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Origin of Internal Symmetries

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1. INTRODUCTION

Symmetry groups in physics seem to belong to two classes: the so-called relativity (or frame) groups, which may be called the *external symmetry groups*, defined by the geometric relations between "inertial" systems for which the laws of physics are the same, and the *internal symmetry groups*. We call the symmetry "internal" because we see only its manifestations; there is no primitive geometric characterization of the symmetry group from any fundamental dynamic principle. We shall try to see to what extent a dynamic principle can be expected to generate a symmetry group.

In this connection, two sets of quantum numbers can be distinguished—the additive quantum numbers (such as the third components of $\vec{J}$ and $\vec{T}$), which are algebraically additive, and the nonadditive ("vector") quantum numbers (such as the total angular momentum $\vec{J}$, total isotopic spin $\vec{T}$, etc.), which obey vector laws of addition and multiplication. One fact worth recalling is that the irreducible representations of a compact group are finite dimensional and are equivalent to unitary representations.

We naturally ask about the properties of particles in interaction. Suppose, for example, we consider the following (virtual) reaction:

$$N \longrightarrow N + \pi$$

From the $(NN\pi)$ vertex, we can write the invariant interactions (by using the Clebsch–Gordan coefficients) and obtain the following relationships between the various $(NN\pi)$ coupling constants (g) and among the various virtual transition probabilities:

$$
\begin{array}{lll}
g_{pp\pi^0} = -g_{nn\pi^0} & & \\
 & \frac{1}{\sqrt{2}} g_{pn\pi^+} & \\
 & \frac{1}{\sqrt{2}} g_{np\pi^-} & \\
\Gamma(n \longrightarrow n + \pi^0) & \Gamma(p \longrightarrow p + \pi^0) & (1) \\
 & \frac{1}{2}\Gamma(p \longrightarrow n + \pi^+) & \\
 & \frac{1}{2}\Gamma(n \longrightarrow p + \pi^-) &
\end{array}
$$

where p and n refer to the proton and neutron, respectively. From these we conclude that for the total widths

$$\Gamma(n \longrightarrow \text{any particle}) = \Gamma(p \longrightarrow \text{any particle})$$

We know that the multiplet structures displayed by the known particles are consequences of the (postulated) existence of an internal symmetry. We therefore ask whether the existence of the multiplet structure conversely implies an internal symmetry.

Recently, there have been a good many attempts to explain the internal symmetry by some direct dynamic calculations. If we start with a multiplet of N vector mesons of equal masses and assume that the interactions among these vector mesons are essentially trilinear in character, we can make a dynamic scheme in terms of a straightforward and self-consistent bootstrap mechanism between these (equally massive) vector mesons. One such attempt was made by Capps,[1] who found that the interactions among these N equally massive vector mesons obey unitary symmetry (i.e., invariance under the group SU_3). Capps was surprised to find this relationship between unitary symmetry and a self-consistent bootstrap calculation. It looked as though unitary symmetry could be derived from first principles. However, it is possible that the symmetry would have emerged from the assumption of the existence of a multiplet degenerate in mass before the interaction and the postulate that this multiplet structure is preserved even in the presence of interactions, so that the particles exhibit the same mass degeneracy even in the presence of the interaction if they have equal masses. Such arguments have been used by Sakurai,[2] who tried to prove that the emergence of the symmetry is not a consequence of a sophisticated dynamic calculation, but rather the immediate consequence of the assumptions:

1. Equality of masses of the particles.
2. Existence of a degenerate multiplet before the interaction.
3. The presence of interaction not altering the equality of masses (or the multiplicity of the particles).

He directly shows, as an example, that if we equate the contributions of certain self-energy diagrams we arrive at the required symmetry.

