RELATIVISTIC PARTICLE SPECTRA AND MASS SPLITTINGS*

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We show how to do three things: to construct a relativistic Lie algebra whose irreducible representations contain infinite sequences of particles with spins and isospins which can be correlated as in the isobar spectrum of strong coupling theory; to obtain a mass spectrum for such isobars from an associative algebra structure; and to show how such an associative algebra has an underlying structure in terms of a finite-parameter Lie algebra.

The problem of understanding the spectrum of particle multiplets with their characteristic spins, parities, internal symmetry labels, and masses, within a group-theoretical framework, has been of much interest during the past few years. Attempts at a direct relativistic generalization [1] of the non-relativistic SU(6) theory encountered certain basic difficulties [2] and have been largely abandoned. In this letter we present a Lie algebraic structure which incorporates relativity and leads in a simple manner, to some general features of the hadron spectrum, including the correlation between spin and internal symmetry as well as the gross features of the mass spectrum. It is possible that the question of mass splittings in supermultiplets is somewhat different from that of generating a spectrum of particle states with the proper sequence of spins and internal symmetry labels. For clarity of presentation we discuss first the case of mass degenerate multiplets, treating in turn three different schemes. We come back to the question of the mass spectrum in the latter part of this letter.

Mass degenerate multiplets. Consider the fourteen-parameter Lie algebra $\mathcal{L}_I$ generated by $P_\mu, J_{\mu\nu}, \eta_\mu$ ($\mu = 0, 1, 2, 3$), with the commutation relations

$$[J_{\mu\nu}, J_{\sigma\tau}] = i\{g_{\nu\sigma}J_{\mu\tau} - g_{\mu\sigma}J_{\nu\tau} + g_{\nu\tau}J_{\sigma\mu} - g_{\mu\tau}J_{\sigma\nu}\} \quad (1)$$

$$[J_{\mu\nu}, P_\sigma] = i\{g_{\nu\sigma}P_\mu - g_{\mu\sigma}P_\nu\} \quad (2)$$

$$P_\mu, P_\nu = 0 \quad (3)$$

$$[J_{\mu\nu}, \eta_\sigma = i\{g_{\nu\sigma}\eta_\mu - g_{\mu\sigma}\eta_\nu\} \quad (4)$$

$$[\eta_\mu, \eta_\nu = 0 \quad (5)$$

$$[\eta_\mu, P_\nu] = 0 \quad (6)$$

The ten elements P_μ, $J_{\mu\nu}$ are identified with the generators of the Poincaré group. The space inversion acts as an outer automorphism on the Poincaré algebra. It is extended to act on all of $\mathcal{L}_I$ by defining η_μ to be a pseudovector.

We now look for particle-like unitary irreducible representations of $\mathcal{L}_I$ and the parity operator. The four independent invariants of $\mathcal{L}_I$ may be chosen to be

$$Q_1 = P^\mu P_\mu \qquad Q_2 = \eta^\mu\eta_\mu \qquad Q_3 = P^\mu\eta_\mu$$
$$Q_4 = \epsilon^{\mu\nu\lambda\sigma}P_\mu\,\eta_\nu J_{\lambda\sigma} \quad (7)$$

As long as η_μ is not represented by zero, the Pauli-Lubanski operator $W_\sigma W^\sigma$, where $W_\sigma = \frac{1}{2}\epsilon_{\sigma\lambda\mu\nu}P^\lambda J^{\mu\nu}$, is not an invariant of $\mathcal{L}_I$. Hence every faithful unitary representation of $\mathcal{L}_I$ contains a spectrum of representations of the Poincaré algebra with various spins (and a common mass !). The invariant Q_3 is pseudoscalar and unless it vanishes, it will give rise to states of both parity in a representation of $\mathcal{L}_I$ and the parity operator. To avoid this doubling we investigate only representations with $Q_3 = 0$. $Q_1 = P^\mu P_\mu$ is the (degenerate) squared mass of the multiplet and is chosen positive: normalizing it to +1, P_μ lies on the future time-like unit hyper boloid.

