TIME EVOLUTION OF COHERENT STATES ‡

C. L. MEHTA ‡‡ and E. C. G. SUDARSHAN
Physics Department, Syracuse University, Syracuse, N.Y., USA

Received 29 July 1966

General conditions under which the states which are initially coherent remain coherent at all times are considered. It is also shown that the eigenvalues of the annihilation operator for interacting systems in general are not analytic signals.

Coherent states which are defined as the eigenstates of the annihilation operator have recently been found very useful in the description of optical coherence and in some other related topics. It is also known that the time-Fourier transforms of the eigenvalues of the annihilation operator for a free field have only positive frequency components so that these eigenvalues are the so-called analytic signals. However, no serious attempt has yet been made to study the time development of coherent states. In this paper we consider general conditions under which coherent states do not change their essential character with time. Explicit solution for the eigenvalue of the annihilation operator is obtained and it is shown that in presence of interactions, the eigenvalues of the annihilation operator need not be analytic signals. Throughout this paper we consider systems with only one degree of freedom.

‡ Supported in part by the U.S. Atomic Energy Commission.

‡‡ Permanent address: Department of Physics and Astronomy, University of Rochester, Rochester, N.Y.

The canonical annihilation and creation operators ‡‡‡ $\hat{a}$ and $\hat{a}^\dagger$ satisfy

$$[\hat{a}, \hat{a}^\dagger] = 1 \ ; \ [\hat{a}, \hat{a}] = [\hat{a}^\dagger, \hat{a}^\dagger] = 0 \ . \tag{1}$$

The normalized coherent states are then given by [1]

$$|v\rangle = \exp\{v\hat{a}^\dagger - v^*\hat{a}\}|0\rangle \tag{2}$$

and satisfy the eigenvalue relation

$$\hat{a}|v\rangle = v|v\rangle \ . \tag{3}$$

Here $|0\rangle$ denotes the vacuum state.

To study the time evolution of the system, one can work either in Heisenberg or in Schrödinger picture. For convenience we use Heisenberg picture in this paper. In the Heisenberg picture, the state is fixed and we consider the time evolution in terms of the time evolution of the operators which satisfy the Heisenberg equation of motion

$$i\dot{\hat{a}}(t) = [\hat{a}(t), \hat{H}(t)] \ .$$

‡‡‡ All symbols with a circumflex, e.g. $\hat{a}$, $\hat{a}^\dagger$ etc., denote operators.

In particular, for very small τ we can write

$$\hat{a}(t+\tau) = \hat{a}(t) - \mathrm{i}[\hat{a}(t), \hat{H}(t)]\tau + \mathrm{O}(\tau^2) . \quad (5)$$

If we require that the state $|v\rangle$ which is an eigenstate of $\hat{a}(t)$ with eigenvalue $\zeta(v, t)$ be also an eigenstate of $\hat{a}(t+\tau)$ with eigenvalue $\zeta(v, t+\tau)$ we must have

$$\hat{a}(t+\tau)|v\rangle = \zeta(v, t+\tau)|v\rangle . \quad (6)$$

Retaining only the terms of the first order in τ we obtain, from (5) and (6), the relation

$$[\hat{a}(t), \hat{H}(t)]|v\rangle = \{\mathrm{i}\,\frac{\partial\zeta(v,t)}{\partial t}\}|v\rangle . \quad (7)$$

Eq. (7) implies that the commutator $[\hat{a}(t), \hat{H}(t)]$ must depend on $\hat{a}(t)$ alone and not an $\hat{a}^{\dagger}(t)$ i.e. that

$$[\hat{a}(t), \hat{H}(t)] = F(\hat{a}(t), t) , \quad (8)$$

where F is some function of $\hat{a}(t)$ and may also depend explicity on t. Condition (8) is also sufficient to guarantee that the state $|v\rangle$ is an eigenstate of $\hat{a}(t)$ for all t, as has recently been shown by Glauber [2].

The function F appearing on the right-hand side of eq. (8) cannot be an arbitrary function because of the restriction that $\hat{H}$ is Hermitian. It can then be seen that the general form of the Hamiltonian with the requirement that states which are initially coherent remain coherent at all times is given by ‡

$$\hat{H}(t) = \omega(t)\hat{a}^{\dagger}(t)\,\hat{a}(t) + f(t)\hat{a}^{\dagger}(t) + f^{*}(t)\,\hat{a}(t) + \beta(t) \quad (9)$$

where $\omega(t)$ and $\beta(t)$ are real.