Thus, if we equate the second-order self-energy of the nucleons (assuming equality of masses before the interaction) as shown in Fig. 1a and that of the pions as shown in Fig. 1b, we obtain

$$2g^2_{pp\pi^0} = 2g^2_{nn\pi^0} = g^2_{pn\pi^+} = g^2_{np\pi^-}$$

In this calculation the multiplets are treated "on the same footing," and the "total width" for each component comes out to be equal. By taking fourth-order diagrams also, we can further deduce

$$g_{pp\pi^0} = -g_{nn\pi^0}$$

Thus, the symmetries may well be explained if we assume the equality of masses and multiplicity of the particles and postulate that these properties remain unchanged even in the presence of interaction.

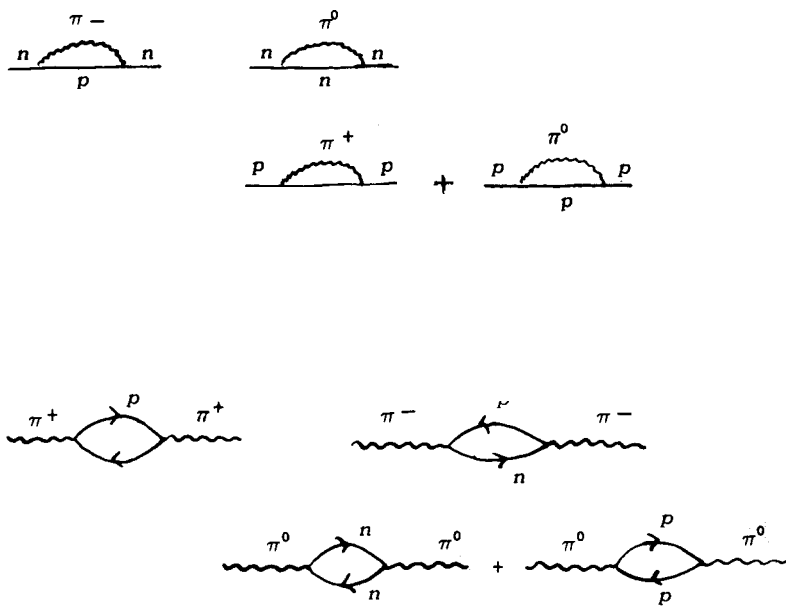

Fig.

2. GAUGE FIELDS

The consequence of the existence of symmetries and the postulated invariance of interactions under the gauge transformations of the second kind is the existence of vector gauge fields coupled linearly to conserved quantities (such as electric charge, etc.).

Unlike the electromagnetic field, which by itself is neutral and interacts only with charged fields (and is thus coupled to the electric current), gauge vector fields may themselves carry the properties. The isospin gauge field, for instance, itself carries the properties of isospin, and it is hence nonlinearly self-coupled. We may even consider a situation in which the gauge vector field alone carries isotopic spin and is consequently self-coupled. Thus, if we can write $L = j^\mu A_\mu$ for the Lagrangian of the electromagnetic interaction, where A_μ is the electromagnetic field and j^μ is the current to which it is coupled, what can we write for the Lagrangian of the interaction of the gauge vector field? Since the gauge vector field is coupled to itself, we naturally expect that the interaction can be written as a product of these fields B. Then how many B can enter the product? The simplest possibility (which we may take to be basic) is the trilinear interaction between vector particles. This is because the current is bilinear in B field and coupled to another B field, making a trilinear vertex.

Cutkosky[3] has given a simple model in which he assumes that there are a number (N) of vector mesons which have the same mass, i.e., he assumes a multiplet structure. Then, with a number of additional plausible assumptions, he shows that a Lie group could be associated with these particles. The assumptions made are:

