Realization of Lie Algebras by Analytic Functions of Generators of a Given Lie Algebra*

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In this paper we discuss the problem of the Poisson bracket realization of various Lie algebras in terms of analytic functions of the generators of a given Lie algebra. We pose and solve the problem of realizing the general $O(4)$, $O(3, 1)$, and $E(3)$ algebras in terms of analytic functions of the generators of a prescribed realization of an $E(3)$ algebra. A similar problem is solved for the symmetric tensor realizations of $SU(3)$ and $SL(3, R)$. Related questions are discussed for $O(n + 1)$, $O(n, 1)$, $E(n)$, $SU(n)$, and $SL(n, R)$. We study in some detail the finite canonical transformations realized by the generators of the various groups. The relation of these results to the reconstruction problem is briefly discussed.

1. INTRODUCTION

THE success of group-theoretic methods in particle physics has led to a re-examination of these methods as applied to elementary dynamical problems in both quantum and classical mechanics. One finds that sometimes the quantum-mechanical and the classical-mechanical problems can be made to correspond to each other in such a fashion that their invariance groups and their noninvariance groups have the same structure. Thus, for example, the isotropic harmonic oscillator in n dimensions has the invariance group $SU(n)$ and the noninvariance groups $SU(n, 1)$ and $SU(n + 1)$, both in classical and in quantum mechanics.[1]

For quantum-mechanical systems, the Lie algebras of the invariance and noninvariance groups are realized by commutation relations between appropriate dynamical operators, whereas for classical systems the Lie algebras are realized by Poisson brackets between appropriate dynamical variables. That such different realizations exhibit a correspondence between them is quite remarkable. It has been known for quite some time that the infinite-dimensional Lie algebras of polynomials in canonical variables for quantum mechanics (commutator brackets) and for classical mechanics (Poisson brackets) have quite different structures.[2] Yet there exists the same local Lie group in both kinds of dynamics. This, then, suggests that in these two different algebraic systems, there are selected subsets of elements whose Lie algebraic structures are isomorphic.

Given any Lie algebra $\mathcal{A}$ of quantities which possess an associative law of multiplication (in addition to the Lie bracket), we can define an enveloping algebra $\mathcal{E}$ whose elements are polynomials in the elements of the primitive Lie algebra $\mathcal{A}$. The enveloping algebra $\mathcal{E}$ can be given an induced Lie algebra structure by imposing the relation

$$\{X_1 \cdot X_2, Y\} = \{X_1, Y\} \cdot X_2 + X_1 \cdot \{X_2, Y\},$$

where the curly bracket stands for the Lie bracket and the dot (which will usually be omitted) stands for the associative product. We can generalize our definition of the enveloping algebra $\mathcal{E}$ by including among its elements the analytic functions of the elements of the primitive Lie algebra $\mathcal{A}$. This enveloping algebra $\mathcal{E}$ is given an induced Lie-algebra structure by imposing the derivation property. Using this rule, we can identify $\mathcal{E}$ with an infinite-dimensional Lie algebra. In classical mechanics it would be a Lie algebra of Poisson brackets, whereas in quantum mechanics it would be a Lie algebra of commutators. The enveloping algebras in the two cases have, in general, quite different structures. However, in both cases there are certain sets of invariant elements which have vanishing Lie brackets with every element of the primitive Lie algebra and, consequently, with every element of the enveloping algebra. For the quantum-mechanical case, these are the well-known invariant operators, which are expressible as functions of the so-called Casimir invariants. If the N elements of the primitive Lie algebra are denoted by $X_1, X_2, \cdots, X_N$, then the Casimir invariants (for both quantum and classical systems) are homogeneous polynomials of the type

$$\mathcal{C}_n = \sum_{\alpha_1=1}^{N} \cdots \sum_{\alpha_n=1}^{N} \mathcal{C}^{\alpha_1\alpha_2\cdots\alpha_n} X_{\alpha_1} \cdots X_{\alpha_n},$$

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[1] See, for instance, N. Mukunda, L. O'Raifeartaigh, and E. C. G. Sudarshan, Phys. Rev. Letters 15, 1041 (1965); R. C. Hwa and J. Nuyts, Phys. Rev. 145, 1188 (1966).

[2] J. E. Moyal, Proc. Cambridge Phil. Soc. 45, 99 (1949).

where $C^{\alpha_1\cdots\alpha_n}$ are some numerical coefficients which are symmetric in the indices $\alpha_1, \cdots, \alpha_n$. It is possible that in some particular realizations these Casimir invariants may degenerate into numbers. (In fact, in the case of irreducible realizations, all these invariant elements reduce to numbers.)

In many cases, it is possible to select, out of the infinite-dimensional enveloping algebra, a finite-dimensional subset of elements which constitute another Lie algebra. We show, in this paper, how such construction can be carried out for the realizations of the Lie algebras by Poisson brackets. We begin in Sec. 2 by giving the structures and the Casimir invariants of the Lie algebras of the groups $E(3)$, $O(4)$, $O(3, 1)$, $SU(3)$, and $SL(3, R)$. A simple realization of all these Lie algebras in terms of three pairs of canonical variables q_a, p_a ($a = 1, 2, 3$) is also given in this section. In Sec. 3, after defining the primitive $E(3)$ Lie algebra and the corresponding generalized enveloping algebra, we give explicit realizations of the $O(4)$, $O(3, 1)$, and other $E(3)$ Lie algebras. The properties of the finite canonical transformations generated by the elements of these Lie algebras are also discussed. A similar problem is discussed for $SU(3)$ and $SL(3, R)$ Lie algebras in Sec. 4. Section 5 deals with a generalization to n dimensions, while in Sec. 6 we discuss briefly the relation of these results to the problem of reconstruction of canonical variables from the generators of a noninvariance group.

2. STRUCTURES AND INVARIANTS OF THE LIE ALGEBRAS OF VARIOUS GROUPS

In this section we briefly outline the structure and invariants of the Lie algebras of $E(3)$, $O(4)$, $O(3, 1)$, $SU(3)$, and $SL(3, R)$.

i. $E(3)$ Lie Algebra

The Lie algebra of the Euclidean group $E(3)$ in three dimensions consists of six elements: J_a, P_a ($a = 1, 2, 3$), and has the following basic Poisson brackets:

$$\{J_a, J_b\} = \epsilon_{abc} J_c, \tag{2.1}$$

$$\{J_a, P_b\} = \epsilon_{abc} P_c, \tag{2.2}$$

$$\{P_a, P_b\} = 0. \tag{2.3}$$

Here ϵ_{abc} is the completely antisymmetric unit tensor of Levi-Civita. Throughout this paper we employ the usual summation convention according to which a summation is implied over repeated dummy indices.

The two quadratic elements

$$P^2 = P_a P_a \tag{2.4}$$

and

$$\mathbf{J} \cdot \mathbf{P} = J_a P_a \tag{2.5}$$

of the enveloping algebra are invariants. We shall assume that the realization is real and faithful so that P^2 is positive and may be normalized to unity. There are, however, two classes of realizations corresponding to vanishing or nonvanishing of $\mathbf{J} \cdot \mathbf{P}$.

ii. $O(4)$ Lie Algebra

The Lie algebra of the real orthogonal group $O(4)$ in four dimensions also consists of six elements: J_a, K_a, having the following basic Poisson brackets ($1 \leq a, b, c \leq 3$):

$$\{J_a, J_b\} = \epsilon_{abc} J_c, \tag{2.6}$$

$$\{J_a, K_b\} = \epsilon_{abc} K_c, \tag{2.7}$$

$$\{K_a, K_b\} = \epsilon_{abc} J_c. \tag{2.8}$$

The two quadratic elements

$$J^2 + K^2 = J_a J_a + K_a K_a \tag{2.9}$$

and

$$\mathbf{J} \cdot \mathbf{K} = J_a K_a \tag{2.10}$$

of the enveloping algebra are invariants. For a real faithful realization $J^2 + K^2$ is positive, whereas $\mathbf{J} \cdot \mathbf{K}$ may be positive, negative, or zero.

iii. $O(3, 1)$ Lie Algebra

The Lie algebra of the real pseudo-orthogonal group $O(3, 1)$ in four dimensions again consists of six elements: J_a, K'_a, and has the following basic Poisson brackets:

$$\{J_a, J_b\} = \epsilon_{abc} J_c, \tag{2.11}$$

$$\{J_a, K'_b\} = \epsilon_{abc} K'_c, \tag{2.12}$$

$$\{K'_a, K'_b\} = -\epsilon_{abc} J_c. \tag{2.13}$$

The two quadratic elements

$$J^2 - K'^2 = J_a J_a - K'_a K'_a \tag{2.14}$$

and

$$\mathbf{J} \cdot \mathbf{K}' = J_a K'_a \tag{2.15}$$

of the enveloping algebra are invariants. For a real faithful realization, either invariant may be positive, negative, or zero.