To enumerate the spin that appear, we examine the (ideal) states with four momentum $P_0 = +1$, $\underline{P} = 0$: all other states are obtained from these by Lorentz transformation. The corres-

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ponding stabilizer Lie algebra ("Little Group" Lie algebra) is generated by $\underline{J}$ and $\underline{\eta}$ (since η_0 vanishes on these states by virtue of $Q_3 = 0$). This is isomorphic to E(3), the algebra of the three dimensional Euclidean group. The unitary irreducible representations of E(3) are well known; the two invariants of E(3) are Q_2 and Q_4, evaluated for $\underline{P} = 0$. Assuming the representation is faithful, η_μ is not identically zero; hence Q_2 is negative and may be normalized to -1. Q_4 is the "helicity" of the E(3) representations; if it is zero, the spectrum consists of all integral values of the spin, from zero to infinity. In any case, these irreducible representations of $\mathcal{L}_\mathrm{I}$ consist of an infinite sequence of particles with equal mass and parity, and with spins starting at a value determined by Q_4, and going up in integral steps. Each spin appears no more than once. This spectrum for half odd integral spins, is thus reminiscent of the isobar spectrum in the strong-coupling limit of the static neutral pseudoscalar meson theory [4].

Maintaining the mass degeneracy, we can add an internal symmetry to this model in a trivial way by means of a direct product structure. In the isospin case, for example, we introduce six generators I_j, F_j (j = 1,2, 3), commuting with all the elements of $\mathcal{L}_\mathrm{I}$, and obeying among themselves the commutation relations of the algebra of SU(2) ⊗ SU(2), T(3) × SU(2), or SL(2,c). Representations of $\mathcal{L}_\mathrm{I}$ and the generators I_j, F_j are obtained as direct products of a representation of $\mathcal{L}_\mathrm{I}$ and one of I_j, F_j. In no representation of this type will there be any correlation between the spins and the isospins. It would be much more interesting to obtain an algebra including internal symmetry, such that in at least some of its representations there exist definite correlation between the spins and isospins of the particles, as in the strong coupling limit of the static symmetric pseudoscalar theory.

To this end, we consider the Lie algebra $\mathcal{L}_\mathrm{II}$ made up of the twenty-five elements P_μ, $J_{\mu\nu}$, I_j, $\pi_{\mu j}$ (μ = 0, 1, 2, 3; j = 1, 2, 3); the only non-vanishing commutators are (1), (2) and the following

$$[I_j, I_k] \quad = \mathrm{i}\,\epsilon_{jkl} I_l \tag{8}$$

$$[I_j, \pi_{\mu k}] \quad = \mathrm{i}\,\epsilon_{jkl}\, \pi_{\mu l} \tag{9}$$

$$[J_{\mu\nu}, \pi_{\sigma k}] = \mathrm{i}\{g_{\nu\sigma}\pi_{\mu k} - g_{\mu\sigma}\pi_{\nu k}\}. \tag{10}$$

$\pi_{\mu j}$ forms an isovector pseudovector. To determine the spin-isospin spectrum of a representation we start from the independent invariants of $\mathcal{L}_\mathrm{II}$:

$$Q_1 = P^\mu P_\mu\,; \qquad Q_2 = \pi_j^\mu \pi_{\mu j}\,; \qquad Q_3 = T_j\, T_j: \tag{11}$$

where

$$T_j = \pi_{\mu j} P^\mu \tag{12}$$

Once again, we choose $Q_1 = +1$ and let P_μ lie on the unit time-like future hyperboloid. To avoid parity doubling in the representation, we choose $Q_3 = 0$, forcing the pseudoscalars T_j to vanish. This time, the stabilizer Lie algebra for the (ideal) state vectors with $P_0 = 1$, $\underline{P}$, = 0 is generated by the elements I_j, $\underline{J}$ and $\underline{\pi}_j$, which obey the commutation rules of the Lie algebra T(9) × × {SU(2) ⊗ SU(2)}. The representations of this algebra have been studied recently [5,6]. Three interesting kinds of representations may be identified:

(i) The sequence with equal spin and isospin, $I = S$ = all integral or all half odd integral values. Each $I = S$ multiplet appears just once. This sequence is familiar from symmetric pseudoscalar meson theory [4].

(ii) The sequence with the spin S taking on all integral or all half odd integral values; for each S, $I = |S-n|$, $|S-n| + 1, \ldots S+n$, where n is any integer or half odd integer characteristic of the representation. (Representations of type (i) correspond to $n = 0$). The case $n = \frac{1}{2}$ corresponds to the strong coupling sequence for strange hyperons [6]. In this case again each spin-isospin multiplet occurs just once. One also has representations of this type with I and S interchanged. (iii) The sequence with I and S varying independently; S taking on all integral or half odd integral values, and independently all I values being integral or half odd integral. There is here no correlation between spin and isospin. A simple way of generating this kind of representation is to take for $\underline{\pi}_{\mu j}$ the product of a pseudovector a_μ and an isovector b_j.