1. The vector mesons arise as self-consistent bound states of pairs of vector mesons.

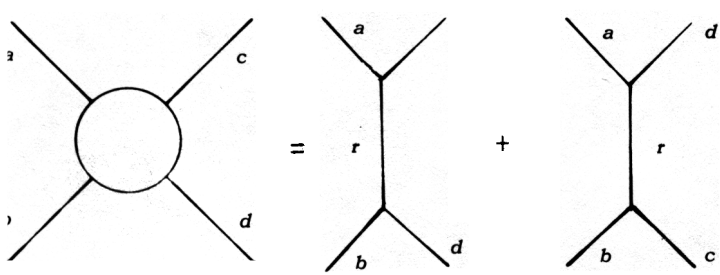

Fig. 2

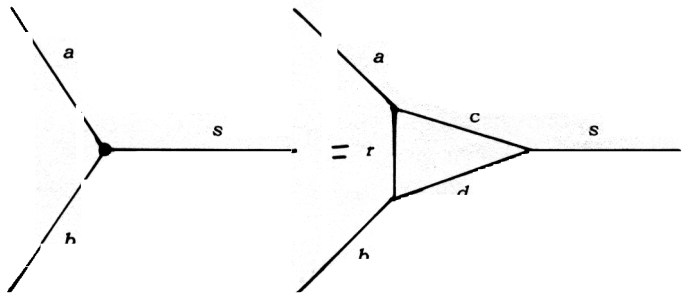

Fig.

2. The binding force is mediated by the exchange of single vector mesons; for example, the long-range part of the force is given by the one-particle exchange diagrams shown in Fig. 2.
3. The renormalized coupling constants are well approximated by the simplest irreducible vertex part, with the "bare-coupling constants" set equal to zero, as shown in Fig. 3.
4. Parity is conserved in strong interactions, and strong interactions are invariant under charge conjugation.
5. Electric charge is conserved.
6. The dependence of the vertex on the internal labels F_{abc} is antisymmetric in all pairs of indices.

If we represent the particles by real vector fields $B_\mu (\mu = 1 \ldots N)$, the invariant interaction has the form

$$F_{abc}\, B_a\, B_b\, B_c$$

and F_{abc} is antisymmetric. We then look for the eigenfunctions of F_{abc}. The Born-approximation scattering amplitude is proportional to

$$V_{ab,cd} = (F_{adr}\, F_{bcr} - F_{acr}\, F_{bdr}) \tag{2}$$

corresponding to the two diagrams in Fig. 2 and taking into account the antisymmetric nature of F.

Since all the particles which are together, and also all the exchanged particles, have the same mass (which we have normalized to unity), it is clear that we can obtain N degenerate bound states only if V has N degenerate eigenvalues. Also, the F themselves must be eigenvectors of V, in view of postulate (3):

$$V_{ab,cd}\, F_{cds} = \lambda F_{abs} \tag{3}$$

where $\lambda > 0$. Since F_{abr} can be interpreted as the internal "wave function" of the particle considered as a bound state of the two particles (a, b), we may normalize F as

$$F_{abr} F_{bas} = \delta_{rS} \tag{4}$$

If the model is self-consistent, we must see that no particle comes out with a mass less than the mass with which we started. Therefore, of all the antisymmetric eigenvectors of V ($V\psi_i = \lambda_i \psi_i$), we should allow only those for which

$$\lambda > \lambda_i \tag{5}$$

that is, we must require that no other vector particles which have a lower mass than the N we started with should arise from the potential; otherwise, the model will not be self-consistent. We then proceed to determine F, satisfying equations (3), (4), and (5). We have from the definition of trace

$$\mathrm{Tr}V^2 = N\lambda^2 + \sum \lambda_i^2 \tag{6}$$

The explicit form of V given by equation (2), together with equations (3) and (4), can be used to calculate the alternative expression

$$\mathrm{Tr}V^2 = 2N - N\lambda \tag{7}$$

Therefore, it follows that

$$\lambda(\lambda + 1) \quad 2 \quad \frac{1}{N} \sum \lambda_i^2 \tag{8}$$

so that

$$\lambda \leq 1 \tag{9}$$

The equality holds when $\sum \lambda_i^2 = 0$. Under the orthogonal transformation

$$B'_a = 0_{ab} B_b$$

equations (3) and (4) are covariant. For infinitesimal transformations of 0_{ab}:

$$0_{ab} = \delta_{ab} + i\epsilon^\alpha G^\alpha_{ab}$$

the F_{abc} transform according to

$$F'_{abc} = F_{abc} + i\epsilon^\alpha f^\alpha_{abc}$$

$$f^\alpha_{abc} = F_{rbc} G^\alpha_{ra} + F_{arc} G^\alpha_{rb} + F_{abr} G^\alpha_{rc} \tag{10}$$