iv. $SU(3)$ Lie Algebra

The Lie algebra of the unimodular unitary group $SU(3)$ in three dimensions consists of eight elements: J_a ($a = 1, 2, 3$) and five linearly independent elements of a symmetric traceless "quadrupole" tensor Q_{ab}:

$$Q_{ab} = Q_{ba}; \quad Q_{aa} = 0. \tag{2.16}$$

In this case, we have the following basic Poisson brackets:

$$\{J_a, J_b\} = \epsilon_{abc} J_c, \tag{2.17}$$

$$\{J_a, Q_{bc}\} = \epsilon_{abc} Q_{dc} + \epsilon_{acd} Q_{bd}, \tag{2.18}$$

$$\{Q_{ab}, Q_{cd}\} = (\epsilon_{ace}\delta_{bd} + \epsilon_{ade}\delta_{bc} + \epsilon_{bce}\delta_{ad} + \epsilon_{bde}\delta_{ac}) J_e. \tag{2.19}$$

There are two basic invariants, one of the second degree,

$$J_a J_a + \tfrac{1}{2} Q_{ab} Q_{ab}, \tag{2.20}$$

and one of the third degree,

$$(\sqrt{3}/2)(3 J_a Q_{ab} J_b - Q_{ab} Q_{bc} Q_{ca}). \tag{2.21}$$

As will be seen in Sec. 4 [see also Eqs. (2.40), (2.41)], there exist some realizations in which the cube of the quadratic invariant equals the square of the cubic invariant.

v. $SL(3, R)$ Lie Algebra

The Lie algebra of the unimodular real linear group $SL(3, R)$ in three dimensions also consists of eight elements: J_a $(a = 1, 2, 3)$ and five linearly independent elements of a symmetric traceless tensor Q'_{ab} $(Q'_{ab} = Q'_{ab};\ Q'_{aa} = 0)$. The basic Poisson-bracket relations are

$$\{J_a, J_b\} = \epsilon_{abc} J_c, \tag{2.22}$$

$$\{J_a, Q'_{bc}\} = \epsilon_{abd} Q'_{dc} + \epsilon_{acd} Q'_{bd}, \tag{2.23}$$

$$\{Q'_{ab}, Q'_{cd}\} = -(\epsilon_{ace}\delta_{bd} + \epsilon_{ade}\delta_{bc} + \epsilon_{bce}\delta_{ad} + \epsilon_{bde}\delta_{ac}) J_e. \tag{2.24}$$

There are again two basic invariants, one of second degree,

$$J_a J_a - \tfrac{1}{2} Q'_{ab} Q'_{ab}, \tag{2.25}$$

and one of the third degree,

$$(\sqrt{3}/2)(3 J_a Q'_{ab} J_b + Q'_{ab} Q'_{bc} Q'_{ca}). \tag{2.26}$$

It will be seen that, with the particular choice of the over-all coefficient in (2.26), in some realizations the cube of the quadratic invariant equals the negative of the square of the cubic invariant.

For each of these Lie algebras, an explicit construction can be given for the appropriate elements in terms of three pairs of canonical variables q_a, p_a which satisfy

$$\{q_a, p_b\} = \delta_{ab}, \quad \{q_a, q_b\} = \{p_a, p_b\} = 0. \tag{2.27}$$

The simplest construction for the elements of the $E(3)$ Lie algebra is given by

$$J_a = (\mathbf{q} \times \mathbf{p})_a = \epsilon_{abc} q_b p_c, \quad P_a = p_a. \tag{2.28}$$

For this realization, the invariant $\mathbf{J} \cdot \mathbf{P}$ vanishes and the other invariant P^2 is given by the dynamical variable $p_a p_a$. This realization is reducible, since the value of P^2 is unchanged by canonical transformations belonging to $E(3)$. One can now ask whether one could construct other realizations of $E(3)$ in terms of the same canonical variables, such that the $O(3)$ subalgebra generated by J_a is unchanged and that the two invariants $\mathbf{J} \cdot \mathbf{P} = \alpha_0$ and $P^2 > 0$ can be assigned arbitrary values. For this purpose we consider

$$J_a = \epsilon_{abc} q_b p_c \tag{2.29}$$

and

$$P_a = f(J^2) p_a + g(J^2) J_a, \tag{2.30}$$

where f and g are some functions of J^2. If we now impose $\mathbf{J} \cdot \mathbf{P} = \alpha_0$ and $P^2 = 1$, we obtain, from (2.30), the following expression for P_a:

$$P_a = \left\{\frac{1}{p^2}\left(1 - \frac{\alpha_0^2}{J^2}\right)\right\}^{\frac{1}{2}} p_a + \frac{\alpha_0}{J^2} J_a. \tag{2.31}$$

It may easily be verified that (2.29) and (2.31) do furnish a realization of $E(3)$.

For the elements of the $O(4)$ Lie algebra, we have the following simple construction:

$$J_a = \epsilon_{abc} q_b p_c, \tag{2.32}$$

$$K_a = (\beta^2 - J^2)^{\frac{1}{2}} (p^2)^{-\frac{1}{2}} p_a. \tag{2.33}$$

In this particular realization the invariant $\mathbf{J} \cdot \mathbf{K}$ vanishes and the other invariant $J^2 + K^2$ has the value β^2. If we choose J_a given by (2.32) and K_a by an expression similar to the one given on the right-hand side of (2.30) and impose the conditions $\mathbf{J} \cdot \mathbf{K} = \alpha\beta$, $J^2 + K^2 = \alpha^2 + \beta^2$, we obtain a realization with arbitrary values of the two invariants:

$$J_a = \epsilon_{abc} q_b p_c, \tag{2.34}$$

$$K_a = \{(\beta^2 - J^2)(J^2 - \alpha^2)/J^2 p^2\}^{\frac{1}{2}} p_a + (\alpha\beta/J^2) J_a. \tag{2.35}$$

It may be checked that this is a solution of (2.8). Without loss of generality we can assume $\beta \geq |\alpha| \geq 0$. The realization given by (2.34), (2.35) is real in the region $\alpha^2 \leq J^2 \leq \beta^2$.

A realization of $O(3, 1)$ is obtained by simply choosing $K'_a = iK_a$ and by analytically continuing α to pure imaginary values, i.e., by putting $\alpha' = i\alpha$, where α' is now real. We thus obtain the following construction for the elements of $O(3, 1)$ Lie algebra:

$$J_a = \epsilon_{abc} q_b p_c, \tag{2.36}$$

$$K'_a = \{(J^2 - \beta^2)(J^2 + \alpha'^2)/J^2 p^2\}^{\frac{1}{2}} p_a + (\alpha'\beta/J^2) J_a. \tag{2.37}$$

The two invariants $\mathbf{J} \cdot \mathbf{K}$ and $J^2 - K^2$ have the values $\alpha'\beta$ and $\beta^2 - \alpha'^2$, respectively, and the realization is real in the region $J^2 \geq \beta^2$.

The simplest construction for the elements of the $SU(3)$ Lie algebra is given by

$$J_a = \epsilon_{abc} q_b p_c, \tag{2.38}$$

$$Q_{ab} = q_a q_b + p_a p_b - \tfrac{1}{3}\delta_{ab}(q^2 + p^2). \tag{2.39}$$

In this case the two invariants have the values

$$J_a J_a + \tfrac{1}{2} Q_{ab} Q_{ab} = \tfrac{1}{3}(q^2 + p^2)^2 \tag{2.40}$$

and

$$(\sqrt{3}/2)(3J_a Q_{ab} J_b - Q_{ab} Q_{bc} Q_{ca}) = -(1/3\sqrt{3})(q^2 + p^2)^3, \tag{2.41}$$

so that for such realizations, the two invariants cannot be assigned independent values [the cube of (2.40) equals the square of (2.41)].

Lastly, a simple construction for the elements of the $SL(3, R)$ Lie algebra in terms of the three pairs of canonical variables q_a, p_a is given by

$$J_a = \epsilon_{abc} q_b p_c, \tag{2.42}$$

$$Q'_{ab} = q_a q_b - p_a p_b - \tfrac{1}{3}\delta_{ab}(q^2 - p^2). \tag{2.43}$$

The two invariants have the values

$$J_a J_a - \tfrac{1}{2} Q'_{ab} Q'_{ab} = -\tfrac{1}{3}(q^2 - p^2)^2 \tag{2.44}$$

and

$$(\sqrt{3}/2)(3J_a Q'_{ab} J_b + Q'_{ab} Q'_{bc} Q'_{ca}) = (1/3\sqrt{3})(q^2 - p^2)^3. \tag{2.45}$$

For such a realization also, the two invariants cannot be assigned independent values [the cube of (2.44) equals the negative of the square of (2.45)].