We have thus exhibited a finite-parameter Lie algebra $\mathcal{L}_\mathrm{II}$ containing the Poincaré algebra, and with unitary irreducible representations which reproduce the isobar spectrum in the strong coupling limit of the symmetric pseudoscalar theory. The extension from isospin to SU_3 is obvious. In accordance with O'Raifeartaigh's theorem [2], there are no mass splittings in such a model.

Mass splittings and Lie algebras. To generate an algebraic framework within which we could obtain mass splittings, we have to construct a richer structure than Lie algebras. We observe that in the strong-coupling approximation to meson theory, the spin-isospin *spectrum* is obtained in the strong coupling limit, but the masses are no longer degenerate for finite coupling strength.

The meson sources have matrix elements between these states of different mass [7]. In the present case we should require that the operators $\underline{\pi}_j$ perform this function in the frame in which the spatial momentum of the isobar is zero. We have to consider a relativistic transcription of this result to allow $\pi_{\mu j}$ to connect states with non-zero spatial momentum. Since the zero three-momentum states essentially determine these matrix elements, the natural method of implementing this transcription is to consider $\pi_{\mu j}$ to commute with the four velocity $M^{-1}P_\mu$ and fail to commute with M in a manner which is completely determined by the matrix elements of the operators between zero three-momentum states.

Accordingly we consider an algebra which includes the elements P_μ, $J_{\mu\nu}$, $\pi_{\mu j}$, I_j and M with the restriction that $M^{-1}P_\mu$ commute with $\pi_{\mu j}$. Following Werle [8], this can be written in this form:

$$P_\mu \pi_{\mu j} M - M \pi_{\nu j} P_\mu = 0 \tag{13}$$

This relation goes beyond the structure of a Lie algebra, as does the relation

$$P^\mu P_\mu - M^2 = 0 \tag{14}$$

To (13) and (14) we add (1), (2), (3), (8), (9), (10) and the commutativity of P_μ, $J_{\mu\nu}$ with I_j. From (14) follows the commutativity of M with P_μ, $J_{\mu\nu}$, I_j. We also require the components of $\pi_{\mu j}$ to commute with one another. At this point we are dealing with an associative algebra $\mathcal{A}$ generated by the twenty six elements P_μ, $J_{\mu\nu}$, $\pi_{\mu j}$, I_j, M modulo the commutation rules and nonlinear relations listed above.

For states at rest, $\underline{P} = 0$, and we may choose $P_0 = M$. The relation (13) states that acting on these states, $\pi_{\mu j}$ reproduces states at rest. Once again we consider the stabilizer algebra of these states. It is generated by the elements $\underline{J}$, $\pi_{\mu j}$ and I_j. Since for the algebra $\mathcal{A}$ the pseudoscalar $T_j = P^\mu \pi_{\mu j}$ may be chosen zero without fear of contradiction, we do so and avoid partity doubling. Then the stabilizer algebra is generated by $\underline{J}$, $\underline{\pi}_j$ and I_j. and this is isomaorphic to the Lie algebra of $T(9) \times \{SU(2) \otimes SU(2)\}$. We have already noted the kinds of spin-isospin spectra that can be obtained from this algebra. It is to be noted that this spectrum has been obtained here without further specification of the mass operator beyond (13).

Having obtained the spin-isospin spectrum, we impose additional nonlinear relationships to obtain a suitable mass spectrum. These relations may be chosen in such a way that $T_j = 0$ is a consequence of them. We require M to obey the following:

$$-iM[\pi_{\mu j}M, P_\lambda]M = b\epsilon_{jkl}M\{I_k, \pi_{\mu l}\}P_\lambda M + a M\{J_{\mu\beta}, \pi_j^\beta\}P_\lambda M + a\{J_{\beta\gamma}, P_\mu\}\pi_j^\beta P^\gamma P_\lambda \tag{15}$$

where a and b are two real constants and $\{\ ,\ \}$ denotes the anti-commutator. By direct computation, we deduce from (15):

$$-iM[\pi_{\mu j}, M]M = b\epsilon_{jkl}M\{I_k, \pi_{\mu l}\}M + aM\{J_{\mu\beta}, \pi_j^\beta\}M + a\{J^{\beta\gamma}, P_\mu\}\pi_j P_\gamma \tag{16}$$