Cutkosky makes assumption (5), that the interactions satisfy a nontrivial additive conservation law (say, conservation of charge at each vertex). This requirement simplifies the analysis, because $F_{\ldots}$ are

invariant under gauge transformations of the first kind if we hold this assumption. (If we had assumed the existence of l independent additive conservation laws, F_{abc} would have been invariant under an l parameter Abelian subgroup of 0_N.) If we denote by G^A_{ab} a generator of the Abelian gauge transformation, then

$$F_{rbc}G^A_{ra} + F_{arc}G^A_{rb} + F_{abr}G^A_{rc} = 0 \tag{11}$$

If we multiply equation (11) by F_{bad}, and use the fact that F is antisymmetric, we get

$$G^A_{cd} = V_{cd,ab}G^A_{ab} \tag{12}$$

The generators G^A_{ab} are eigenvectors of V with unit eigenvalue. Hence, from equations (5) and (8) it follows that all $\lambda_i = 0$; consequently, the completeness of the eigenvectors of V then allows us to write

$$V_{ab,cd} = F_{abr}F_{dcr}$$

which can be written as

$$F_{abr}F_{cdr} + F_{bcr}F_{adr} + F_{car}F_{bdr} = 0 \tag{13}$$

using the definition of V given by equation (2). Equations (13) and (15) are the necessary and sufficient conditions for the F_{abc} to be the structure constants of a compact Lie group. The association is necessarily with a particular representation of the group, the adjoint representation.

Quite recently, Weinberg[4] stated that charge conservation seems to play a crucial role in generating continuous symmetries. He observed that any discrete (or continuous) symmetry arising from dynamics will always be transmuted into a full-fledged Lie group by the condition of charge conservation, provided that the electric charge operator is not invariant under the original symmetry. If U is any member of the group of physical symmetries, then also is

$$U^{-1}[\exp(i\theta Q)]U = \exp[i\theta(U^{-1}QU)]$$

where Q is the charge operator. Hence, both Q and $U^{-1}QU$ belong to the Lie algebra of the physical symmetry group. When U does not commute with Q, then we can generate a larger symmetry group.

However, it seems to be really possible that we may relax the condition of charge conservation in obtaining internal symmetries. In a model calculation,[5] we find that charge conservation comes out naturally and need no longer be imposed. This is a direct self-energy calculation such as that of Sakurai[2] with no condition on electric charge conservation. We can also try to relax the condition of charge conservation from Cutkosky's calculation.[3]

Let us first consider the modified Cutkosky model. The potential established by the exchange of single vector mesons between pairs of vector mesons (a, b) and (c, d) in the Born approximation can be written as

$$V_{ab,cd} = (F_{acr}F_{bdr} - F_{adr}F_{bcr}) \tag{14}$$

corresponding to the two diagrams given in Fig. 2. This potential to a very good degree of approximation is also the scattering amplitude which will contain pole terms corresponding to single particle exchanges to the direct channel. We omit higher intermediate particle contributions to the scattering amplitude, i.e., in the pole approximation

$$\text{Scattering amplitude} = \sum_r \frac{F_{abr}F_{cdr}}{s - \mu^2} \tag{15}$$

which corresponds to the pole diagram shown in Fig. 4. Here s is the total center of mass energy squared and μ is the mass of the particle exchange. Thus, the potential

$$\begin{aligned} V_{ab,cd} &= -F_{acr}F_{bdr} + F_{adr}F_{bcr} \\ &= \chi F_{abr}F_{cdr} \end{aligned} \tag{16}$$