All these general constructions are made in terms of one set of three canonical pairs of variables. It is shown in the following sections that it is possible to express the elements of one realization as functions of the elements of another realization. In general, the functional forms involve algebraic functions, rather than polynomials. We therefore have to work with the elements of the generalized enveloping algebra which contains analytic functions (not only polynomials) of the generators. Within such a framework we show, in the following sections, that the generators of the Lie algebras of $O(4)$, $O(3, 1)$, $E(3)$, $SU(3)$, and $SL(3, R)$ can be expressed in terms of the elements of a generalized enveloping algebra of a given $E(3)$ realization.

3. REALIZATION OF $O(4)$, $O(3, 1)$, AND $E(3)$ LIE ALGEBRAS BY ANALYTIC FUNCTIONS OF P AND J

We begin by defining the $E(3)$ representation in terms of which we will construct $O(4)$, $O(3, 1)$, $SU(3)$, $SL(3, R)$, and other $E(3)$ generators. We consider two three-dimensional vectors **P** and **J** obeying the following basic Poisson-bracket relations:

$$\{J_a, J_b\} = \epsilon_{abc} J_c, \tag{3.1}$$

$$\{J_a, P_b\} = \epsilon_{abc} P_c, \tag{3.2}$$

$$\{P_a, P_b\} = 0. \tag{3.3}$$

Restriction to an irreducible realization of $E(3)$ implies that the two invariants $P^2 > 0$ and $\mathbf{J} \cdot \mathbf{P}$ are given by two preassigned real numbers. Since the multiplication of P_a by a constant does not change the Poisson-bracket relations (3.1)–(3.3), we can normalize P_a such that $P_a P_a = 1$. We then define the "phase space of $E(3)$" as the set of all pairs of real vectors **P** and **J** obeying the constraints

$$P^2 = P_a P_a = 1; \quad \mathbf{J} \cdot \mathbf{P} = J_a P_a = \alpha_0. \tag{3.4}$$

Because of these two constraints the phase space is now a four-dimensional space. However, we will continue to label the points of this phase space by two vectors **P** and **J**. The finite canonical transformation generated by J_a and P_a are mappings of this phase space on to itself, and provide a realization of the group $E(3)$.

Under a finite canonical transformation generated by J_a, the point (P_a, J_a) is mapped to a new point (P'_a, J'_b) as follows:

$$\begin{aligned} P'_a &= \exp(\widetilde{\mathbf{n} \cdot \mathbf{J}}) P_a = R_{ba}(\mathbf{n}) P_b, \\ J'_a &= \exp(\widetilde{\mathbf{n} \cdot \mathbf{J}}) J_a = R_{ba}(\mathbf{n}) J_b. \end{aligned} \tag{3.5}$$

By $\exp(\tilde{\phi}) \cdot f$ we mean the following infinite series

$$\exp(\tilde{\phi}) \cdot f = f + \{\phi, f\} + (1/2!)\{\phi, \{\phi, f\}\} + \cdots,$$

where ϕ and f are arbitrary functions of the dynamical variables. Here the vector $\mathbf{n}$ specifies the transformation and $R(\mathbf{n})$ is the real orthogonal matrix corresponding to a rotation by an angle $|\mathbf{n}| = n$ about an axis in the direction of $\mathbf{n}$:

$$R_{ba}(\mathbf{n}) = \cos n \delta_{ba} + \frac{(1 - \cos n)}{n^2} n_b n_a + \frac{\sin n}{n} \epsilon_{abc} n_c$$

It is easy to verify that P'_a, J'_a obey the same Poisson-bracket relations as P_a, J_a, so that the above transformation is in fact a canonical transformation. We denote this transformation by $(\mathbf{0}, R)$. Next consider a finite transformation generated by P_a:

$$\begin{aligned} P'_a &= \exp(\widetilde{\boldsymbol{\lambda} \cdot \mathbf{P}}) P_a = P_a, \\ J'_a &= \exp(\widetilde{\boldsymbol{\lambda} \cdot \mathbf{P}}) J_a = J_a + \epsilon_{abc} P_b \lambda_c. \end{aligned} \tag{3.7}$$

We denote this transformation by $(\boldsymbol{\lambda}, 1)$. It can easily be verified that this transformation is also canonical. A general element of the group $E(3)$ is represented by the canonical transformation $(\boldsymbol{\lambda}, R)$ obtained by performing $(\mathbf{0}, R)$ first and $(\boldsymbol{\lambda}, \mathbf{1})$ next:

$$(\boldsymbol{\lambda}, R) \equiv (\boldsymbol{\lambda}, \mathbf{1})(\mathbf{0}, R),$$
$$P'_a = R_{ba}P_b, \qquad (3.8)$$
$$J'_a = R_{ba}(J_b + \epsilon_{bcd}P_c\lambda_d).$$

We have the following composition law:

$$(\boldsymbol{\lambda}', R')(\boldsymbol{\lambda}, R) = (\boldsymbol{\lambda}' + R'\boldsymbol{\lambda}, R'R), \qquad (3.9)$$

where $R'\boldsymbol{\lambda}$ is the vector whose components are $R_{ab}\lambda_b$, $(a = 1, 2, 3)$. These finite canonical transformations map the entire phase space of $E(3)$ onto itself, and there is no region in the phase space which is invariant under all the transformations $(\boldsymbol{\lambda}, R)$. Further, we may verify that every transformation $(\boldsymbol{\lambda}, R)$ preserves the restriction $J^2 \geq \alpha_0^2$, which is implied by (3.4) and the reality of P_a and J_a.

We are now interested in constructing generators for $O(4)$ and other Lie algebras as analytic functions of P_a and J_a. We will find that, in general, these generators will be real in some regions of the underlying $E(3)$ phase space and complex in others. When we consider the finite canonical transformations arising from these generators, we can impose the following requirement: There exists some region in the phase space, which is mapped into itself under these canonical transformations; i.e., given any real point (P_a, J_a) in this region, the transformed point (P'_a, J'_a) also lies in this region, with P'_a and J'_a being real. We then obtain a representation of the group elements by means of finite real canonical transformations operating within this region of phase space. As is seen later, this requirement will, in general, impose further restrictions on the generators.

Let us begin with the construction of the $O(4)$ generators in items of the $E(3)$ generators. We choose to leave the $O(3)$ subalgebra unaltered, so that the first three generators are J_1, J_2, J_3. The most general form of the other three generators K_a $(a = 1, 2, 3)$ is given by

$$K_a = f_1P_a + f_2\epsilon_{abc}J_bP_c + gJ_a, \qquad (3.10)$$

where f_1, f_2, and g are functions of J^2, to be determined. Equations (2.6) and (2.7) are automatically satisfied because of (3.1)–(3.3), whereas if we impose (2.8), we get a set of first-order differential equations[3] involving f_1, f_2, and g in their dependence on J^2. For this purpose, we rewrite (2.8) in a slightly different, but equivalent, form:

$$\epsilon_{abc}\{K_b, K_c\} = 2J_a. \qquad (3.11)$$

[3] We follow the method of calculation to be found in H. Bacry, Nuovo Cimento 41A, 221 (1966).

If we substitute (3.10) in (3.11) and also use (1.1) and (3.1)–(3.4), we obtain, after some long but straightforward calculations, the following relation:

$$2J_a = 2J_a\{g^2 - f_2^2 - 2(f_1f_1' + J^2f_2f_2') - 2\alpha_0(f_1g' - \alpha_0f_2f_2')\} + 4P_af_1\{g + J^2g' + \alpha_0f_1'\} + 4\epsilon_{abc}J_bP_cf_2\{g + J^2g' + \alpha_0f_1'\},$$

where primes denote differentiation with respect to J^2. If we multiply this last equation by $(\alpha_0J_a - J^2P_a)$, $(J_a - \alpha_0P_a)$, and $\epsilon_{aef}J_eP_f$ and sum over a in the resulting three equations, we obtain

$$f_1(g + J^2g' + \alpha_0f_1') = 0, \qquad (3.12)$$

$$f_2(g + J^2g' + \alpha_0f_1') = 0, \qquad (3.13)$$

$$1 = g^2 - f_2^2 - 2(f_1f_1' + J^2f_2f_2') - 2\alpha_0(f_1g' - \alpha_0f_2f_2'), \qquad (3.14)$$

where we have assumed $J^2 - \alpha_0^2 \neq 0$. Apart from the trivial solution $f_1 = f_2 = 0$, $g = 1$ (i.e., $\mathbf{K} = \pm\mathbf{J}$), we obtain on solving (3.12)–(3.14) the following functional forms of f_1, f_2, and g:

$$f_1(J^2) = \left\{\frac{(\beta^2 - J^2)(J^2 - \alpha^2)}{(J^2 - \alpha_0^2)}\right\}^{\frac{1}{2}}\cos\{\Theta(J^2)\}, \qquad (3.15)$$

$$f_2(J^2) = \left\{\frac{(\beta^2 - J^2)(J^2 - \alpha^2)}{\alpha_0^2)}\right\}^{\frac{1}{2}}\sin\{\Theta(J^2)\}, \qquad (3.16)$$

$$g(J^2) = \frac{\alpha\beta}{J^2} \qquad \left\{\frac{(\beta^2 - J^2)(J^2 - \alpha^2)}{(J^2 - \alpha_0^2)}\right\}^{\frac{1}{2}}\cos\{\Theta(J^2)\} \qquad (3.17)$$