The equation of motion for the operator $\hat{a}(t)$ is now given by (cf. eqs. (4) and (8))

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{a}(t) = -\mathrm{i}\,\omega(t)\hat{a}(t) - \mathrm{i}f(t) , \quad (10)$$

whose solution can be immediately written in a closed form. We thus obtain

$$\hat{a}(t) = \exp[-\mathrm{i}\psi(t)]\hat{a}(0) + \\ - \mathrm{i}\exp[-\mathrm{i}\psi(t)]\int_0^t \mathrm{d}t'\, f(t')\exp[\mathrm{i}\psi(t')] , \quad (11)$$

where

$$\psi(t) = \int_0^t \omega(t')\,\mathrm{d}t' . \quad (12)$$

The eigenvalue $\zeta(v, t)$ of the annihilation operator $\hat{a}(t)$ is thus given by

‡ This conclusion concerning the form of the Hamiltonian is the same as required for the dynamical brackets of the Hamiltonian with arbitrary dynamical variables to reduce to the Poisson brackets [3], so that the quantum and classical equations of motion are identical [e.g. 4].

$$\zeta(v, t) = \exp[-\mathrm{i}\psi(t)]\,v + \\ - \mathrm{i}\exp[-\mathrm{i}\psi(t)]\int_0^t \mathrm{d}t'\, f(t')\exp(\mathrm{i}\psi(t')) . \quad (13)$$

It is interesting to look at the frequency spectrum of the annihilation operator $\hat{a}(t)$:

$$\hat{b}(\nu) = \int_{-\infty}^{\infty} a(t)\exp[2\pi\mathrm{i}\nu t)\,\mathrm{d}t . \quad (14)$$

We shall consider some simple cases:

Case 1. Free Hamiltonian: $\omega(t) \equiv \omega > 0$; $f(t) \equiv 0$. In this case we have from eqs. (11), (12) and (14)

$$\hat{b}(\nu) = \delta\left(\nu - \frac{\omega}{2\pi}\right)\hat{a}(0) , \quad (15)$$

so that we have a single positive frequency at $\nu = \omega/2\pi$. This means that we can refer the time dependent annihilation operator as the "positive frequency" component of the canonical variable $\hat{q} = (\hat{a} + \hat{a}^{\dagger})/\sqrt{2\omega}$. This is not true in general when the system begins to interact as will be seen by following other cases.

Case 2. Forced oscillator: $\omega(t) \equiv \omega > 0$; $f(t) = \int_{-\infty}^{\infty} \mathrm{d}\nu\, g(\nu)\exp[-2\pi\mathrm{i}\nu t)$. In this case we obtain

$$\hat{b}(\nu) = \delta\left(\nu - \frac{\omega}{2\pi}\right)\left\{\hat{a}(0) + \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{g(\nu')\,\mathrm{d}\nu'}{\nu - \nu'}\right\} - \frac{g(\nu)}{\omega - 2\pi\nu} . \quad (16)$$

Thus we see that depending on whether $g(\nu)$ does or does not vanish for negative frequencies, $\hat{b}(\nu)$ also vanishes or does not vanish for negative frequencies. However still only C-number part of $\hat{a}(t)$ can contain negative frequency components.

Case 3. Oscillator with time dependent frequency and $f(t) \equiv 0$. In this case

$$\hat{b}(\nu) = \int_{-\infty}^{\infty}\hat{a}(t)\exp[2\pi\mathrm{i}\nu t]\,\mathrm{d}t = \\ = \int_{-\infty}^{\infty}\exp[-\mathrm{i}\psi(t)]\exp[2\pi\mathrm{i}\nu t]\,\mathrm{d}t\,\hat{a}(0) \quad (17)$$

and we have, in general, both positive and negative frequency components. If we require that only positive frequency components should occur, it implies that $\exp[-\mathrm{i}\psi(t)]$ regarded as a function of the complex variable t to be analytic in the lower half of the complex t-plane. Now since $\omega(t)$ is real, so also is $\psi(t)$ and hence the most general form of the unimodular analytic signal $\exp[-\mathrm{i}\psi(t)]$ is given by [5]

$$\exp[-\mathrm{i}\psi(t) = \exp[-\mathrm{i}\alpha - \mathrm{i}\omega_0 t]\prod_k \frac{t - \alpha_k + \mathrm{i}\beta_k}{t - \alpha_k - \mathrm{i}\beta_k} ; \\ \omega_0 \geqslant 0;\ \beta_k \geqslant 0 , \quad (18)$$

or

$$\psi(t) = \alpha + \omega_0 t - 2\sum_k \arctan[\beta_k/(t - \alpha_k)] . \quad (19)$$

From (19) we thus obtain

$$\omega(t) = \frac{\mathrm{d}\psi(t)}{\mathrm{d}t} = \omega_0 + 2\sum_k \frac{\beta_k}{(t-\alpha_k)^2 + \beta_k^2} . \qquad (20)$$

Hence we conclude that the only case when we can refer the annihilation operator $\hat{a}(t)$ as the positive frequency component of the canonical operator $\hat{q}(t)$ is the case when $\omega(t)$ is given by (20). In all other cases the eigenvalue of the annihilation operator $\hat{a}(t)$ is not an analytic signal.

The problem treated in this paper can obviously be generalized for systems having more than one degree of freedom.

References

1. J.R.Klauder, Ann.Phys.(N.Y.) 11 (1960) 123.
2. R.J.Glauber, Physics Letters 21 (1966) 650.
3. S.F.Edwards and G.B.Parrent Jr., Opt.Acta. 6 (1959) 367;
   N.G.Van Kampen, Phys.Rev.89 (1953) 1072.
4. J.E.Moyal, Proc.Cambridge Phil.Soc. 45 (1949) 99;
   T.F.Jordan and E.C.G.Sudarshan, Rev.Mod.Phys. 33 (1961) 515;
   C.L.Metha, J.Math.Phys. 5 (1964) 677.

* * * * *