where χ is a constant which does not depend on r. (This equation can of course be never true as it stands; the left-hand side terms have poles in the momentum transfer variables t and u, while the right-hand side term has a pole in the energy variable. However, if we iterate either the left-hand (or the right or both!) terms, we will change their dependence on s, t, u. But it is possible that their dependence on the internal labels a, b, c, d is unaltered. We consider this possibility here only, and after the solution is obtained we can in fact verify that a "horizontal" iteration leaves the a, b, c, d dependence essentially unchanged, since a, b, c, d turns out by virtue of equation (18) to be a constant multiple of a projection matrix. We short-circuit these essential dynamic points in the sequence of arguments in the text. Equation (16) is an identity which we can write as

$$\chi F_{abr}F_{cdr} - F_{adr}F_{bcr} + F_{acr}F_{bdr} \equiv 0 \tag{16a}$$

Making all permutations b, c, d in equation (16a), taking into account the antisymmetric nature of the F, and adding all such equations, we can obtain the equation

$$(2 - \chi)[F_{abr}F_{cdr} + F_{acr}F_{bdr} + F_{adr}F_{bcr}] \equiv 0 \tag{17}$$

But χ is a function of the invariant energy and momentum transfer

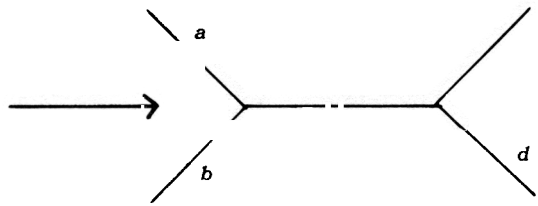

Fig. 4

variables, and therefore almost everywhere the factor (2 — nonzero. Hence

$$F_{abr}F_{cdr} + F_{acr}F_{bdr} + F_{adr}F_{bcr} \equiv 0 \tag{18}$$

Equation (18) is just the Jacobi identity obeyed by the structure constants of a Lie group. Consequently, from the group property, there should be at least one Abelian subgroup corresponding to conservation of an additive quantum number, which we choose to be the charge. Thus, charge conservation comes out of the calculation.

Next we shall see that charge conservation can in fact be relaxed from Sakurai's calculation of self-energy contributions.[5] Consider now the self-energy diagrams of pion and nucleons shown in Fig. 4. Here the labels $\alpha, \beta \ldots$ correspond to the mesons and r, s, t correspond to nucleons. Let us at this stage state the generalized Smushkevich principle: "The (dressed) propagators of the component fields of a multiplet are the same."

A more useful and (possibly) equivalent statement of the Smushkevich principle is the following: "Topologically identical self-energy diagrams should give equal contributions to the propagators of component fields of a multiplet."

Suppose we write down the nucleon self-energy contribution from Fig. 5. This is equal to

$$\sum_{\alpha,t} g_{rt\alpha}g_{ts\alpha}$$

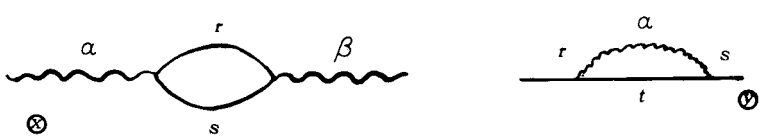

Fig. 5

where $g_{rt\alpha}$ is the coupling constant corresponding to the vertex $(rt\alpha)$. Let us define the matrices C by

$$g_{rt\alpha} = C^{\alpha}_{rt} \tag{20}$$

so that equation (19) becomes

$$\begin{aligned} \sum_{\alpha,t} g_{rt\alpha} g_{ts\alpha} &= \sum_{\alpha,t} C^{\alpha}_{rt} C^{\alpha}_{ts} \\ &= \sum_{\alpha} (C^{\alpha} C^{\alpha})\, rs\text{-th element} \\ &= A\delta_{rs} \end{aligned} \tag{21}$$