Here $\Theta(J^2)$ is an arbitrary function of J^2, and α and β are two real constants. This is the most general solution of (3.10), since the invariants

$$J^2 + K^2 = \alpha^2 + \beta^2 \qquad (3.18)$$

and

$$\mathbf{J}\cdot\mathbf{K} = \alpha\beta \qquad (3.19)$$

can be assigned independently. Without loss of generality, we can assume $\beta \geq |\alpha| \geq 0$. In analogy with the quantum-mechanical case, we call the $O(4)$ representation with $\mathbf{J}\cdot\mathbf{K} = 0$, the "symmetric traceless tensor representation."

We now discuss the above solution in some detail. We note first that, according to (3.18) and (3.19), the two invariants are entirely independent of the function Θ. It should therefore be possible to trace the arbitrariness associated with Θ to a freedom in the choice of the form of the generators. This is in fact true. The arbitrary function Θ simply reflects the freedom to make canonical transformations generated by arbitrary functions $\phi(J^2)$ of J^2:

$$\begin{aligned} J_a \to J'_a &= \exp[\widetilde{\phi(J^2)}]J_a = J_a, \\ P_a \to P'_a &= \exp[\widetilde{\phi(J^2)}]P_a, \\ &= \cos[2(J^2)^{\frac{1}{2}}\phi']P_a \\ &+ \frac{1-\cos[2(J^2)^{\frac{1}{2}}\phi']}{J^2}(\mathbf{J}\cdot\mathbf{P})J_a \\ &+ \frac{\sin[2(J^2)^{\frac{1}{2}}\phi']}{(J^2)^{\frac{1}{2}}}\epsilon_{abc}P_bJ_c \qquad (3.20) \end{aligned}$$

By a proper choice of the function ϕ, we can eliminate Θ altogether and write the generators K_a in the form

$$K_a = \left\{\frac{(\beta^2 - J^2)(J^2 - \alpha^2)}{(J^2 - \alpha_0^2)}\right\}^{\frac{1}{2}} P_a + \left[\alpha\beta - \alpha_0\left\{\frac{(\beta^2 - J^2)(J^2 - \alpha^2)}{(J^2 - \alpha_0^2)}\right\}^{\frac{1}{2}}\right]\frac{J_a}{J^2}. \qquad (3.21)$$

Next we consider the reality properties of K_a and of the canonical transformations generated by K_a. Since $J^2 \geq \alpha_0^2$ in the phase space of $E(3)$, we must choose $\beta^2 > \alpha_0^2$ in order to have some region,

$$\beta^2 \geq J^2 \geq \max(\alpha^2, \alpha_0^2),$$

where K_a is real. To discuss the nature of the canonical transformations $(P_a, J_a) \to (P'_a, J'_a)$, we first rewrite (3.21) in the following compact form:

$$\mathbf{K} = \left\{\frac{(\beta^2 - J^2)(J^2 - \alpha^2)}{(J^2 - \alpha_0^2)}\right\}^{\frac{1}{2}} \frac{(\mathbf{J}\times\mathbf{P}\times\mathbf{J})}{J^2} + \frac{\alpha\beta}{J^2}\mathbf{J}, \qquad (3.22)$$

from which we can express $\mathbf{P}$ in terms[4] of $\mathbf{K}$ and $\mathbf{J}$:

$$\mathbf{P} = \left\{\frac{(J^2 - \alpha_0^2)}{(\beta^2 - J^2)(J^2 - \alpha^2)}\right\}^{\frac{1}{2}} \frac{(\mathbf{J}\times\mathbf{K}\times\mathbf{J})}{J^2} + \frac{\alpha_0}{J^2}\mathbf{J}. \qquad (3.23)$$

Thus, given real vectors $\mathbf{J}$, $\mathbf{P}$ with

$$\beta^2 \geq J^2 \geq \max(\alpha^2, \alpha_0^2)$$

(and $P^2 = 1$, $\mathbf{J}\cdot\mathbf{P} = \alpha_0$), $\mathbf{K}$ is real; and conversely, given real vectors $\mathbf{J}$, $\mathbf{K}$ with $\beta^2 \geq J^2 \geq \max(\alpha^2, \alpha_0^2)$ (and $J^2 + K^2 = \alpha^2 + \beta^2$, $\mathbf{J}\cdot\mathbf{K} = \alpha\beta$), $\mathbf{P}$ is real. The finite canonical transformations generated by $\mathbf{J}$ and $\mathbf{K}$ can be shown to represent orthogonal rotations on the two vectors $\mathbf{J} \pm \mathbf{K}$:

$$\begin{aligned} J_a \pm K_a \to J'_a \pm K'_a &= \exp(\widetilde{\mathbf{n}\cdot\mathbf{J}})(J_a \pm K_a) \\ &= R_{ba}(\mathbf{n})(J_b \pm K_b), \qquad (3.24) \\ J_a \pm K_a \to J'_a \pm K'_a &= \exp(\widetilde{\boldsymbol{\lambda}\cdot\mathbf{K}})(J_a \pm K_a) \\ &= R_{ba}(\pm\boldsymbol{\lambda})(J_b \pm K_b), \qquad (3.25) \end{aligned}$$

where R_{ba} is the orthogonal matrix given by (3.6). The magnitudes of these vectors are thus fixed:

$$(\mathbf{J} \pm \mathbf{K})^2 = (\beta \pm \alpha)^2.$$

Therefore, if we start with a point (P_a, J_a) with $\beta^2 \geq J^2 \geq \max(\alpha^2, \alpha_0^2)$, then, by choosing an appropriate canonical transformation in $O(4)$ of the type (3.25), we get an image (P'_a, J'_a) with J'^2 lying anywhere between the values[5] β^2 and α^2 (i.e., $\beta^2 \geq J'^2 \geq \alpha^2$). However, if P_a is real, we also have $J'^2 \geq \max(\alpha^2, \alpha_0^2)$, and therefore $\alpha^2 \geq \alpha_0^2$.

We have then the following result: Given the phase space of $E(3)$ with a certain α_0, the generators K_a of (3.21), for $\beta \geq |\alpha| \geq |\alpha_0|$, are real in the region $\beta^2 \geq J^2 \geq \alpha^2$. The finite canonical transformation generated by J_a and K_a map this region into itself and provide a representation of the group $O(4)$. If $|\alpha| < |\alpha_0|$, the K_a are real in the region $\beta^2 \geq J^2 \geq \alpha_0^2$; however, in this case there is no region in the phase space which is invariant under the transformation generated by K_a. [Alternatively, for every finite transformation generated by K_a, there exist some real points (P_a, J_a) which are carried into points (P'_a, J'_a) with complex P'_a.] As a particular case we see that, in order to obtain a real representation of $O(4)$ of the "symmetric traceless tensor" type with $\alpha = 0$, we must start with $\alpha_0 = 0$.

Let us now consider the region of the phase space where the generators (3.21) are not real. The generators K_a become complex outside the region $\beta^2 \geq J^2 \geq \alpha^2$ (with $\beta \geq |\alpha| > |\alpha_0|$), and we do not have a real realization of any Lie algebra. If, however, we make an "analytic continuation" of the parameter α to pure imaginary values, we can generate real realizations of the $O(3, 1)$ Lie algebra in the region $J^2 \geq \beta^2$. We define $K'_a = iK_a$ and simultaneously put $\alpha' = i\alpha$ (with α' real), and obtain, from (3.21), the following expression for K'_a:

$$K'_a = \left\{\frac{(J^2 - \beta^2)(J^2 + \alpha'^2)}{(J^2 - \alpha_0^2)}\right\}^{\frac{1}{2}} P_a + \left[\frac{\alpha'\beta}{J^2} - \frac{\alpha_0}{J^2}\left\{\frac{(J^2 - \beta^2)(J^2 + \alpha'^2)}{(J^2 - \alpha_0^2)}\right\}^{\frac{1}{2}}\right] J_a, \qquad (3.27)$$

which is real for $J^2 \geq \beta^2$ (remember also that $\beta^2 \geq \alpha_0^2$). Since J_a and K'_a now satisfy the Poisson-bracket relations (2.11)–(2.13), they generate the

[4] Since we have obtained an expression for $\mathbf{P}$ in terms of $\mathbf{K}$ and $\mathbf{J}$, we can, in all of the present discussion, replace $\mathbf{P}$ by (3.23) and thus obtain realizations of various Lie algebras in terms of analytic functions of $\mathbf{K}$ and $\mathbf{J}$, the generators of $O(4)$.