and

$$T_j \equiv \pi_{\mu j}P^\mu = 0 \tag{17}$$

We may further show that in virtue of (17), the element

$$Z \equiv M - a W_\sigma W^\sigma - b I_j I_j$$
$$W_\sigma = \tfrac{1}{2} M^{-1} \epsilon_{\sigma\lambda\mu\nu} P^\lambda J^{\mu\nu} \tag{18}$$

commutes with all elements and must be represented by a number M_0 in every irreducible representation of the algebra $\mathcal{A}$ (in which (17) holds). We thus deduce the mass formula

$$M = M_0 + aS^2 + bI^2 \tag{19}$$

where $I^2 = \mathcal{I}(\mathcal{I}+1)$ and $S^2 = \mathcal{S}(\mathcal{S}+1)$ are the eigenvalues of the squares of isospin and spin.

The restriction to isotopic spin is of course for convenience in illustration. One can generalize the above construction to incorporate SU(3) as the internal symmetry, and choose a non-linear relation analogous to (15) so as to lead to a mass operator showing spin dependence, the octet type symmetry breaking, and SU(3) representation mixing. The strong coupling spectrum in the SU(3) case is also known [9].

Application to mesons. In a similar framework, using an associative algebra with certain non-linear relations postulated, one can arrive at a mass squared formula for the mesons. At the isospin level, such a formula reads:

$$M^2 = M_0^2(Y, SU(3)) + a\,S(S+1) - b\,I(I+1) \tag{20}$$

Here a and b are fixed constants characteristic of the algebra, while M_0^2 contains the dependence on hypercharge Y and the SU(3) representation labels (viz, octet, 27-plet, etc.). To compare with experiment, we tentatively choose $b \approx \frac{1}{2}a \approx 0.1415$ $(\mathrm{GeV})^2$.

The π, ρ, A^2, η, E, S(1930) and U(2380) mesons can be treated as one "multiplet" under this mass formula, with $Y = 0$, SU(3) octet. The spin-isospin quantum numbers for these mesons are not all established but we take them to be $(S,I) = (0,1), (1,1), (2,1), (0,0), (2,0), (3,1)$ and $(4,1)$ respectively. We use the pion mass to fix M_0^2, which gives $M_0^2 \approx 0.30$ (GeV)^2. We then predict the squared masses of the remaining mesons, in the order listed, to be 0.58, 1.74, 0.30, 2.03, 3.5 and 5.8 (GeV)^2. The corresponding experimental [10] values are respectively 0.58, 1.75, 0.30, 2.02, 3.7 and 5.7.

The $Y = +1$, SU(3) octet mesons K, K*(891), K*(1400), and K*(1800) can be grouped together, with a common M_0^2. Using the kaon mass, we find now $M_0^2 \approx 0.35$ (GeV)^2. We then predict the squared masses of the K*'s listed above to be $\approx$ 0.81, 1.99 and 3.6 (GeV)^2; this is in fair agreement with experimental values which give respectively 0.79, 1.99 and 3.4.

The consideration of a mass formula with a more specific dependence on Y and SU(3), and which allows SU(3) representation mixing, will be taken up elsewhere.

Remarks. In the construction of algebras leading to a mass spectrum, we have generated an associative algebra $\mathcal{A}$ with an infinite number of elements. This algebra is not a finite parameter Lie algebra. It is possible, however, by a factorization process, to provide a simple underlying structure in terms of a finite parameter Lie algebra, as follows: Consider the twenty-five parameter Lie algebra $\mathcal{L}_{\text{II}}$ generated by R_μ, $J_{\mu\nu}$, I_j, $\pi_{\mu j}$, write

$$M = M_0 + a\, V_\sigma V^\sigma + b\, I_j I_j \qquad V = \tfrac{1}{2}\, \epsilon_{\sigma\lambda\mu\nu} R^\lambda J^{\mu\nu} \tag{21}$$

where M_0, a, b are constants. If we now construct P_μ according to

$$P_\mu = M R_\mu \quad (R^\mu R_\mu = +1) \tag{22}$$

we obtain the algebra $\mathcal{A}$. Thus while the algebra $\mathcal{A}$ is not itself a finite parameter Lie algebra, it can be simply generated from a finite parameter Lie algebra $\mathcal{L}_{\text{II}}$.

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