by Smushkevich's principle, where A is a constant independent of r and s. Thus

$$\sum_{\alpha} (C^{\alpha} C^{\alpha}) = AI \tag{22}$$

Similarly, the contribution from Fig. 4 to the pion self-energy is equal to

$$\begin{aligned} \sum_{r,s} g_{rs\alpha} g_{sr\beta} &= \sum_{r,s} C^{\alpha}_{rs} C^{\beta}_{sr} \\ &= \sum_{r} C^{\alpha}_{r} C^{\beta}_{r} \\ &= \mathrm{Tr}(C^{\alpha} C^{\beta}) \\ &= B\delta_{\alpha\beta} \end{aligned} \tag{23}$$

by the Smushkevich principle, where B is a constant independent of α and β. Thus

$$\mathrm{Tr}\, C^{\alpha} C^{\beta} = B\delta_{\alpha\beta} \tag{24}$$

We can easily show that $3B = 2A$, but we shall not use this result. Therefore, our task is to find the matrices C which satisfy equations (22) and (24). Since the pion fields retain their properties under real orthogonal transformations:

$$C^{\alpha} \longrightarrow \sum_{\beta} 0^{\alpha\beta} C_{\beta} \tag{25}$$

and the nucleon fields retain their properties under arbitrary unitary transformation

$$C^{\alpha} \longrightarrow U C^{\alpha} U^{-1}$$

we can show that the C are the matrices that we want, namely, τ_1, τ_2, and τ_3. It is always possible, by an orthogonal transformation, to have two of the matrices C_1, C_2, and C_3 be traceless. It may be noted that the C are Hermitian, i.e., we can have

$$\begin{aligned} \mathrm{Tr}\, C_1 &= 0 \\ \mathrm{Tr}\, C_2 &= 0 \end{aligned} \tag{26}$$

Since any traceless Hermitian matrix can be made a multiple of τ_1 by a unitary transformation, let us choose C_1. [Incidentally, it is clear that $\mathrm{Tr}(\tau_1 \tau_1) = 2$ and therefore satisfies equation (24) if we choose $B = 2$.] That is,

$$C_1 = \tau_1 \tag{27}$$

[Otherwise, we get $C_1 = (B/2)^{1/2} \tau_1$.] C_2, which is to be traceless and Hermitian, can be expanded as

$$C_2 = a\tau_1 + b\tau_2 + c\tau_3 \tag{28}$$

where a, b, and c are real coefficients. In this, we exclude the unit matrix by virtue of condition (26). Let us now use equations (22) and (24) to evaluate the coefficients a, b, and c.

$$\mathrm{Tr} C_2 C_2 = 2 = 2(b^2 + c^2)$$

therefore

$$b^2 + c^2 = 1$$

$$\mathrm{Tr}(C_1 C_2) = 0 = 2a \Longrightarrow a = 0 \tag{29}$$

We can choose $b = \cos\theta$ and $a = -\sin\theta$. Thus

$$C_2 = \tau_2 \cos\theta - \tau_3 \sin\theta \tag{30}$$

By a rotation around the τ_1 axis with our choice of C_1 unaltered, we can get C_2 as

$$\exp(\tfrac{1}{2}\boldsymbol{\tau}_1 \cdot \boldsymbol{\theta}) C_2 \exp(-\tfrac{1}{2}\boldsymbol{\tau}_1 \cdot \boldsymbol{\theta}) = \tau_2 \tag{31}$$

Thus, having found C_1 and C_2, let us now find C_3. In general

$$C_3 = d\tau_1 + e\tau_2 + f\tau_3 + gI \tag{32}$$

The unit matrix is included since we do not need C_3 to be traceless. Coefficients d, e, f, and g are real. We now use equations (22) and (24) and the choice of C_1 and C_2 to find d, e, f, and g:

$$\mathrm{Tr} C_1 C_3 = 0 : \quad \rightarrow d = 0$$

$$\mathrm{Tr} C_2 C_3 = 0 : \quad \Rightarrow e = 0$$

$$\mathrm{Tr} C_3 C_3 = 2 : \quad 2(f^2 + g^2)$$

$$f^2 + g^2 \quad : 1$$

therefore

$$C_3 = f\tau_3 + gI \tag{33}$$

If we use equation (22)