[5] The fact that β^2 and α^2 are the maximum and minimum values, respectively, attained by J^2 under the finite canonical transformations generated by $\mathbf{K}$ can also be seen from the relation

$$0 = \delta J^2 = \exp(\widetilde{\delta\boldsymbol{\lambda}\cdot\mathbf{K}})J^2 - J^2 = (\mathbf{J}\times\mathbf{K})\cdot\delta\boldsymbol{\lambda}$$

and (3.22).

$O(3, 1)$ Lie algebra. The two invariants in this case are

$$J^2 - K'^2 = \beta^2 - \alpha'^2, \quad (3.28)$$

$$\mathbf{J} \cdot \mathbf{K}' = \beta\alpha'. \quad (3.29)$$

We can evaluate the minimum value attained by J^2 under finite canonical transformation generated by $\mathbf{K}'$. Let (P_a, J_a) be the point with a minimum value of J^2. Under an infinitesimal transformation generated by $\mathbf{K}'$, we have

$$\delta(J^2) = 2\mathbf{J} \cdot \delta\mathbf{J} = 2J_a\{\delta\boldsymbol{\lambda} \cdot \mathbf{K}', J_a\} = 2\delta\boldsymbol{\lambda} \cdot (\mathbf{J} \times \mathbf{K}'). \quad (3.30)$$

For J^2 to be minimum, $\delta(J^2) = 0$, and we then find from (3.28)–(3.30) that this minimum value is β^2. Thus, under the finite canonical transformations generated by $\mathbf{J}$ and $\mathbf{K}'$, the region $J^2 \geq \beta^2$ is mapped into itself and we have a real realization of the group $O(3, 1)$.

The above discussion also holds for the case $\alpha = \alpha_0 = 0$, i.e., the generators J_a and K_a' given by

$$K_a' = \{J^2 - \beta^2\}^{\frac{1}{2}} P_a \quad (3.31)$$

provide a real realization of the group $O(3, 1)$ in the region $J^2 \geq \beta^2$. We note, however, that if we replace β^2 by $-\beta^2$ in (3.31), we obtain a realization of the $O(3, 1)$ Lie algebra, real over the entire phase space of $E(3)$. In this case, the entire phase space is mapped into itself under all finite canonical transformations generated by $\mathbf{J}$, or $\mathbf{K}' = \{J^2 + \beta^2\}^{\frac{1}{2}} P_a$.

To summarize, then, starting with an $E(3)$ realization with a given α_0 and with the parameters β, α obeying $\beta \geq |\alpha| \geq |\alpha_0|$, J_a and K_a of (3.21) generate a real realization of the group $O(4)$ in the region $\beta^2 \geq J^2 \geq \alpha^2$. Outside this region, the K_a are complex. One can analytically continue α to $\alpha' = i\alpha$ with α' real, and obtain K_a' of Eq. (3.27), which together with J_a generate a real realization of the group $O(3, 1)$ in the region $J^2 \geq \beta^2$. If $\alpha = \alpha_0 = 0$, we also have a realization of the group $O(3, 1)$, real over the entire phase space.

We conclude this section by considering the "contraction" of the $O(4)$ generators of the equation (3.21) to yield new generators of $E(3)$. For this purpose we set

$$K_a = \beta \bar{P}_a \quad (1 \geq a \geq 3), \quad (3.32)$$

and take the limit $\beta \to \infty$, keeping α fixed. If we substitute (3.32) in the Poisson-bracket relations (2.6)–(2.8) and take the limit $\beta \to \infty$, we find that J_a and $\bar{P}_a$ satisfy (2.1)–(2.3), required of the generators of $E(3)$. Thus J_a and

$$\bar{P}_a = \left\{\frac{(J^2 - \alpha^2)}{(J^2 - \alpha_0^2)}\right\}^{\frac{1}{2}} P_a + \left[\frac{\alpha}{J^2} - \frac{\alpha_0}{J^2}\left\{\frac{(J^2 - \alpha^2)}{(J^2 - \alpha_0^2)}\right\}^{\frac{1}{2}}\right] J_a, \quad (3.33)$$

obtained from (3.32) and (3.21) in the limit $\beta \to \infty$, provide a realization of the $E(3)$ Lie algebra. Thus, starting with a realization of $E(3)$ with the generator P_a, J_a and invariants $P^2 = 1$, $\mathbf{J} \cdot \mathbf{P} = \alpha_0$, we have exhibited a realization of $E(3)$ Lie algebra by $\bar{P}_a$, J_a with arbitrary value α for $\mathbf{J} \cdot \bar{\mathbf{P}}$ (and $\bar{P}^2 = 1$). It must be noted that since the value of the invariant $\mathbf{J} \cdot \mathbf{P}$ is changed, (3.33) does not represent a canonical transformation generated by any function of $\mathbf{J}$ or $\mathbf{P}$. It may be seen that the minimum value reached by J^2 under finite canonical transformation generated by $\bar{\mathbf{P}}$ is α^2. Hence, if $|\alpha| \geq |\alpha_0|$, the region $J^2 \geq \alpha^2$, where $\bar{P}_a$ is real, is mapped into itself by the finite canonical transformation generated by $\bar{\mathbf{P}}$ or $\mathbf{J}$, and thus we obtain a real realization of $E(3)$ in this region. On the other hand, if $|\alpha| < |\alpha_0|$, for every finite canonical transformation generated by $\bar{\mathbf{P}}$, there are some real points (P_a, J_a) which are carried into image points (P_a', J_a') with complex P_a'.

4. REALIZATION OF $SU(3)$ AND $SL(3, R)$ LIE ALGEBRAS BY ANALYTIC FUNCTIONS OF P AND J

In this section we discuss the realization of $SU(3)$ and $SL(3, R)$ Lie algebras from the generalized enveloping algebra of $E(3)$. The $O(3)$ subalgebra of $SU(3)$, $SL(3, R)$, and $E(3)$ will be taken to be identical, i.e., three of the generators of $SU(3)$ and $SL(3, R)$ are chosen to be J_a ($a = 1, 2, 3$). The other five generators of $SU(3)$ and $SL(3, R)$ (i.e., the symmetric traceless tensors Q_{ab} and Q_{ab}', respectively,) are to be determined as functions of $\mathbf{P}$ and $\mathbf{J}$.

It is known that, in general, the unitary irreducible (matrix) representations of the group $SU(3)$ are not only reducible with respect to $O(3)$, but the same $O(3)$ representation may appear more than once.[6] However, for the special class of "completely symmetric tensor" representations, there is no such multiplicity, and only states with the same parity occur. We restrict our considerations to Poisson-bracket "symmetric tensor realizations" of $SU(3)$ and $SL(3, R)$ with even parity [i.e., $Q_{ab}(\mathbf{P}) = Q_{ab}(-\mathbf{P})$ and $Q_{ab}'(\mathbf{P}) = Q_{ab}'(-\mathbf{P})$] from amongst the elements of the generalized enveloping algebra of $E(3)$.

We show below [Eq. (4.26)] that, in any real representation of the group $SU(3)$ with the generator J_a, Q_{ab}, the minimum value of J^2 is always zero. A similar result also holds for $SL(3, R)$. From the examples discussed up to now, we can then conclude that we may restrict ourselves to an underlying $E(3)$ realization with

$$\mathbf{J} \cdot \mathbf{P} = \alpha_0 = 0; \quad P^2 = 1. \quad (4.1)$$

[6] See, for instance, G. Racah, Rev. Mod. Phys. 21, 494 (1949); V. Bargmann and M. Moshinsky, Nucl. Phys. 23, 177 (1961).

