akad. Nauk. SSSR 71, 825, 1017 (1950).
8. T. D. Newton, Annals of Math. 51, 730 (1950).
9. V. Bargmann, Annals of Math. 48, 568 (1947).
10. I. M. Gelfand and N. Ya.Vilenkin, "Generalized Functions" Vol. V, English Translation by E. J. Saletan, Academic Press (in press). I thank Professor Saletan for making this translation available to me.
11. R. E. Marshak, S. Okubo and E. C. G. Sudarshan, Phys. Rev. 106, 599 (1957).
12. S. Okubo and others, Phys. Rev. 112, 665 (1958).
13. M. Gell-Mann, Physics 1, 63 (1964).
14. M. Suzuki, Phys. Rev. Letters 15, 986 (1965).
15. H. Sugawara, Phys. Rev. Letters 15, 870 (1965); erratum 15, 997 (1965).

DISCUSSION

Biedenharn - As one of the people who is fond of group theory, perhaps it might be forgiven if I say that now that everyone has succumbed to what used to be called the "gruppenpest", one ought to realize very firmly that group theory is a subset of quantum mechanics and not the other way around. It goes back to Wigner who noted quite some time ago when he was dealing with the rotation group, that this defined a differential equation whose solutions were far more inclusive than the group theory itself could permit. This was in his lectures on the special functions of mathematical physics. I like very much the idea of the hydrogen atom and also note that I made a relativistic extension of the symmetry properties which has been duly forgotten in the literature; however, I had a physical problem for it, I would like to say that the point that Sudarshan raised, namely, the fact that there is a discrete spectrum, which is, of course, a compact group R(4), and a continuous spectrum, which is of course homologous to the Lorentz group, should be taken very seriously because it shows us that if you wish to make a group theory the basis of your quantum mechanics, it is too tight of a mold. It is a Procrustian bed that just won't work, because we see here that there is no group that contains this system; no matter how you slice it you just can't get away with it. Now you can of course generalize, and the way you generalize is by analytical extension, and in order to prove this by a reductio ad absurdum, I took the Wigner coefficient and I gave the most general possible example of any possible extension which is a double contour Pockhammer integral, which was defined for every possible value analytically and uniquely for all the values of its observables. And what was the use of this beast? I haven't found it. But you can get around these difficulties of the so-called degenerate or discrete series. By means of this horrible beast you can construct every possible representation of any group that has the complex algebra of SU(2) (or an extension of it) and you can find with this Wigner coefficient an analytic function which gives a completely satisfactory answer in one and the same Hilbert space wherever it is possible to find a group with this many parameters. But I don't believe that this is physics and I think that many of the colleagues would agree with me. May I make one more remark? I have a colleague who is extremely clever, Jean Nuyts, who is now at the Institute for Advanced Study. He posed the question that Prof. Sudarshan raised on the reduction of O(2,1), actually SU(1,1), with respect to a noncompact subgroup, to the Jauch Seminar in Geneva, and I believe that Jauch gave the answer that Professor Sudarshan gave here. These are very interesting problems and they show me something quite different, namely that this is the wrong way to proceed.

Sudarshan - I would like to make two remarks. The first remark is that obviously some quantum mechanical systems do not admit of nice group theoretic description and some groups do not provide nice quantum mechanical treatment. The study of the noncompact group as a model for particles has been attempted in the series of papers that were initiated by Pryce and Eddington, and in a more exact and satisfactory form by Newton and Wigner, about the notion of position operators and canonical variables in relativistic theories. It was shown that canonical variables are very good for describing most relativistic systems, but they are certainly not good for describing zero mass, spin larger than one-half, particles. Now we should ask, should we stick with the canonical variables or should we stick with the representation of the Lorentz group? Obviously we do not stick with either of these things; we avoid the question and go on to more glorious things by considering a relativistic field, which happens to have some quantum states which happen to have these properties. The second thing is that quantum theory in which you do not have an additional group structure is more general than a quantum theory which in addition to the operator structure, contains also the structure which is associated with the Lie algebra. If you have a field theory in which you have local observables which are defined in an appropriate sense in a mathematical theory, which satisfy only causal commutation relations, then that theory is definitely more general than the theory in which you further require that equal-time commutation relations of some of these objects must themselves be new objects which belong to the same algebra-the current algebra kind of commutation relations. Now obviously, a theory which does not allow for the second kind of thing is more general, but the question is do we want such a theory? It is my belief at the present time that we have much more to gain by using the group theoretic framework, because that seems to be the general pattern into which many things can be put. One cannot just say that this is not the right method of procedure; Dr. Fronsdal mentioned yesterday that obviously these studies do not lead to anything. The only way to proceed is to see what happens a few years from now.

Fronsdal - I would like to point out one thing that you said here about the representation of noncompact groups which I have not said before, is that $f(\lambda)$ goes into $f(\lambda \pm i)$. This is wrong! Everything you said is right and has been said before but I am sure that the very last statement that f has to be an analytic function of λ is false! I would like to see your derivation of it if you really believe it.

Sudarshan - If he wants to see a derivation, he will see it.

Hamermesh - If you want to see a derivation, you can look up a paper in the Bulletin of the Academy of Sciences of the U.S.S.R. by Gelfand in which this problem is treated completely.

Sudarshan - Much of this now appears in the English translation of the book by Gelfand and Shilov and I have had the proof sheets of this thing; so I just don't buy this statement that Fronsdal made.

Guth - I would like to come back very briefly again to the hydrogen atom. We can still learn from the hydrogen atom though it may not be directly relevant to elementary particles. In the case of hydrogen one sees clearly that the occurance of the so-called accidental degeneracy group is accidental, so that this type of theory cannot be generalized for every system. That group gives, for the Kepler problem, the $\frac{1}{n^2}$ of the energy levels. On the other hand, the so-called noninvariance or dynamical group, just mediates the transition from one level to another, so I don't quite see how it leads to dynamics. In that respect I agree with Larry Biedenharn. However, considerations like Sudarshan's might be fruitful in trying to develop a sort of generalized quantum mechanics where you are guided by group considerations but don't take everything that you now know in quantum mechanics.