$$C_1 C_1 + C_2 C_2 + C_3 C_3 \quad AI$$

so that

$$+ 1 + f^2 + g^2 + 2fg\tau_3 \quad AI$$

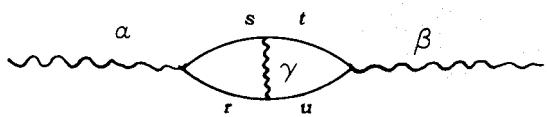

Fig. 6

Since each term on the left-hand side has to be a multiple of the unit matrix, the only term which is not a multiple of identity should vanish:

$$fg = 0 \Longrightarrow f = 0 \qquad \text{or} \qquad g = 0$$

Thus we are left with two choices:

$$C_3 = \tau_3 \qquad \text{or} \qquad C_3 = I \tag{34}$$

In the following, we will show that the choice (τ_1, τ_2, I) for the C contradicts the Smushkevich principle given in equation (24). If we go to the next higher order self-energy diagram for the meson (Fig. 6), the contribution is

$$\begin{aligned} &\sum_{\substack{r,s,t,u \\ \gamma}} g_{rs\alpha} g_{st r} g_{ut\beta} g_{ur\gamma} \\ &\quad = \sum_{\substack{r,s,t,u \\ \gamma}} C^{\alpha}_{rs} C^{\gamma}_{st} C^{\beta}_{tu} C^{\gamma}_{ur} \\ &\quad = \sum_{\gamma} \mathrm{Tr}(C^{\alpha} C^{\gamma} C^{\beta} C^{\gamma}) \\ &\quad = B' \delta_{\alpha\beta} \end{aligned} \tag{35}$$

by the Smushkevich principle, where B' is another constant independent of α and β.

If we choose the basis (τ_1, τ_2, I) for the C, then for $\alpha = \beta = 1$ or 2 we really get

$$\sum_{\gamma} \mathrm{Tr}(C^{\alpha} C^{\gamma} C^{\beta} C^{\gamma}) = 2$$

and hence equation (24) is satisfied. However, for $\alpha = \beta = 3$, we get

$$\begin{aligned} \sum_{\gamma} \mathrm{Tr}\,(C^{\alpha} C^{\gamma} C^{\beta} C^{\gamma}) &= \mathrm{Tr}\,(I\tau_1 I\tau_1) \\ &\quad + \mathrm{Tr}\,(I\tau_2 I\tau_2) \\ &\quad + \mathrm{Tr}\,(IIII) \\ &= 2 + 2 + 2 \end{aligned}$$

and thus equation (35) is not satisfied. Hence, we are left with the unique choice (τ_1, τ_2, τ_3) for the C. The ambiguity about the sign (i.e., choice $\pm\tau_1, \pm\tau_2, \pm\tau_3$) is there, but this can be eliminated by a redefinition of the pion fields. Thus we end up with a $(\boldsymbol{\tau} \cdot \boldsymbol{\phi})$ interaction

for the π-N system. In other words, the interaction is such that the charge is conserved.

Therefore, we have seen that the idea of charge conservation can be suppressed in both Cutkosky's and Sakurai's calculations. The question is whether we can generalize this statement to all interactions. The answer seems to be no, since we have a counterexample for the $(\Sigma\Sigma\pi)$ system. A similar fate befalls Sakurai's method, which includes electric charge conservation for higher isospins.

It appears then that no particular axiom is crucial in the "derivation of an internal symmetry," since Sakurai's calculation relaxes self-consistency, and we have shown that even charge conservation can be relaxed. The question is how many axioms we need in order to specify the internal symmetry. The point of view advanced here is that the Smushkevich principle, when augmented by suitable auxiliary restrictions, is the really basic one in understanding the internal symmetry. The origin of internal symmetries is still open, although we have some indication as to the direction in which a correct solution may lie. These demonstrations have now been extended[5] to a variety of systems—in particular, to a system of two multiplets of n components each coupled to a multiplet of $(n^2 - 1)$ members to deduce invariance under the special unitary group $SU(n)$, again without assuming electric charge conservation.

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