If the symmetric traceless tensor Q_{ab} is to be constructed from the vectors **P** and **J** satisfying (4.1), then the most general form for Q_{ab} with even parity $Q_{ab}(\mathbf{P}) = Q_{ab}(-\mathbf{P})$ is given by

$$Q_{ab} = f_0 \cdot \{P_aP_b - \tfrac{1}{3}\delta_{ab}\} + g_0 \cdot \left(\frac{J_aJ_b}{J^2} - \tfrac{1}{3}\delta_{ab}\right) + h_0\{(\mathbf{J}\times\mathbf{P})_aP_b + P_a(\mathbf{J}\times\mathbf{P})_b\}, \quad (4.2)$$

where f_0, g_0, and h_0 are some functions of J^2 to be determined. The absence of a term proportional to

$$(\mathbf{J}\times\mathbf{P})_a(\mathbf{J}\times\mathbf{P})_b - \tfrac{1}{3}J^2\delta_{ab} \equiv \epsilon_{acd}\epsilon_{bef}J_cP_dJ_eP_f - \tfrac{1}{3}J^2\delta_{ab} \quad (4.3)$$

in (4.2) is no loss of generality, since such a term can be re-expressed in terms of δ_{ab}, J_aJ_b, and P_aP_b, which are included in (4.2). This can be seen immediately, if we make use of the identity

$$\epsilon_{acd}\epsilon_{bef} = \delta_{ab}\delta_{ce}\delta_{df} + \delta_{ae}\delta_{cf}\delta_{db} + \delta_{af}\delta_{cb}\delta_{de} - \delta_{ab}\delta_{cf}\delta_{de} - \delta_{ae}\delta_{cb}\delta_{df} - \delta_{af}\delta_{ce}\delta_{db}. \quad (4.4)$$

In order to determine the three functions f_0, g_0, h_0 we must impose the Poisson-bracket relations (2.19). [The relations (2.16)–(2.18) are automatically satisfied with the choice (4.2).] However, it is possible to simplify Q_{ab} first by means of a canonical transformation of the form (3.20), leaving J_a invariant. As we show below, a proper choice of the function $\phi(J^2)$ in (3.20) can eliminate the term proportional to

$$(\mathbf{J}\times\mathbf{P})_aP_b + (\mathbf{J}\times\mathbf{P})_bP_a$$

in (4.2). For this purpose, we rewrite (4.2) with P_a, J_a replaced by P'_a, J'_a:

$$Q_{ab} = f_0(P'_aP'_b - \tfrac{1}{3}\delta_{ab}) + g_0((J'_aJ'_b/J^2) - \tfrac{1}{3}\delta_{ab}) + h_0\{(\mathbf{J}'\times\mathbf{P}')_aP'_b + (\mathbf{J}'\times\mathbf{P}')_bP'_a\}, \quad (4.5)$$

where $\mathbf{P}'$ and $\mathbf{J}'$ are given by (3.20), with $\mathbf{J}\cdot\mathbf{P}$, i.e.,

$$J'_a = \exp\{\widetilde{\phi(J^2)}\}J_a = J_a,$$
$$P'_a = \exp\{\widetilde{\phi(J^2)}\}P_a, = \cos\Theta P_a - [\sin\Theta/(J^2)^{\frac{1}{2}}](\mathbf{J}\times\mathbf{P})_a, \quad (4.6)$$

where $\Theta = 2(J^2)^{\frac{1}{2}}\, d\phi(J^2)/d(J^2)$. If we substitute (4.6) in (4.5), we obtain

$$Q_{ab} = f(P_aP_b - \tfrac{1}{3}\delta_{ab}) + g\{(J_aJ_b/J^2) - \tfrac{1}{3}\delta_{ab}\} + h\{(\mathbf{J}\times\mathbf{P})_aP_b + (\mathbf{J}\times P)_bP_a\}, \quad (4.7)$$

where

$$f = f_0\cos 2\Theta + 2h_0(J^2)^{\frac{1}{2}}\sin 2\Theta, \quad (4.8)$$

$$g = g_0 - f_0\sin^2\Theta + h_0(J^2)^{\frac{1}{2}}\sin 2\Theta, \quad (4.9)$$

$$h = h_0\cos 2\Theta - [f_0/2(J^2)^{\frac{1}{2}}]\sin 2\Theta. \quad (4.10)$$

We now choose $\Theta = \tfrac{1}{2}\tan^{-1}[2h_0(J^2)^{\frac{1}{2}}/f_0]$ so that $h = 0$, and (4.7) then shows that we can restrict our considerations to the case when Q_{ab} is of the form

$$Q_{ab} = f(P_aP_b - \tfrac{1}{3}\delta_{ab}) + g[(J_aJ_b/J^2) - \tfrac{1}{3}\delta_{ab}]. \quad (4.11)$$

The functional dependence of f and g on J^2 is to be determined from the Poisson-bracket relation (2.19). We deduce from (2.19) the essentially equivalent relation

$$\epsilon_{hde}\{Q_{ah}, Q_{cd}\} = 4\delta_{ac}J_e - \delta_{ce}J_a - \delta_{ae}J_c \quad (4.12)$$

and substitute (4.11) in (4.12). If we also use Eqs. (1.1), (3.1)–(3.3), and (4.1), we find, after long but straightforward calculations, that

$$\begin{aligned}4\delta_{ac}J_e - (\delta_{ce}J_a + \delta_{ae}J_c) &= -\tfrac{4}{3}f(f' + g')\delta_{ac}J_e \\ &\quad + [4/(J^2)^2](J^2fg' - fg + g^2)J_aJ_cJ_e \\ &\quad + \left\{\tfrac{1}{3}f(f' + g') + \frac{fg}{J^2} - \frac{g^2}{J^2} - 2fg'\right\} \\ &\quad \times (\delta_{ce}J_a + \delta_{ae}J_c) + \tfrac{4}{3}(2fg' - ff') \\ &\quad \times \{P_aP_cP_e + (P_aJ_c + J_aP_c)J_e\}, \quad (4.13)\end{aligned}$$

where primes again denote differentiation with respect to J^2. Equating the coefficients of the various terms, we find that (4.13) is satisfied if and only if

$$f(f' + g') = -3, \quad (4.14)$$

$$f(f' - 2g') = 0, \quad (4.15)$$

and

$$g(g - f) = J^2. \quad (4.16)$$

It is possible to find a solution of the three equations (4.14)–(4.16) for the two functions f, g:

$$f = (\beta^2 - 4J^2)^{\frac{1}{2}}, \quad (4.17)$$

$$g = \tfrac{1}{2}(\beta^2 - 4J^2)^{\frac{1}{2}} \pm \tfrac{1}{2}\beta, \quad (4.18)$$

where β is some arbitrary constant. From (4.11), (4.17), and (4.18), we thus obtain

$$Q_{ab} = (\beta^2 - 4J^2)^{\frac{1}{2}}[P_aP_b + (J_aJ_b/2J^2) - \tfrac{1}{2}\delta_{ab}] \pm \tfrac{1}{2}\beta[(J_aJ_b/J^2) - \tfrac{1}{3}\delta_{ab}]. \quad (4.19)$$

This solution for Q_{ab} obeys the Poisson-bracket relation (2.19). The parameter β and the ambiguity of the sign in (4.19) are related to the quadratic and cubic invariants [cf. (2.25), (2.26)]

$$J_aJ_a + \tfrac{1}{2}Q_{ab}Q_{ab} = \tfrac{1}{3}\beta^2, \quad (4.20)$$

$$\tfrac{1}{2}\sqrt{3}\,\{3J_aQ_{ab}J_b - Q_{ab}Q_{bc}Q_{ca}\} = \pm(\beta^3/3\sqrt{3}). \quad (4.21)$$

As mentioned in Sec. 2, we note that the cube of the quadratic invariant is equal to the square of the cubic invariant.

The generators Q_{ab} in (4.19) are real in the region $J^2 \le \tfrac{1}{4}\beta^2$. In order to discuss the finite transformations generated by J_a and Q_{ab}, we first evaluate the maximum and minimum values attained by J^2 under these transformations.

Let us evaluate the minimum value of J^2. If we just use the Poisson-bracket relations (2.17)–(2.19), we see that, under the finite canonical transformations generated by J_a or Q_{ab}, these quantities go over into certain linear combinations of themselves. In fact, J_a and Q_{ab} transform according to the eight-dimensional adjoint representation of $SU(3)$.[7] To exhibit these transformations we introduce three antisymmetric Hermitian (3×3) matrices Λ^a ($a = 1, 2, 3$):

$$\Lambda^1 = i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \Lambda^2 = i\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\Lambda^3 = i\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.22}$$

and the Hermitian symmetric traceless matrices Λ^{ab}:

$$\Lambda^{ab} = \tfrac{1}{2}(\Lambda^a\Lambda^b + \Lambda^b\Lambda^a) - \tfrac{2}{3}\delta_{ab}\mathbf{1}, \tag{4.23}$$

of which only five are linearly independent. Together, Λ^a and Λ^{ab} form a basis for traceless Hermitian (3×3) matrices. Given the variables J_a, Q_{ab}, we form the Hermitian matrix

$$A = J_a\Lambda^a + \tfrac{1}{2}Q_{ab}\Lambda^{ab}. \tag{4.24}$$

Now let U be any unitary unimodular matrix. The matrix $A' = UAU^{-1}$ can also be expanded linearly in terms of Λ^a and Λ^{ab}, i.e.,

$$A' = UAU^{-1} = J_a'\Lambda^a + \tfrac{1}{2}Q_{ab}'\Lambda^{ab}, \tag{4.25}$$

where the coefficients J_a' and Q_{ab}' are linear combinations of J_a and Q_{ab}. By choosing all possible matrices U, we get precisely all those linear combinations J_a', Q_{ab}' that are obtained by performing all possible finite canonical transformations generated by J_a and Q_{ab} on themselves. Now, given any A, we can choose a U such that A' is diagonal. In that case J_a vanishes, since Λ^a are antisymmetric and Λ^{ab} are symmetric matrices. Thus, starting with any value of J_a, Q_{ab}, there exists a particular transformation generated by J_a, Q_{ab} which takes

$$J_a \to J_a' = 0. \tag{4.26}$$

This proves the statement that in any realization of $SU(3)$, the value $J^2 = 0$ is always attained and this is, of course, the minimum value.

Let us next evaluate the maximum value of J^2. Let (P_a, J_a) be the point where J^2 is maximum. We first perform an orthogonal rotation (3.5), generated by J_a, which leaves J^2 unchanged, such that

$$J_1 = J_2 = 0; \quad J_3 = (J^2)^{\frac{1}{2}}. \tag{4.27}$$

Now under an arbitrary infinitesimal transformation generated by Q_{ab},

$$J_a \to \exp\,(\widetilde{\delta\lambda_{bc}\cdot Q_{bc}})J_a, \tag{4.28}$$

where the $\delta\lambda_{bc}$ are arbitrary, we must have

$$\delta(J^2) = 0. \tag{4.29}$$

Using the Poisson-bracket relation (2.18), we obtain

$$\begin{aligned}\delta(J^2) &= 2J_a(\delta J_a) = 2J_a\{\delta\lambda_{bc}Q_{bc}, J_a\} \\ &= 2J_a[\epsilon_{bad}Q_{dc} + \epsilon_{cad}Q_{bd}]\delta\lambda_{bc},\end{aligned} \tag{4.30}$$

so that (4.28) is satisfied if and only if

$$J_a(\epsilon_{bad}Q_{dc} + \epsilon_{cad}Q_{bd}) = 0. \tag{4.31}$$

If we use (4.27) and the fact that $Q_{ab} = Q_{ba}$, $Q_{cc} = 0$, in (4.31), we obtain the following form for Q_{ab}:

$$Q_{11} = Q_{22} = -\tfrac{1}{2}Q_{33} = q; \quad Q_{ab} = 0, \quad a \neq b, \tag{4.32}$$

where q is some constant. Using the special forms of J_a and Q_{ab}, as given by (4.27) and (4.32) in (4.20) and (4.21), we get the following two equations in J^2 and q:

$$J^2 + 3q^2 = \tfrac{1}{3}\beta^2, \tag{4.33}$$

$$-J^2q + q^3 = \pm\tfrac{1}{27}\beta^3, \tag{4.34}$$

with the solution $J^2 = 0$, $q = \pm\frac{1}{3}\beta$, or $J^2 = \frac{1}{4}\beta^2$, $q = \pm\frac{1}{6}\beta$. The former solution corresponds to the minimum value of J^2 [note that (4.29) is also satisfied when J^2 is minimum], which we already derived earlier, as the latter solution corresponds to the maximum value of J^2.

It is thus seen that the maximum and minimum values of J^2 are $\frac{1}{4}\beta^2$ and 0, respectively, which are just the boundaries of the region where Q_{ab} are real. We therefore conclude that not only are the Q_{ab} real in the region $0 \leq J^2 \leq \frac{1}{4}\beta^2$ of the $E(3)$ phase space, but, in fact, the finite canonical transformation generated by J_a, Q_{ab} carries this region into itself and provides a real realization of the group $SU(3)$.

We conclude this section by a brief discussion of the realization of the $SL(3, R)$ Lie algebra. We see from (4.19) that Q_{ab} become complex in the region $J^2 > \frac{1}{4}\beta^2$. We can, however, redefine the generators $Q_{ab}' = iQ_{ab}$ and simultaneously analytically continue β to $\beta' = i\beta$

[7] For any Lie group, the generators transform according to the adjoint representation of the group; for $SU(3)$, this is the octet or eight-dimensional representation. The matrices Λ^a, Λ^{ab} are the Hermitian generators of the three-dimensional representation of $SU(3)$. The generators Λ^a of the $O(3)$ subgroup correspond to the spin-1 representation of $O(3)$, and are here represented in the Cartesian form.

(with β' real) to get

$$Q'_{ab} = (\beta'^2 + 4J^2)^{\frac{1}{2}}\{P_aP_b + \tfrac{1}{2}(J_aJ_b/J^2) - \tfrac{1}{2}\delta_{ab}\} \pm \tfrac{1}{2}\beta((J_aJ_b/J^2) - \tfrac{1}{3}\delta_{ab}). \quad (4.35)$$

J_a and Q_{ab} now obey the Poisson-bracket relations (2.22)–(2.24) corresponding to the unimodular real linear group in three dimensions, $SL(3, R)$. The two invariants of this representation are

$$J_aJ_a \quad \tfrac{1}{2}Q'_{ab}Q'_{ab} = -\tfrac{1}{3}\beta'^2 \quad (4.36)$$

and

$$\tfrac{1}{2}\sqrt{3}\,(3J_aQ'_{ab}J_b + \tfrac{1}{3}Q'_{ab}Q'_{bc}Q'_{ca}) = \mp\beta^3/3\sqrt{3}. \quad (4.37)$$

The generators Q'_{ab} are real over the entire region $J^2 \geq 0$.

Let us now evaluate the limiting values attained by J^2 under finite canonical transformations generated by J_a and Q'_{ab}. Let (P_a, J_a) be the point where J^2 is stationary and let us first perform an orthogonal rotation generated by J_a (leaving J^2 unchanged) such that J_1, J_2, J_3 are given by (4.27). By following a strictly similar argument as was used to obtain the relation (4.32), we obtain in this case

$$Q'_{11} = Q'_{22} = -\tfrac{1}{2}Q''_{33} = q', \quad Q'_{ab} = 0, \quad a \neq b, \quad (4.38)$$

where q' is some constant. If we use the special forms of J_a and Q'_{ab} as given by (4.27) and (4.38) in (4.36) and (4.37), we obtain the following two equations in J^2 and q':

$$J^2 - 3q'^2 = \tfrac{1}{3}\beta'^2, \quad (4.39)$$

$$J^2q' + q'^3 = \pm\tfrac{1}{27}\beta^3, \quad (4.40)$$

with the only real solution $J^2 = 0$, $q' = \pm\frac{1}{3}\beta'$.

Hence we conclude that J_a and Q'_{ab} generate real finite canonical transformations mapping the entire phase space into itself and providing a realization of the group $SL(3, R)$.

5. GENERALIZATION TO n DIMENSIONS

So far we have restricted our attention to the realization of some Lie algebras in terms of the elements of a generalized enveloping algebra of $E(3)$. In this section, we wish to make some comments about its generalization to n dimensions. There are some special features of the $E(3)$ and $O(4)$ algebras which do not generalize to higher dimensions. However, the symmetric-tensor-type realizations permit an immediate extension to arbitrary dimensions.

We start with the symmetric-tensor realization of $E(n)$, $(n \geq 3)$, with the generators P_a, J_{ab}, $[J_{ab} = -J_{ba}; a, b = 1, 2, \cdots, n]$, which obey the Poisson-bracket relations

$$\{J_{ab}, J_{cd}\} = \delta_{ac}J_{bd} + \delta_{bd}J_{ac} - \delta_{bc}J_{ad} - \delta_{ad}J_{bc}, \quad (5.1)$$

$$\{J_{ab}, P_c\} = \delta_{ac}P_b - \delta_{bc}P_a, \quad (5.2)$$

$$\{P_a, P_b\} = 0. \quad (5.3)$$

We will restrict our discussion to the case when the totally antisymmetric tensors

$$H_{abc} = P_aJ_{bc} + P_bJ_{ca} + P_cJ_{ab} \quad (5.4)$$

and

$$G_{abcd} = J_{ab}J_{cd} + J_{ac}J_{db} + J_{ad}J_{bc} \quad (5.5)$$

identically vanish. For $n = 3$, the constraint (5.5) is empty, whereas the constraint (5.4) reduces to $\mathbf{J}\cdot\mathbf{P} = 0$, which implies that the "helicity" is zero. We also choose the normalization such that

$$P^2 = 1. \quad (5.6)$$

which is always permissible since the multiplication by a constant does not change the Poisson-bracket relations (5.1)–(5.3).

The generators J_{ab} and K_a of $O(n + 1)$ obey the Poisson-bracket relations

$$\{J_{ab}, K_c\} = \delta_{ac}K_b - \delta_{bc}K_a, \quad (5.7)$$

$$\{K_a, K_b\} = J_{ab}, \quad (5.8)$$

and the Poisson bracket of J_{ab} with J_{cd} is given by (5.1).

From (3.21), if we set $\alpha = \alpha_0 = 0$, we can immediately write down the generators K_a in terms of P_a and J_{ab} $(1 \leq a \leq n)$:

$$K_a = (\beta^2 - J^2)^{\frac{1}{2}}P_a, \quad (5.9)$$

where

$$J^2 = \tfrac{1}{2}J_{ab}J_{ab} \quad (5.10)$$

and β is related to the invariant

$$J^2 + K^2 = \tfrac{1}{2}J_{ab}J_{ab} + K_aK_a = \beta^2. \quad (5.11)$$

By direct calculations, it may be verified that the Poisson-bracket relations (5.7) and (5.8) are satisfied and therefore J_{ab} and K_a do serve as generators of $O(n + 1)$. Of course, the form of K_a can be changed, by finite canonical transformations generated by some function of J^2, to

$$K_a = (\beta^2 \quad J^2)^{\frac{1}{2}}\left\{P_a \cos\Theta + \frac{J_{ab}P_b}{J^2}\sin\Theta\right\}$$

The generator K_a is real in the region $J^2 \leq \beta^2$. For $J^2 > \beta^2$, K_a becomes pure imaginary. In this region

$$K'_a = iK_a = (J^2 - \beta^2)^{\frac{1}{2}}P_a \quad (5.13)$$

and J_{ad} generate a representation of the pseudo-orthogonal group $O(n, 1)$. The generators $K'_a = (J^2 + \beta^2)^{\frac{1}{2}}\,P_a$ and J_{ab} generate a representation of

$O(n, 1)$ which is real over the entire phase space. [The Poisson-bracket relations of K_a and J_{ab} are similar to (5.7) and (5.8) except for a change of sign in (5.8).]

The $SU(n)$ generators J_{ab}, Q_{ab} obey the Poisson-bracket relations

$$\{J_{ab}, Q_{cd}\} = +\delta_{ac}Q_{bd} + \delta_{ad}Q_{cb} - \delta_{cb}Q_{ad} - \delta_{bd}Q_{ca}, \tag{5.14}$$

$$\{Q_{ab}, Q_{cd}\} = \delta_{bd}J_{ac} + \delta_{ad}J_{bc} + \delta_{bc}J_{ad} + \delta_{ac}J_{bd}. \tag{5.15}$$

In analogy with (4.19) we can again write down Q_{ab} in terms of P_a and J_{ab}:

$$Q_{ab} = (\beta^2 - 4J^2)^{\frac{1}{2}}\left(P_aP_b - \frac{1}{n}\delta_{ab}\right) + \tfrac{1}{2}[(\beta^2 - 4J^2)^{\frac{1}{2}} \pm \beta]\left(\frac{2}{n}\delta_{ab} - \frac{J_{ac}J_{bc}}{J^2}\right), \tag{5.16}$$

where the parameter β is related to the invariant

$$J^2 + \tfrac{1}{2}Q_{ab}Q_{ab} = \tfrac{1}{2}(1 - n^{-1})\beta^2 \tag{5.17}$$

and J^2 is again given by (5.10). From the structure of Q_{ab} in (5.16), it is evident that the Poisson-bracket relation (5.14) is obeyed. It is expected that (5.15) also holds.

The generators Q_{ab} are real in the region $J^2 \leq \tfrac{1}{4}\beta^2$, and together with J_{ab} provide a real realization of $SU(n)$ in this region. For $J^2 > \tfrac{1}{4}\beta^2$, Q_{ab} of (5.16) become complex.

Defining $Q'_{ab} = iQ_{ab}$ and choosing $\beta' = i\beta$ (β' real) as before, we obtain from (5.16) the generators of a noncompact group $SL(n, R)$:

$$Q'_{ab} = (\beta'^2 + 4J^2)^{\frac{1}{2}}\left(P_aP_b - \frac{J_{ac}J_{bc}}{2J^2}\right) \pm \tfrac{1}{2}\beta\left(\frac{2}{n}\delta_{ab} - \frac{J_{ac}J_{bc}}{J^2}\right) \tag{5.18}$$

[Q'_{ab} and J_{ab} obey similar Poisson-bracket relations as (5.14), (5.15), except a change of sign in (5.15).] This realization is real over the entire phase space $J^2 > 0$.

It may be noted that for "symmetric tensor" realizations of $O(n + 1)$ and $SU(n)$ and similarly for $O(n, 1)$ and $SL(n, R)$, the quadratic invariants (5.11) and (5.17), respectively, essentially determine the realizations, except for the automorphism

$$J_{ab} \to J_{ab}, \quad K_a \to -K_a, \quad Q_{ab} \to -Q_{ab}, \tag{5.19}$$

which leave all even-degree invariants unaltered, but change the signs of all odd-degree invariants.

6. CONCLUDING REMARKS

We have discussed the realization of certain Lie algebras in terms of the generalized enveloping algebras of certain other prescribed Lie algebras. We now make a few comments on the realizations of Lie algebras in terms of canonical variables. Such constructions are important in connection with explicit realizations of the symmetry groups of Hamiltonian systems as well as in the noninvariance-group description of dynamical systems. The possibility of identifying a dynamical system (with particular emphasis on the quantum-theoretic formulation of particle physics) with the generalized enveloping algebra of a suitable Lie algebra has been discussed elsewhere by one of the authors.[8] The problem of the recovery of the canonical variables for the system is also an essential dynamical problem.

To give an example, consider the special, familiar case of the Kepler problem. The generators of the Euclidean noninvariance group $E(4)$ are, in this case, given by[9]

$$J_{ab} = \epsilon_{abc}J_c \quad (a, b, c = 1, 2, 3), \tag{6.1}$$

$$J_{a4} = B'_a \quad (a = 1, 2, 3), \tag{6.2}$$

$$P_a = K_a \quad (a = 1, 2, 3),$$

and

$$P_4 = S',$$

where

$$J_a = \epsilon_{abc}q_bp_c,$$

$$B'_a = [(-2H)^{\frac{1}{2}}q \cos\{(-2H)^{\frac{1}{2}}(\mathbf{q}\cdot\mathbf{p})\} - (\mathbf{q}\cdot\mathbf{p})\sin\{(-2H)^{\frac{1}{2}}(\mathbf{q}\cdot\mathbf{p})\}]p_a + [(1/q)\sin\{(-2H)^{\frac{1}{2}}(\mathbf{q}\cdot\mathbf{p})\}]q_a,$$

$$K_a = (-2H)^{-\frac{1}{2}}[(\mathbf{q}\cdot\mathbf{p})p_a - p^2q_a + (e/q)q_a],$$

$$S' = -(-2H)^{\frac{1}{2}}(\mathbf{q}\cdot\mathbf{p})\sin[(-2H)^{\frac{1}{2}}(\mathbf{q}\cdot\mathbf{p})] + (1 + 2Hq)\cos[(-2H)^{\frac{1}{2}}(\mathbf{q}\cdot\mathbf{p})],$$

$$q = (q_aq_a)^{\frac{1}{2}}; \quad H = \tfrac{1}{2}p_ap_a - e/q.$$

Given these ten generators, one could construct the primitive dynamical variables. These points are discussed in more detail in a paper by two of us.[9] We only mention that such a construction yields

[8] E. C. G. Sudarshan, "Currents, Algebras and Dynamical Systems," invited paper at the Eastern Theoretical Physics Conference, Stony Brook, Long Island, New York, 1965.

[9] E. C. G. Sudarshan and N. Mukunda, in *Lectures in Theoretical Physics* (University of Colorado Press, Boulder, Colorado, 1966), Vol. VIII-B, p. 407.

expressions which are undefined (singular) when the Hamiltonian H vanishes. The Hamiltonian H only plays an auxiliary role in the construction of the canonical variables. We could equally well write down some functions of these ten generators which satisfy canonical Poisson-bracket relations. In another publication,[10] one of the authors has discussed the construction of n pairs of canonical variables from the generalized enveloping algebra of the classical groups $SU(n + 1)$ and $O(n + 2)$.

[10] N. Mukunda, J. Math. Phys. 8, 1069 (1967).